S©MR ISSN 1813-3304

СИБИРСКИЕ ЭЛЕКТРОННЫЕ
МАТЕМАТИЧЕСКИЕ ИЗВЕСТИЯ

Siberian Electronic Mathematical Reports
http://semr.math.nsc.ru

Том 13, стр. 1271–1282 (2016) УДК 512.542.7
DOI 10.17377/semi.2016.13.099 MSC 20B25

Special issue: Graphs and Groups, Spectra and Symmetries — G2S2 2016

AUTOMORPHISM GROUPS OF CYCLOTOMIC SCHEMES OVER FINITE NEAR-FIELDS

D.V. CHURIKOV, A.V. VASIL'EV

Abstract. We prove that apart from a finite number of known exceptions the automorphism group of a nontrivial cyclotomic scheme over a finite near-field $\mathbb{K}$ is isomorphic to a subgroup of the group $\mathrm{A\Gamma L}(1, \mathbb{F})$, where $\mathbb{F}$ is a field with $|\mathbb{F}| = |\mathbb{K}|$. Moreover, we obtain that the automorphism group of such a scheme is solvable if the base group of the scheme is solvable.

Keywords: near-field, cyclotomic scheme, automorphism group of a scheme, 2-closure of a permutation group, $\frac{3}{2}$-transitive permutation groups.

1. Introduction

An algebraic structure $\mathbb{K} = \langle \mathbb{K}, +, \circ \rangle$ is called a (right) *near-field* if $\mathbb{K}^+ = \langle \mathbb{K}, + \rangle$ is a group, $\mathbb{K}^\times = \langle \mathbb{K}\backslash\{0\}, \circ \rangle$ is a group, $(x+y)\circ z = x \circ z + y \circ z$ for all $x, y, z \in \mathbb{K}$, and $x \circ 0 = 0$ for all $x \in \mathbb{K}$. Zassenhaus [1] established that finite near-fields are in one-to-one correspondence with finite 2-transitive Frobenius permutation groups. Namely, for every finite near-field $\mathbb{K}$, the permutation group $\mathbb{K}^+ \rtimes \mathbb{K}^\times$ on $\mathbb{K}$, where $\mathbb{K}^+$ acts by right multiplications and $\mathbb{K}^\times$ acts by conjugations, is 2-transitive and Frobenius with kernel $\mathbb{K}^+$ and complement $\mathbb{K}^\times$; and vice versa, for every finite 2-transitive Frobenius group, its kernel and complement give the additive and multiplicative groups of some near-field. This correspondence allowed Zassenhaus to

Churikov, D.V., Vasil'ev A.V., Automorphism groups of cyclotomic schemes over finite near-fields.

The second author was supported by International Mathematical Center of Novosibirsk State University.

Received October, 7, 2016, published December, 23, 2016.

obtain the complete classification of finite near-fields. It turns out that apart from 7 exceptional near-fields (now they are called *Zassenhaus near-fields*), each finite near-field is a *Dickson near-field*, i. e., it can be naturally constructed via an appropriate finite field $\mathbb{F}$ with $\mathbb{F}^+ = \mathbb{K}^+$. We refer to this field $\mathbb{F}$ as the *field associated with* $\mathbb{K}$ (see Sect. 2.1).

Let $\mathbb{K}$ be a finite near-field. Given $K \leq \mathbb{K}^\times$, define $\mathcal{R}_K = \{R_K(a) \mid a \in \mathbb{K}\}$, where

$$R_K(a) = \{(x, y) \in \mathbb{K}^2 \mid y - x \in K \circ a\}.$$

As easily seen, $\mathcal{R}_K$ coincides with the set $\mathrm{Orb}_2(G)$ of the 2-orbits of the group

$$G = G(\mathbb{K}, K) = \{x \mapsto x \circ b + c, x \in \mathbb{K} \mid b \in K, c \in \mathbb{K}^+\} \simeq \mathbb{K}^+ \rtimes K.$$

It follows that the pair $\langle \mathbb{K}, \mathcal{R}_K \rangle$ forms an association scheme in the sense of [2] (that is, commutative but not necessarily symmetric, cf. also [3]). Following [4], we call this scheme the *cyclotomic scheme* over the near-field $\mathbb{K}$ with the *base group* K and denote it by $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$. The rank of $\mathcal{C}$ is equal to $|\mathcal{R}_K|$ and a scheme is called *trivial* if its rank equals 2. Obviously, $\mathcal{C}$ is trivial if and only if its base group coincides with $\mathbb{K}^\times$.

For a given cyclotomic scheme $\mathcal{C}$, all permutations on $\mathbb{K}$ that fix setwise all elements of $\mathcal{R}_K$ form the *automorphism group* $\mathrm{Aut}(\mathcal{C})$ of the cyclotomic scheme $\mathcal{C}$, namely,

$$\mathrm{Aut}(\mathcal{C}) = \{g \in \mathrm{Sym}(\mathbb{K}) \mid R^g = R \text{ for all } R \in \mathcal{R}_K\}.$$

By definition, $\mathrm{Aut}(\mathcal{C})$ coincides with the 2-closure of the permutation group $G(\mathbb{K}, K)$ (see Sect. 2.2 and 2.3).

If a cyclotomic scheme $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$ is trivial, then $\mathcal{R}_K$ consists of two relations (the diagonal of $\mathbb{K}^2$ and its complement), so $\mathrm{Aut}(\mathcal{C}) = \mathrm{Sym}(\mathbb{K})$. Further we are interested only in nontrivial schemes.

If $\mathbb{K} = \mathbb{F}$ is a field, then we come to cyclotomic schemes introduced by P. Delsarte in [2]. A consequence of the main theorem in [5] is that the automorphism group of a nontrivial scheme over a field $\mathbb{F}$ is a subgroup of

$$\mathrm{A\Gamma L}(1, \mathbb{F}) = \{x \mapsto x^\sigma \cdot b + c, x \in \mathbb{F} \mid \sigma \in \mathrm{Aut}(\mathbb{F}), b \in \mathbb{F}^\times, c \in \mathbb{F}\}$$

(see, e. g., the remark after Prop. 12.7.5 in [6], where it was also proved that such schemes are pseudocyclic).

Cyclotomic schemes over near-fields (which are also pseudocyclic due to [7, Theorem 1.1] and the fact that they arise from Frobenius groups) were introduced in [4]. The main result [4, Theorem 1.1] of that paper yields that $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{AGL}(V_\mathbb{K})$, where $V_\mathbb{K}$ is the vector space over the prime field lying in the center of $\mathbb{K}$ and $V_\mathbb{K} = \mathbb{K}^+$ (note that $\mathbb{K}^+$ is always elementary abelian for finite near-fields). Moreover, for Dickson near-fields $\mathbb{K}$ with some restrictions on the order of the base group K, it was proved [4, Theorem 1.3] that $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, \mathbb{F})$, where $\mathbb{F}$ is the field associated with $\mathbb{K}$. It was also conjectured that the same is true for all finite near-fields apart from a finite number of possible exceptions. The goal of the present paper is to validate this conjecture.

Theorem. *Let* $\mathbb{K}$ *be a finite near-field,* K *a proper subgroup of* $\mathbb{K}^\times$*. If* $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$ *is the cyclotomic scheme over* $\mathbb{K}$ *with the base group* K*, then exactly one of the following hold:*

(1) $\mathbb{K}$ *is a Dickson near-field and* $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, \mathbb{F})$*, where* $\mathbb{F}$ *is the finite field associated with* $\mathbb{K}$*.*

(2) $\mathbb{K}$ *is the Dickson near-field of order* 7^2, $K = \langle a, b\rangle \simeq 3 \times Q_8$, *and* $\mathrm{Aut}(\mathcal{C}) = \mathbb{K}^+ \rtimes H$, *where* $H = \langle K, c\rangle \simeq 3 \times \mathrm{SL}(2,3)$, *and the action of* a, b, *and* c *on* $\mathbb{K}^+$ *is represented respectively by the matrices*

$$\begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

(3) $\mathbb{K}$ *is a Zassenhaus near-field,* K *is a subgroup of* M, *where* M *is a maximal solvable subgroup of* $\mathbb{K}^\times$, $\mathrm{Aut}(\mathcal{C})$ *is a subgroup of* $\mathbb{K}^+ \rtimes H$, *where* $K \leq M \leq H$. *The groups* $\mathbb{K}^+$, M, H, *and generators of* M *and* H *are listed in Table 4.1 of Appendix.*

(4) $\mathbb{K}$ *is a Zassenhaus near-field of order either* 29^2 *or* 59^2, $K \simeq \mathrm{SL}(2,5)$, *and either* $\mathrm{Aut}(\mathcal{C}) = 29^2 \rtimes (\mathrm{SL}(2,5) \rtimes 2)$ *or* $\mathrm{Aut}(\mathcal{C}) = 59^2 \rtimes \mathrm{SL}(2,5)$, *respectively. The groups* $\mathbb{K}^+$, K, H, *and generators of* K *and* H *are listed in Table 4.2 of Appendix.*

In particular, if the base group K *is solvable, then so is* $\mathrm{Aut}(\mathcal{C})$.

It is worth mentioning that the main reason why we were able to proceed further than the authors of [4] is the recently completed classification of $\frac{3}{2}$-transitive permutation groups [8] (see Sect. 2.4)

As the automorphism group of a scheme $\mathrm{Cyc}(\mathbb{K}, K)$ coincides with the 2-closure of the permutation group $G(\mathbb{K}, K)$, our result can be considered in the context of the computational 2-closure problem: Given a finite permutation group G, find its 2-closure $G^{(2)}$. This problem was solved for nilpotent groups [9] and groups of odd order [10] in time polynomial in the degree of G by using a technique from [11] and the fact that the 2-closures of such groups are solvable [12, § 5]. In our case, for every proper subgroup G of a 2-transitive Frobenius group of degree m containing its kernel, we found the subgroup H of $\mathrm{Sym}(m)$ that has order polynomial in m and contains $G^{(2)}$. This implies that using the same technique one can find H and, consequently, $G^{(2)}$ in time polynomial in m.

Notations. In the paper, $A \times B$, $A \rtimes B$, and $A.B$ stand for a direct product, a semidirect product, and a non-split extension of a group A by a group B, respectively. We denote the (generalized) quaternion group and quasidihedral group of order n by Q_n and QD_n, respectively. For a prime p and a positive integer n, we briefly write p^n for the elementary abelian p-group of order p^n. We also use the notation $(n)_p$ for the p-part of n, that is the largest power of p dividing n.

2. Preliminaries

2.1. **Finite near-fields.** Let $\mathbb{K}$ be a finite near-field. Then the group $\mathbb{K}^+$ is always elementary abelian, and the group $\mathbb{K}^\times$ is abelian if and only if $\mathbb{K}$ is a field (as to near-fields theory, we refer to [13]).

A finite near-field $\mathbb{K}$ is called a *Dickson near-field*, if exists a finite field $\mathbb{F}_0$ of order $q_0 = p^l$ and its extension $\mathbb{F}$ of order q_0^n such that $\mathbb{F}^+ = \mathbb{K}^+$ and the multiplication in $\mathbb{K}$ can be represented in the following way:

$$y \circ x = y^{\sigma_x} \cdot_{\mathbb{F}} x, \quad x, y \in \mathbb{K}, \sigma_x \in \mathrm{Aut}(\mathbb{F}/\mathbb{F}_0),$$

where $\cdot_{\mathbb{F}}$ denotes the multiplication in $\mathbb{F}$. The field $\mathbb{F}$ is said to be *associated with* $\mathbb{K}$. It is known that such a near-field exists whenever the pair (q_0, n) forms a *Dickson pair*, i. e., every prime factor of n is a divisor of $q_0 - 1$ and $4|n$ implies $4|(q_0 - 1)$. In fact, for every Dickson pair (q_0, n), there are $\varphi(n)/k$ nonisomorphic Dickson near-fields, where k is the order of p modulo n.

As mentioned, apart from the Dickson near-fields there are 7 exceptional so-called Zassenhaus finite near-fields. For every such near-field, Table 2.1 contains the order of $\mathbb{K}$ and generators of $\mathbb{K}^\times$ as linear transformations of $\mathbb{K}^+$ (see, e.g., [13, p. 373].

Table 2.1. Zassenhaus near-fields

$\lvert\mathbb{K}\rvert$	generators of $\mathbb{K}^\times$
5^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix}$
11^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ -5 & -2 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
7^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$
23^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -6 \\ 12 & -2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
11^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 1 & -3 \end{pmatrix}$
29^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -7 \\ 12 & -2 \end{pmatrix}, \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$
59^2	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 9 & 15 \\ -10 & -10 \end{pmatrix}, \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

Thus, the classification of finite near-fields is given by the following

Lemma 1 ([1])**.** *A finite near-field is either a Dickson near-field, or a Zassenhaus near-field.*

2.2. **2-closure of permutation groups.** Let G be a permutation group on a finite set Ω. The action of G on Ω induces the componentwise action on Ω^2: $(\alpha,\beta)^g = (\alpha^g, \beta^g)$ for all α and β from Ω and $g \in G$. Denote the set of orbits of this action on Ω^2 by $\mathrm{Orb}_2(G)$. The elements of $\mathrm{Orb}_2(G)$ are called the 2-*orbits* of G. The largest subgroup of $\mathrm{Sym}(\Omega)$ with the same 2-orbits as G is called the 2-*closure* of G and denoted by $G^{(2)}$. Equivalently, $G^{(2)}$ is the automorphism group of $\mathrm{Orb}_2(G)$:

$$G^{(2)} = \mathrm{Aut}(\mathrm{Orb}_2(G)) = \{g \in \mathrm{Sym}(\Omega) \mid O^g = O, O \in \mathrm{Orb}_2(G)\}.$$

In particular, $G \leq G^{(2)}$. A group G is called 2-*closed* if $G = G^{(2)}$.

If G is transitive then $\{(\alpha,\alpha) \mid \alpha \in \Omega\}$ is a 2-orbit. We refer to such an orbit as *trivial* and to any other 2-orbits as *nontrivial*.

Lemma 2 ([12, Theorem 5.7])**.** *If $H \leq G \leq \mathrm{Sym}(\Omega)$ then $H^{(2)} \leq G^{(2)}$.*

There is a natural bijection between the 2-orbits and the orbits of a point stabilizer.

Lemma 3 ([12, p. 4])**.** *If G is a transitive permutation group on Ω, then there exists a bijection between the 2-orbits of G and the orbits of a point stabilizer G_α, namely, for $\alpha, \beta \in \Omega$, the 2-orbit $(\alpha,\beta)^G$ of G corresponds to the orbit β^{G_α} of G_α, and $|(\alpha,\beta)^G| = |\Omega||\beta^{G_\alpha}|$.*

Recall that a transitive permutation group G is called a *Frobenius* group if a one-point stabilizer is nontrivial and the stabilizer of any two distinct points is trivial.

Lemma 4. *Let G be a permutation group such that $G^{(2)}$ is a Frobenius group. Then G is 2-closed.*

Proof. Lemma 3 yields that

$$|G_\alpha : G_{\alpha\beta}| = |\beta^{G_\alpha}| = \frac{|(\alpha,\beta)^G|}{|\Omega|} = \frac{|(\alpha,\beta)^{G^{(2)}}|}{|\Omega|} = |\beta^{(G^{(2)})_\alpha}| = |(G^{(2)})_\alpha : (G^{(2)})_{\alpha\beta}|$$

for $\alpha, \beta \in \Omega$ with $\alpha \neq \beta$. Since $G^{(2)}$ is Frobenius and $G \leq G^{(2)}$, both G and $G^{(2)}$ are transitive. Hence $|G : G_\alpha| = |\Omega| = |G^{(2)} : (G^{(2)})_\alpha|$. Furthermore, $G_{\alpha\beta} \leq (G^{(2)})_{\alpha\beta} = 1$, so $|G| = |G^{(2)}|$. Thus $G = G^{(2)}$, as required. □

Lemma 5 ([14, Theorem 2.5.8])**.** *An imprimitive Frobenius group is 2-closed.*

2.3. **Cyclotomic schemes over finite near-fields and their automorphisms.** Let $\mathbb{K}$ be a finite near-field, $K \leq \mathbb{K}^\times$, $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K) = \langle \mathbb{K}, \mathcal{R}_K \rangle$. As observed, $\mathcal{R}_K$ coincides with $\mathrm{Orb}_2(G)$, where $G = G(\mathbb{K}, K)$. It follows that $G \leq \mathrm{Aut}(\mathcal{C}) = \mathrm{Aut}(\mathrm{Orb}_2(G)) = G^{(2)}$, so to find the automorphism group of a cyclotomic scheme $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$ it suffices to find the 2-closure of the group $G = G(\mathbb{K}, K)$.

Zassenhaus [1] proved that $G(\mathbb{K}, \mathbb{K}^\times)$ is a 2-transitive Frobenius group for every near-field $\mathbb{K}$. So the 2-closure of $G(\mathbb{K}, \mathbb{K}^\times)$, as well as $\mathrm{Aut}(\mathcal{C}(\mathbb{K}, \mathbb{K}^\times))$, is equal to $\mathrm{Sym}(\mathbb{K})$. Furthermore, if K is a non-trivial subgroup of $\mathbb{K}^\times$ then $G(\mathbb{K}, K)$ is also Frobenius.

Lemma 6. *Let $\mathbb{K}$ be a Dickson near-field and $K \leq \mathbb{K}^\times$. Then $G(\mathbb{K}, K) \leq \mathrm{A\Gamma L}(1, \mathbb{F})$, where $\mathbb{F}$ is the field associated with $\mathbb{K}$.*

Proof. By the definition of the multiplication in the Dickson near-fields,

$$G = G(\mathbb{K}, K) = \{x \mapsto x \circ b + c, x \in \mathbb{K} \mid b \in K, c \in \mathbb{K}^+\} =$$

$$= \{x \mapsto x^{\sigma_b} \cdot_{\mathbb{F}} b + c \mid x \in \mathbb{K}, b \in K, c \in \mathbb{K}^+, \sigma_b \in \mathrm{Aut}(\mathbb{F}/\mathbb{F}_0)\} \leq \mathrm{A\Gamma L}(1, \mathbb{F}),$$

as required. □

The next lemma follows from the main result in [5].

Lemma 7. *If $\mathcal{C} = \mathrm{Cyc}(\mathbb{F}, K)$ is a nontrivial cyclotomic scheme over a finite field $\mathbb{F}$, then $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, \mathbb{F})$.*

Lemma 8. *Let $\mathbb{K}$ be a Dickson near-field. Suppose that a non-trivial subgroup K of $\mathbb{K}^\times$ is abelian and $G = G(\mathbb{K}, K)$ is a primitive permutation group. Then $\mathcal{C} = \mathcal{C}(\mathbb{K}, K)$ is a cyclotomic scheme over the field $\mathbb{F}$ associated with $\mathbb{K}$.*

Proof. Since G is a primitive Frobenius group, the base group K acts irreducibly on $\mathbb{K}^+$ as a group of linear transformations. Now the lemma follows from [4, Theorems 2.2 and 2.4]. □

2.4. **$\frac{3}{2}$-transitive permutation groups.** Recall that a permutation group on Ω is $\frac{1}{2}$-*transitive* if all its orbits on Ω have the same size greater than one. A permutation group on Ω is $\frac{3}{2}$-*transitive* if it is transitive and a stabilizer of a point $\alpha \in \Omega$ is $\frac{1}{2}$-transitive on $\Omega \setminus \{\alpha\}$ (see [15, § 10]).

Lemma 9. *Let $\mathbb{K}$ be a near-field and $1 \neq K \leq \mathbb{K}^\times$. Then $G = G(\mathbb{K}, K)$ and $G^{(2)}$ are $\frac{3}{2}$-transitive.*

Proof. The groups G and $G^{(2)}$ are transitive, because G is Frobenius and $G \leq G^{(2)}$. Since $\mathcal{R}_K = \mathrm{Orb}_2(G) = \mathrm{Orb}_2(G^{(2)})$ and all non-trivial relations from $\mathcal{R}_K$ are of the same size, Lemma 3 implies that all the orbits of G_α and, consequently, all orbits of $(G^{(2)})_\alpha$ on $\mathbb{K} \setminus \{\alpha\}$ have the same size. They cannot be of size one, because K is non-trivial. □

The next two assertions from [8] give the classification of finite $\frac{3}{2}$-transitive permutation groups and $\frac{1}{2}$-transitive linear groups.

Lemma 10 ([8, Corollary 3])**.** *Let G be a $\frac{3}{2}$-transitive permutation group of degree n. Then one of the following holds:*

(a) *G is 2-transitive,*
(b) *G is a Frobenius group,*
(c) *G is almost simple: either*
 (i) *$n = 21$, $G = A_7$ or S_7 acting on the set of pairs in $\{1, ..., 7\}$, or*
 (ii) *$n = \frac{1}{2}q(q-1)$, where $q = 2^f \geq 8$, and either $G = \mathrm{PSL}(2, q)$, or $G = \mathrm{P\Gamma L}(2, q)$ with f prime,*
(d) *G is of affine type: $G = NH \leq \mathrm{AGL}(V)$, where N is group of translations of vector space V, $H \leq \mathrm{GL}(V)$, and H is a $\frac{1}{2}$-transitive group, given by Lemma 11.*

Lemma 11 ([8, Corollary 2])**.** *If $H \leq \mathrm{GL}(V) = \mathrm{GL}(d, p)$ is $\frac{1}{2}$-transitive on $V^\sharp = V \setminus \{\bar{0}\}$, then one of the following holds:*

(a) *H is transitive on $V^\sharp$,*
(b) *$H \leq \Gamma\mathrm{L}(1, p^d)$,*
(c) *H is a Frobenius complement acting semiregularly on $V^\sharp$,*
(d) *$H = S_0(p^{d/2})$ with p odd,*
(e) *H is solvable and $p^d = 3^2, 5^2, 7^2, 11^2, 17^2$ or 3^4,*
(f) *$\mathrm{SL}(2, 5) \lhd H \leq \Gamma\mathrm{L}(2, p^{d/2})$, where $p^{d/2} = 9, 11, 19, 29$ or 169.*

In the lemma, $S_0(p^{d/2})$ stands for the subgroup of $\mathrm{GL}(2, p^{d/2})$ of order $4(p^{d/2}-1)$ consisting of all monomial matrices of determinant ± 1.

Lemma 12. [15, Theorem 10.4] *If G is a finite $\frac{3}{2}$-transitive permutation group, then G is either primitive, or Frobenius.*

The socle $\mathrm{Soc}(G)$ of a group G is the subgroup generated by all its minimal normal subgroups.

Lemma 13. *Let G be a $\frac{3}{2}$-transitive but not 2-transitive permutation group of affine type. Then $\mathrm{Soc}(G) = \mathrm{Soc}(G^{(2)})$.*

Proof. According to Lemma 12, every $\frac{3}{2}$-transitive group is either primitive or Frobenius. If G is an imprimitive Frobenius group, then $G = G^{(2)}$ due to Lemma 5, so the lemma follows trivially. If G is primitive, then $\mathrm{Soc}(G) = \mathrm{Soc}(G^{(2)})$ by [4, Theorem 3.2]. □

3. Proof of the theorem

Let $\mathbb{K}$ be a finite near-field of order q, $q = p^d$, p a prime, K a proper subgroup of $\mathbb{K}^\times$, $G = G(\mathbb{K}, K)$, and $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$.

Recall that $\mathrm{Aut}(\mathcal{C}) = G^{(2)}$. If $K = 1$ then G is regular, so $G^{(2)} = G \simeq \mathbb{K}^+$ and the theorem obviously holds. Therefore, we may further assume that K is nontrivial. By Lemma 9, both G and $G^{(2)}$ are $\frac{3}{2}$-transitive. Furthermore, they cannot be 2-transitive because $K \neq \mathbb{K}^\times$, so Lemma 13 yields $\mathrm{Soc}(G^{(2)}) = \mathrm{Soc}(G) \simeq \mathbb{K}^+$. It follows that both $G = \mathbb{K}^+ \rtimes K$ and $G^{(2)} = \mathbb{K}^+ \rtimes H$ are of affine type, where $K, H \leq \mathrm{GL}(V_\mathbb{K})$, so K and H are $\frac{1}{2}$-transitive on $\mathbb{K}^+ \setminus \{0\}$. Moreover, $K \leq H$ because $G \leq G^{(2)}$.

Lemma 1 implies that $\mathbb{K}$ is either a Dickson near-filed, or a Zassenhaus near-field, and we deal with these two cases separately.

3.1. **Dickson near-fields.** Let $\mathbb{K}$ be a Dickson near-field corresponding to a Dickson pair (q_0, n), where $q_0 = p^l$, $\mathbb{F}_0$ the central subfield of $\mathbb{K}$ of order q_0, and $\mathbb{F}$ the field of order $q = q_0^n$ associated with $\mathbb{K}$ (see Sect. 2.1). Here we are going to prove that apart from the one particular exception, $\mathrm{Aut}(\mathcal{C}) = G^{(2)} \leq \mathrm{A\Gamma L}(1, \mathbb{F})$. Observe that if G is imprimitive, then $G^{(2)}$, being Frobenius by Lemma 12, is 2-closed, so in this case $G^{(2)}$ lies in $\mathrm{A\Gamma L}(1, \mathbb{F})$ due to Lemma 6. Further we suppose that G is primitive. We may also assume that K is nonabelian, otherwise we are done by Lemmas 7 and 8. In particular, we may assume that $\mathbb{K}$ is not a field, so $q_0 < q$ and $n > 1$.

The inclusion $G^{(2)} \leq \mathrm{A\Gamma L}(1, \mathbb{F})$ holds if and only if $H \leq \Gamma\mathrm{L}(1, \mathbb{F})$. Since H acts $\frac{1}{2}$-transitively on $\mathbb{K}^+ \setminus \{0\}$, it appears in one of the cases of Lemma 11. Let us consider them in turn.

(a) If H is transitive on $\mathbb{K}^+ \setminus \{0\}$, then both $G^{(2)}$ and G are 2-transitive, so $K = \mathbb{K}^\times$, a contradiction.

(b) If $H \leq \Gamma\mathrm{L}(1, \mathbb{F})$ then we are obviously done.

(c) If H is a Frobenius complement acting semiregularly on $\mathbb{K}^+ \setminus \{0\}$, then $G^{(2)}$ is a Frobenius group, so it is 2-closed in view of Lemma 4. Therefore, $G^{(2)} = G \leq \mathrm{A\Gamma L}(1, \mathbb{F})$, as required, where the last inclusion follows from Lemma 6.

(d) Let $H = S_0(u) \leq \mathrm{GL}(2, u)$, $u = p^c$, $q = u^2$, p odd. The group H is solvable, has the order $4(u-1)$ and, as a subgroup of $\mathrm{GL}(2, \mathbb{F}_u)$, can be represented in the following way (see [16]):

$$H = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \pm\alpha^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ \pm\alpha^{-1} & 0 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{F}_u^\times \right\}.$$

As easily seen, each nontrivial orbit of H on $\mathbb{K}^+$ is of size $2(u-1)$.

Since K, as a Frobenius complement, acts semiregularly on $\mathbb{K}^+$, it is a proper subgroup of H of order $2(u-1)$. Furthermore, K does not contain the element

$$t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which fixes the non-zero vector $(1, 1)$ (we consider this vector as an element of $\mathbb{K}^+$).

Since $t \in H$, the vectors $(1, 0)$ and $(0, 1)$, as elements of $\mathbb{K}^+$, lie in the same orbit of H, so K contains the elements

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } -s.$$

It follows that K, as a subgroup of H, can be represented as follows:

$$K = \left\langle \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

where θ generates $\mathbb{F}_u^\times$.

If $u = p = 3$ then $q = 9$. However, if $q = 9$ then K is a cyclic group of order 4, so $G^{(2)} \leq \mathrm{A\Gamma L}(1, \mathbb{F})$ and we are done. Therefore, we may assume that $q > 9$, whence K is nonabelian and $Z(K)$ is of order 2.

The central subfield $\mathbb{F}_0$ of $\mathbb{K}$ has the order q_0 (see, e. g., [13, pp. 370–371]), so $|Z(\mathbb{K}^\times)| = q_0 - 1 = p^l - 1$. Suppose that r is an odd prime dividing $p^{(l,c)} - 1 = (q_0 - 1, u - 1)$. Then $(u - 1)_r = (q - 1)_r$, so a Sylow r-subgroup of K is a Sylow r-subgroup of $\mathbb{K}^\times$. Since $Z(K) \geq K \cap Z(\mathbb{K}^\times)$, it follows that r divides $|Z(K)|$, which is impossible. Thus, p is a Fermat prime, and either $(l, c) = 1$, or $p = 3$ and $(l, c) = 2$.

If $p > 3$ or $(l, c) = 2$, then 4 divides $u - 1$, so $2(u - 1)_2 = (q - 1)_2$ and a Sylow 2-subgroup of K is a Sylow 2-subgroup of $\mathbb{K}^\times$. It follows that $Z(K)$ is a multiple of 4, a contradiction. Thus, $p = 3$ and $(l, c) = 1$. Together with the equalities $3^{ln} = q_0^n = q = u^2 = 3^{2c}$, this leaves us with two possibilities: either $l = 1$ and $n = 2c$, or $l = 2$ and $n = c$. Taking into account that (q_0, n) is a Dickson pair (see Sect. 2.1), we easily derive that $l = 1$ and $n = 2$, so $q = 9$, which contradicts our assumption.

(e) H is solvable and q is one of the numbers $3^2, 3^4, 5^2, 7^2, 11^2$, and 17^2.

We are going to check the inclusion $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, \mathbb{F})$. As mentioned above, this is always true when K is abelian, so we may restrict ourself to the case of nonabelian base groups. Moreover, if the inclusion holds for all maximal subgroups of $\mathbb{K}^\times$, then it holds for all proper subgroups of $\mathbb{K}^\times$ in view of Lemma 2. Since we have only finite number of possibilities for $\mathbb{K}$, our strategy will be as follows.

Using MAGMA [17], we obtain permutation group $\mathbb{K}^+ \rtimes \mathbb{K}^\times$. Then applying the package IRREDSOL for GAP [18], we get $\mathbb{K}^\times$ as a group of linear transformations of $V_\mathbb{K}$. In the next step, with the help of GAP, we construct the group $Q \leq \mathrm{GL}(V_\mathbb{K})$ such that $Q \simeq \Gamma\mathrm{L}(1, \mathbb{F})$ and $\mathbb{K}^\times \leq Q$. Then we find all (up to conjugation) nonabelian maximal subgroups M of $\mathbb{K}^\times$ and for every such M, we construct the permutation group $\mathbb{K}^+ \rtimes M$ by the right regular action. Finally, using the package COCO for GAP [19], we find the 2-closure of $\mathbb{K}^+ \rtimes M$ and check its inclusion in $\mathbb{K}^+ \rtimes Q$.

Let $|\mathbb{K}| = 3^2$. There is only one Dickson field of this order and it corresponds to the Dickson pair $(3, 2)$. In this case, $\mathbb{K}^\times \simeq Q_8$. All maximal subgroups of Q_8 are abelian, so for every proper subgroup of $\mathbb{K}^\times$, the inclusion $\mathrm{Aut}(\mathcal{C}) \leq \mathrm{A\Gamma L}(1, \mathbb{F})$ holds.

Suppose $|\mathbb{K}| = 3^4$. Since $(3, 4)$ is not a Dickson pair, again there is just one Dickson field $\mathbb{K}$ of such order and it corresponds to the pair $(9, 2)$. In this case, $\mathbb{K}^\times \simeq 5 \rtimes 16$ and all maximal subgroups of $\mathbb{K}^\times$ are abelian.

Since $n = 2$ for all remaining cases, we always have the only Dickson near-field for each possible q.

If $q = 5^2$ then $\mathbb{K}^\times \simeq 3 \rtimes 8$. Again all maximal subgroups of $\mathbb{K}^\times$ are abelian.

Assume that $q = 7^2$. In this case, $\mathbb{K}^\times \simeq 3 \times Q_{16}$. Representatives of the conjugacy classes of maximal subgroups of $\mathbb{K}^\times$ are M_1, M_2, and M_3, where $M_1 \simeq Q_{16}$, $M_2 \simeq M_3 \simeq 3 \times Q_8$ (in fact, M_2 and M_3, as well as their 2-closures, are conjugated in $\mathrm{GL}(V_\mathbb{K})$). If $G = G(\mathbb{K}, M_1)$, then $G^{(2)} \simeq 7^2 \rtimes (Q_{16} \rtimes 2)$ and $G^{(2)} \leq \mathrm{A\Gamma L}(1, \mathbb{F})$. In the case $G = G(\mathbb{K}, M_2)$ (or $G = G(\mathbb{K}, M_3)$), we have $G^{(2)} = 7^2 \rtimes (3 \times \mathrm{SL}(2, 3))$, and the statement (2) of the theorem holds. Note that $G^{(2)} \not\leq \mathrm{A\Gamma L}(1, \mathbb{F})$, as the order of $G^{(2)}$ does not divide the order of $\mathrm{A\Gamma L}(1, 7^2)$. It is worth mentioning that this

2-closure appears as a subgroup of index 2 in the exceptional solvable 2-transitive group of degree 49 (all such groups were classified in [20]). If K is a proper subgroup of either M_2 or M_3, then it is conjugated to a subgroup of M_1, so $G^{(2)} \leq \mathrm{A\Gamma L}(1, \mathbb{F})$ for all such K.

Let $q = 11^2$. In this case, $\mathbb{K}^\times \simeq 5 \times (3 \rtimes Q_8)$. Representatives of the conjugacy classes of nonabelian maximal subgroups and the corresponding 2-closures are as follows:

$M_1 \simeq 5 \times (3 \rtimes 4)$ and $G^{(2)} = G$;
$M_2 \simeq M_1$ and $G^{(2)} = G$;
$M_3 \simeq 3 \rtimes Q_8$ and $G^{(2)} = 11^2 \rtimes (24 \rtimes 2) \leq \mathrm{A\Gamma L}(1, 11^2)$;
$M_4 \simeq 5 \times Q_8$ and $G^{(2)} = G$.

If $q = 17^2$ then $\mathbb{K}^\times \simeq 9 \rtimes 32$. There is only one class of nonabelian maximal subgroups of $\mathbb{K}^\times$ with the representative $M \simeq 3 \rtimes 32$ and $G^{(2)} = G$ for it.

(f) $\mathrm{SL}(2, 5) \lhd H \leq \Gamma\mathrm{L}(2, p^{d/2})$, where $p^{d/2} = 9, 11, 19, 29$, or 169. Similarly, for all possible Dickson near-fields $\mathbb{K}$ and all nonabelian maximal subgroups M of $\mathbb{K}^\times$, we construct $G = G(\mathbb{K}, M)$ and find $G^{(2)}$, thereby proving that $G^{(2)} = \mathbb{K}^+ \rtimes H$ is solvable, so H does not contain $\mathrm{SL}(2, 5)$.

If $q = p^d$ is equal to 3^4 or 11^2, then the required assertion was proved above.

Suppose that $q = 19^2$. In this case, $\mathbb{K}^\times \simeq 9 \times (5 \rtimes Q_8)$. Representatives of the conjugacy classes of nonabelian maximal subgroups of $\mathbb{K}^\times$ and the corresponding 2-closures are as follows:

$M_1 \simeq 9 \times (5 \rtimes 4)$ and $G^{(2)} = G$;
$M_2 \simeq M_1$ and $G^{(2)} = G$;
$M_3 \simeq 3 \times (5 \rtimes Q_8)$ and $G^{(2)} = 19^2 \rtimes (3 \times (40 \rtimes 2))$;
$M_4 \simeq 9 \times Q_8$ and $G^{(2)} = G$.

For $q = 29^2$, we have $\mathbb{K}^\times \simeq 7 \times (15 \rtimes 8)$. Representatives of the conjugacy classes of nonabelian maximal subgroups and the corresponding 2-closures are:

$M_1 \simeq 15 \rtimes 8$ and $G^{(2)} = 29^2 \rtimes (120 \rtimes 2)$;
$M_2 \simeq 7 \times (5 \rtimes 8)$ and $G^{(2)} = G$;
$M_3 \simeq M_2$ and $G^{(2)} = G$.

Let $q = 13^4$. There exist three nonisomorphic Dickson near-fields of this order.

For the Dickson pair $(13, 4)$, there are 2 nonisomorphic near-fields $\mathbb{K}_1$ and $\mathbb{K}_2$. However, $\mathbb{K}_1^\times \simeq \mathbb{K}_2^\times \simeq 3 \times (595 \rtimes 16)$, and $M_i^1 \simeq M_i^2$, $i \in \{1..5\}$ for the corresponding representatives of the conjugacy classes of nonabelian maximal subgroups. Here we have

$M_1^1 \simeq M_1^2 \simeq 21 \times (85 \rtimes 8)$ and $G^{(2)} = 13^4 \rtimes (3 \times (4760 \rtimes 4))$;
$M_2^1 \simeq M_2^2 \simeq 595 \rtimes 16$ and $G^{(2)} = 13^4 \rtimes (9520 \rtimes 4)$;
$M_3^1 \simeq M_3^2 \simeq 3 \times (119 \rtimes 16)$ and $G^{(2)} = G$;
$M_4^1 \simeq M_4^2 \simeq 3 \times (85 \rtimes 16)$ and $G^{(2)} = 13^2 \rtimes (3 \times ((85 \rtimes 16) \rtimes 2))$;
$M_5^1 \simeq M_5^2 \simeq 3 \times (35 \rtimes 16)$ and $G^{(2)} = G$.

For the Dickson pair $(169, 2)$, $\mathbb{K}^\times \simeq 21 \times (85 \rtimes 16)$. Representatives of the conjugacy classes of nonabelian maximal subgroups and the corresponding 2-closures are:

$M_1 \simeq 7 \times (85 \rtimes 16)$ and $G^{(2)} = 13^4 \rtimes (9520 \rtimes 4)$;
$M_2 \simeq 3 \times (85 \rtimes 16)$ and $G^{(2)} = 13^4 \rtimes (3 \times (1360 \rtimes 16))$;
$M_3 \simeq 21 \times (17 \rtimes 16)$ and $G^{(2)} = G$;
$M_4 \simeq 21 \times (5 \rtimes 16)$ and $G^{(2)} = G$.

All 2-closures that we found here turn out to be solvable, so H does not contain $\mathrm{SL}(2,5)$, as stated.

Finally, if $\mathbb{K}$ is a Dickson near-field, K is a proper subgroup of $\mathbb{K}^\times$, then statements (1) and (2) of the theorem hold. In particular, the automorphism group of the nontrivial scheme $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$ is solvable for every Dickson near-field $\mathbb{K}$.

3.2. **Zassenhaus near-fields.** Let $\mathbb{K}$ be a Zassenhaus near-field. Suppose first that K is a solvable subgroup of $\mathbb{K}^\times$. By Lemma 2, it suffices to find the 2-closures of the groups $G = G(\mathbb{K}, M)$, where M is a maximal solvable subgroup of $\mathbb{K}^\times$ (a solvable subgroup that is not contained in any other proper solvable subgroup). For every such M (up to conjugation), we obtain the permutation group G using MAGMA and find its 2-closure with the help of the package COCO for GAP, thereby proving the statement (3) of the theorem. Results are listed in Table 4.1 of Appendix. In particular, it follows that for every finite near-field $\mathbb{K}$ and every proper solvable subgroup K of $\mathbb{K}^\times$, the automorphism group of the scheme $\mathcal{C} = \mathrm{Cyc}(\mathbb{K}, K)$ is solvable.

Let, finally, consider the cyclotomic schemes $\mathrm{Cyc}(\mathbb{K}, K)$ with a nonsolvable base group K. In fact, there are only two such possibilities.

If K is the Zassenhaus near-field of order 29^2 then $\mathbb{K}^\times \simeq 7 \times \mathrm{SL}(2,5)$. There is, up to conjugacy, only one proper nonsolvable subgroup $K \simeq \mathrm{SL}(2,5)$ of $\mathbb{K}^\times$, and we have $G^{(2)} = 29^2 \rtimes (\mathrm{SL}(2,5) \rtimes 2)$.

If $\mathbb{K}$ is the Zassenhaus near-field of order 59^2 then $\mathbb{K}^\times \simeq 29 \times \mathrm{SL}(2,5)$. The only, up to conjugacy, proper nonsolvable subgroup K of $\mathbb{K}^\times$ is isomorphic to $\mathrm{SL}(2,5)$. In this case, $G^{(2)} = G$.

Thus, the statement (4) of the theorem holds (we summarize these results in Table 4.2 of Appendix). This completes the proof of the theorem.

4. Appendix

Table 4.1. The automorphism groups of schemes over Zassenhaus near-fields with maximal solvable base group

$\mathbb{K}^+$	M	H	Generators of M	Generators of H
5^2	Q_8	$(4 \times 2) \rtimes 2$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$	$M, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
5^2	6	D_{12}	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$M, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
11^2	$5 \times Q_8$	$5 \times Q_8$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	M
11^2	$\mathrm{SL}(2,3)$	$\mathrm{GL}(2,3)$	$\begin{pmatrix} 5 & -2 \\ -1 & 5 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$M, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
11^2	30	30	$\begin{pmatrix} -5 & 1 \\ 2 & -1 \end{pmatrix}$	M
7^2	$\mathrm{SL}(2,3)$	$3 \times \mathrm{SL}(2,3)$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -2 & -2 \\ -1 & 2 \end{pmatrix}$	$M, \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}$
7^2	Q_{16}	QD_{32}	$\begin{pmatrix} 2 & 2 \\ 1 & -2 \end{pmatrix}, \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}$	$M, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
7^2	$3 \rtimes 4$	$(6 \times 2) \rtimes 2$	$\begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ -3 & 2 \end{pmatrix}$	$M, \begin{pmatrix} 1 & 0 \\ -3 & -1 \end{pmatrix}$
to be continued				

Table 4.1, continued

$\mathbb{K}^+$	M	H	Generators of M	Generators of H
23^2	$11{\times}\mathrm{SL}(2,3)$	$11{\times}\mathrm{SL}(2,3)$	$\begin{pmatrix}-8 & 10\\ -2 & -4\end{pmatrix}, \begin{pmatrix}-10 & 9\\ -1 & 10\end{pmatrix}$	M
23^2	$2.S_4$	$2.S_4$	$\begin{pmatrix}-7 & -6\\ 11 & 6\end{pmatrix}, \begin{pmatrix}9 & 1\\ 10 & -9\end{pmatrix}$	M
23^2	$11 \times Q_{16}$	$11 \times Q_{16}$	$\begin{pmatrix}-10 & 5\\ 2 & -8\end{pmatrix}, \begin{pmatrix}-2 & 6\\ 6 & 2\end{pmatrix}$	M
23^2	$11 \times (3 \rtimes 4)$	$11 \times (3 \rtimes 4)$	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}, \begin{pmatrix}7 & -1\\ -8 & -7\end{pmatrix}$	M
11^2	$\mathrm{SL}(2,3)$	$\mathrm{GL}(2,3)$	$\begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}-5 & -2\\ 2 & 5\end{pmatrix}$	$M, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
11^2	$5 \rtimes 4$	$(10 \times 2) \rtimes 2$	$\begin{pmatrix}-1 & -1\\ 2 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 3\end{pmatrix}$	$M, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
11^2	$3 \rtimes 4$	$3 \rtimes 4$	$\begin{pmatrix}-4 & 1\\ 5 & 4\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 1 & 0\end{pmatrix}$	M
29^2	$7 \times \mathrm{SL}(2,3)$	$7 \times \mathrm{SL}(2,3)$	$\begin{pmatrix}7 & 6\\ 13 & 2\end{pmatrix}, \begin{pmatrix}9 & -6\\ 4 & -9\end{pmatrix}$	M
29^2	$7 \times (5 \rtimes 4)$	$7 \times (5 \rtimes 4)$	$\begin{pmatrix}8 & 0\\ 10 & -8\end{pmatrix}, \begin{pmatrix}-6 & -1\\ 1 & 0\end{pmatrix}$	M
29^2	$7 \times (3 \rtimes 4)$	$7 \times (3 \rtimes 4)$	$\begin{pmatrix}0 & 1\\ -1 & -1\end{pmatrix}, \begin{pmatrix}-3 & -9\\ -6 & 3\end{pmatrix}$	M
59^2	$29{\times}\mathrm{SL}(2,3)$	$29{\times}\mathrm{SL}(2,3)$	$\begin{pmatrix}-1 & 22\\ -23 & -14\end{pmatrix}, \begin{pmatrix}4 & 29\\ -25 & -4\end{pmatrix}$	M
59^2	$29 \times (5 \rtimes 4)$	$29 \times (5 \rtimes 4)$	$\begin{pmatrix}-18 & 26\\ 22 & 18\end{pmatrix}, \begin{pmatrix}-26 & -1\\ 1 & 0\end{pmatrix}$	M
59^2	$29 \times (3 \rtimes 4)$	$29 \times (3 \rtimes 4)$	$\begin{pmatrix}-24 & 6\\ 16 & 23\end{pmatrix}, \begin{pmatrix}19 & -16\\ -15 & -19\end{pmatrix}$	M

Table 4.2. The automorphism groups of schemes over Zassenhaus near-fields with nonsolvable base group

$\mathbb{K}^+$	M	H	Generators of M	Generators of H
29^2	$\mathrm{SL}(2,5)$	$\mathrm{SL}(2,5) \rtimes 2$	$\begin{pmatrix}2 & -5\\ 1 & -2\end{pmatrix}, \begin{pmatrix}12 & 4\\ 3 & -11\end{pmatrix}$	$M, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$
59^2	$\mathrm{SL}(2,5)$	$\mathrm{SL}(2,5)$	$\begin{pmatrix}4 & -1\\ -1 & -29\end{pmatrix}, \begin{pmatrix}-29 & 10\\ -5 & 28\end{pmatrix}$	M

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Dmitriy Vladimirovich Churikov
Novosibirsk State University
ul. Pirogova, 2,
630090, Novosibirsk, Russia
E-mail address: `churikovdv@gmail.com`

Andrey Victorovich Vasil'ev
Sobolev Institute of Mathematics,
pr. Koptyuga, 4,
630090, Novosibirsk, Russia
Novosibirsk State University,
ul. Pirogova, 2,
630090, Novosibirsk, Russia
E-mail address: `vasand@math.nsc.ru`