The singular value decomposition of tomography operators

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WOMEN IN MATHEMATICS
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Basic definitions, 2D case

Let $B = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = \sqrt{x^2 + y^2} < 1\}$ be the unit disk,

$\partial B = \{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| = 1\}$ be its boundary,

$Z = \{(s, \xi) \mid s \in \mathbb{R}, \xi \in \partial B\}$ be a cylinder.

Basic definitions, 2D case

The space $L_2(B)$ consists of functions, which are square integrable in B.

The weighted space $L_2(Z,\rho)$ with a non-negative weight function ρ is also used. The inner product in the space $L_2(Z,\rho)$ is defined as

$$(f,g)_{L_2(Z,\rho)} = \int\limits_Z f(z)g(z)\rho(z)dz. \qquad (1)$$

The space of m-tensor fields in B is denoted by $S^m(B)$.
The spaces $H^k(B)$, $H^k(S^m(B))$ are the Sobolev spaces.

Basic definitions, 2D case

The operators of *inner derivation* d and *inner $\perp$-derivation* $\mathrm{d}^\perp$ are the compositions of operators of covariant derivation and symmetrization

$$\mathrm{d}, \mathrm{d}^\perp : H^k(S^m(B)) \to H^{k-1}(S^{m+1}(B))$$

and act on a function f and a vector field v by the formulas

$$(\mathrm{d}f)_i = \frac{\partial f}{\partial x_i}, \qquad (\mathrm{d}v)_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \tag{2}$$

$$(\mathrm{d}^\perp f)_i = (-1)^i \frac{\partial f}{\partial x_{3-i}}, \quad (\mathrm{d}^\perp v)_{ij} = \frac{1}{2}\left((-1)^j \frac{\partial v_i}{\partial x_{3-j}} + (-1)^i \frac{\partial v_j}{\partial x_{3-i}}\right). \tag{3}$$

Basic definitions, 2D case

The divergence operator

$$\mathrm{div} : H^k(S^m(B)) \to H^{k-1}(S^{m-1}(B))$$

acts on a vector field v and on a symmetric 2-tensor field w by the rules

$$\mathrm{div}\, v = \sum_{i=1}^{2} \frac{\partial v_i}{\partial x_i}, \qquad (\mathrm{div}\, w)_j = \sum_{i=1}^{2} \frac{\partial w_{ji}}{\partial x_i}. \tag{4}$$

Basic definitions, 2D case

A vector field u is called potential, if there is a function φ, such that

$$u = \mathrm{d}\varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right).$$

A vector field v is called solenoidal, if its divergence is equal to 0,

$$\mathrm{div}\, v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0.$$

In other words, there is a function ψ, such that

$$v = \mathrm{d}^{\perp}\psi = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right).$$

An arbitrary vector field w can be uniquely decomposed as the sum of potential and solenoidal part

$$w = \mathrm{d}\varphi + \mathrm{d}^{\perp}\psi, \qquad \varphi, \psi|_{\partial B} = 0. \tag{5}$$

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Basic definitions, 2D case

Analogously, there exist decomposition of a symmetric 2-tensor field w on a sum of three terms

$$w = \mathrm{d}^2\varphi + \mathrm{d}\mathrm{d}^{\perp}\phi + (\mathrm{d}^{\perp})^2\psi, \qquad (6)$$

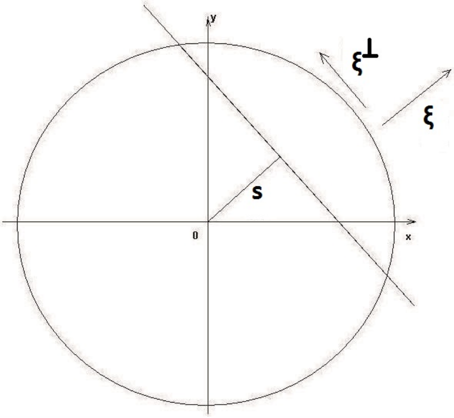
where

$$\varphi \in H_0^2(B), \quad \phi \in H^2(B), \quad \mathrm{d}^{\perp}\phi \in H_0^1(S^1(B)), \quad \psi \in H^2(B).$$

Basic definitions, 2D case

The Radon transform $\mathcal{R}f : L_2(B) \to L_2(Z)$ of a function f is defined by the formula

$$(\mathcal{R}f)(s,\xi) = \int_B f(\mathbf{x})\,\delta(\langle \xi, \mathbf{x}\rangle - s)\,d\mathbf{x}. \tag{7}$$



The unit vector ξ is typically characterized by $\xi = (\cos\alpha, \sin\alpha)$ with angle $\alpha \in [0, 2\pi]$.

The unit vector $\xi^{\perp} = (-\sin\alpha, \cos\alpha)(= \eta)$ specifies the direction of integration.

Basic definitions, 2D case

The transverse ray transform

$$\mathcal{P}^{\perp} : L_2(S^1(B)) \to L_2(Z)$$

acting on the vector field w is given by the formula

$$(\mathcal{P}^{\perp} w)(s, \xi) = \int_B \langle w(\mathbf{x}), \xi \rangle \, \delta(\langle \xi, \mathbf{x} \rangle - s) \, d\mathbf{x}. \qquad (8)$$

The longitudinal ray transform

$$\mathcal{P} : L_2(S^1(B)) \to L_2(Z)$$

of the vector field w is defined as

$$(\mathcal{P} w)(s, \xi) = \int_B \langle w(\mathbf{x}), \xi^{\perp} \rangle \, \delta(\langle \xi, \mathbf{x} \rangle - s) \, d\mathbf{x}. \qquad (9)$$

Basic definitions, 2D case

The longitudinal $\mathcal{P}$, transverse $\mathcal{P}^{\perp}$ and mixed $\mathcal{P}^{\star}$ ray transforms

$$\mathcal{P},\, \mathcal{P}^{\perp},\, \mathcal{P}^{\star} : L_2(S^2(B)) \to L_2(Z)$$

of a symmetric 2-tensor field $w = (w_{11}, w_{12}, w_{22})$:

$$[\mathcal{P}w](s,\xi) = \int_B \langle w(\mathbf{x}), \eta^2 \rangle \, \delta(\langle \xi, \mathbf{x} \rangle - s) \, d\mathbf{x}, \qquad (10)$$

$$[\mathcal{P}^{\perp}w](s,\xi) = \int_B \langle w(\mathbf{x}), \xi^2 \rangle \, \delta(\langle \xi, \mathbf{x} \rangle - s) \, d\mathbf{x}, \qquad (11)$$

$$[\mathcal{P}^{\star}w](s,\xi) = \int_B \langle w(\mathbf{x}), \xi\eta \rangle \, \delta(\langle \xi, \mathbf{x} \rangle - s) \, d\mathbf{x}. \qquad (12)$$

Basic definitions, 2D case

The operators of longitudinal and transverse ray transforms (vector case) have nonzero kernels, namely

$$(\mathcal{P}\,\mathrm{d}\varphi)\,(s,\xi) = (\mathcal{P}^{\perp}\,\mathrm{d}^{\perp}\varphi)(s,\xi) = 0, \qquad \varphi|_{\partial B} = 0.$$

Also for 2-tensor case:

$$[\mathcal{P}\,(\mathrm{d}^{\perp})^2\varphi](s,\xi) = [\mathcal{P}\,\mathrm{d}\mathrm{d}^{\perp}\varphi](s,\xi) = 0,$$
$$[\mathcal{P}^{\perp}\,\mathrm{d}^2\varphi](s,\xi) = [\mathcal{P}^{\perp}\,\mathrm{d}\mathrm{d}^{\perp}\varphi](s,\xi) = 0, \qquad \varphi|_{\partial B} = 0,$$
$$[\mathcal{P}^{\star}\,\mathrm{d}^2\varphi](s,\xi) = [\mathcal{P}^{\star}\,(\mathrm{d}^{\perp})^2\varphi](s,\xi) = 0.$$

Moreover, there are connections between the ray transforms and the Radon transform of the same potential :

$$\left(\mathcal{P}\,\mathrm{d}^{\perp}\varphi\right)(s,\xi) = \left(\mathcal{P}^{\perp}\,\mathrm{d}\varphi\right)(s,\xi) = \frac{\partial(\mathcal{R}\varphi)}{\partial s}(s,\xi), \qquad \varphi|_{\partial B} = 0.$$

$$[\mathcal{P}\,(\mathrm{d}^{\perp})^2\varphi](s,\xi) = [\mathcal{P}^{\perp}\,\mathrm{d}^2\varphi](s,\xi) = 2[\mathcal{P}^{\star}\,\mathrm{d}\mathrm{d}^{\perp}\varphi](s,\xi) = \frac{\partial^2(\mathcal{R}\varphi)}{\partial s^2}(s,\xi).$$

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Statement of the vector and 2-tensor tomography problem

The vector tomography problem reads as follows:

Let the longitudinal ray transform $\mathcal{P}w$ and (or) the transverse ray transform $\mathcal{P}^{\perp}w$ of a vector field w be known for all $(s, \xi) \in Z$.

From these data, we want to determine the unknown vector field $w(\mathbf{x})$, $\mathbf{x} \in B$.

The 2-tensor tomography problem can be defined analogously.

Singular value decomposition method

In other words, one has to solve operator equations

$$Af = g, \quad A : H \to K.$$

Here A is a linear, bounded operator. In the operator equation g is a known right hand-side (data of tomographic measurements), and f is an unknown vector (or 2-tensor) field to be determined.

Singular value decomposition method

The singular value decomposition of operator A is

$$Af = \sum_{k=1}^{\infty} \sigma_k (f, u_k)_H v_k, \tag{13}$$

with (u_k), (v_k) — orthonormal bases in initial and image space of operator A respectively, $\sigma_k > 0$ are called singular values of operator A.

If there is singular value decomposition of A, then

$$A^{-1} g = \sum_{k=1}^{\infty} \sigma_k^{-1} (g, v_k)_K u_k. \tag{14}$$

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Earlier

scalar tomography problem

Louis A.K. *Orthogonal function series expansions and the null space of the Radon transform.* SIAM, J. Mathematical Analysis **15**,621-633 (1984)

2D vector tomography problem

Derevtsov E. Yu., Efimov A. V., Louis A. K., Schuster T. *Singular value decomposition and its application to numerical inversion for ray transforms in 2D vector tomography.* J. Inverse Ill-Posed Problems, **19**(4-5), 689–715 (2011)

Earlier

2D 2-tensor tomography problem

A. Polyakova, E.Yu.Derevtsov *Solving of 2-tensor fields integral geometry problem with usage of SVD.* Vestnik NSU, **3**, 73–94 (2012)

3D vector tomography problem

Polyakova, A.P. *Reconstruction of a vector field in a ball from its normal Radon transform.* Journal of Mathematical Sciences **205**(3), 418–439 (2015)

3D 2-tensor tomography problem

Polyakova, A.P. *Singular value decomposition of the normal Radon transform operator acting on 3D symmetric 2-tensor fields.* (in press)

SVD of ray transform operators

We consider a family of functions

$$\Phi_{k,n}^{\cos,\sin}(x,y) = (1-x^2-y^2)^2 H_k^{\cos,\sin}(x,y) P_n^{(k+3,k+1)}(x^2+y^2),$$
$$k,n = 0,1,2,\dots, \tag{15}$$

in polar coordinates,

$$\left\{ \begin{array}{c} \tilde{\Phi}^{\cos} \\ \tilde{\Phi}^{\sin} \end{array} \right\}_{k,n} (r,\varphi) = (1-r^2)^2 r^k \left\{ \begin{array}{c} \cos k\varphi \\ \sin k\varphi \end{array} \right\} P_n^{(k+3,k+1)}(r^2). \tag{16}$$

Applying operators d and $\mathrm{d}^{\perp}$, we obtain 2-tensor fields:

$$\left(T_{k,n}^{\cos,\sin}\right)^{sol}(x,y), \left(T_{k,n}^{\cos,\sin}\right)^{pot_1}(x,y), \left(T_{k,n}^{\cos,\sin}\right)^{pot_2}(x,y) \tag{17}$$

SVD of ray transform operators

Theorem.
System of 2-tensor fields (??) form orthogonal system in $L_2(S^2(B))$ with norm

$$\|T_{k,n}\| = \frac{8\pi(n+1)^2(n+2)^2}{(k+2n+3)(C^k_{n+k})^2} \qquad (18)$$

SVD of ray transform operators

Proposition. (Louis, 1984)
Let $k, n \geqslant 0$, $-1 \leqslant s \leqslant 1$, $0 \leqslant \beta < 2\pi$, and

$$\Psi(\beta, s) = (1 - s^2)^{5/2} C^{(3)}_{k+2n}(s) Y_k(\beta),$$

where $C^{(3)}_{k+2n}(s)$ — Gegenbauer polynomials and $Y_k(\beta)$ — spherical harmonics on ∂B. Then $\Phi = \mathcal{R}^{-1}\Psi$ is given by

$$\Phi(\beta, r) = c(k, n)(1 - r^2)^2 r^k P_n^{(k+3,k+1)}(r^2) Y_k(\beta)$$

with $P_n^{(p,q)}$ — Yakobi polynomials of degree n and indices p, q, and

$$c(k, n) = (-1)^n 2^{-5} \frac{\Gamma(k + 2n + 6)\Gamma(n + 1)(k + n)!}{\Gamma(k + 2n + 1)\Gamma(3)\Gamma(n + 3)k!n!}.$$

SVD of ray transform operators

Theorem.
System of function

$$\left(\mathcal{P}\left\{\begin{array}{c}\mathbf{T}^{\cos}\\ \mathbf{T}^{\sin}\end{array}\right\}^{sol}_{k,n}(x,y)\right)(\alpha,s) = a(k,n)\sqrt{1-s^2}\,C^{(1)}_{k+2n+2}(s)\left\{\begin{array}{c}\cos k\alpha\\ \sin k\alpha\end{array}\right\}$$
$$=: \left\{\begin{array}{c}\Psi^{\cos}\\ \Psi^{\sin}\end{array}\right\}_{k,n}(\alpha,s),$$

with $a(k,n) = (-1)^n 2\sqrt{\frac{2}{\pi(k+2n+3)}}$, forms orthogonal system in space $L_2(Z,(1-s^2)^{-1/2})$ of images of longitudinal (transverse, mixed) ray transform. The norms are

$$\|\Psi^{\cos,\sin}_{k,n}\|^2_{L_2(Z,(1-s^2)^{-1/2})} = \frac{4\pi}{k+2n+3}.$$

SVD of ray transform operators

The SVD-decomposition of the operator $\mathcal{P}$ is

$$\mathcal{P}\mathbf{V} = \sum_{k,n=0,1,2,\dots}^{\infty} \sigma_{k,n} \Bigg(\left(\mathbf{V}, (\mathbf{T}_{k,n}^{\cos})\right)_{L_2(S^2(B))} G_{k,n}^{\cos} + (1-\delta_{k,0}) \left(\mathbf{V}, (\mathbf{T}_{k,n}^{\sin})\right)_{L_2(S^2(B))} G_{k,n}^{\sin} \Bigg), \qquad (19)$$

where $\sigma_{k,n} = 2\sqrt{\frac{\pi}{k+2n+3}}$ are the singular values.

The required 2-tensor field is calculated using the inverse operator by the formula

$$\mathbf{V} = \mathcal{P}^{-1} g = \sum_{k,n=0,1,2,\dots}^{\infty} \sigma_{k,n}^{-1} \Bigg(\left(g, G_{k,n}^{\cos}\right)_{L_2(Z,(1-s^2)^{-1/2})} \mathbf{T}_{k,n}^{\cos} + (1-\delta_{k,0}) \left(g, G_{k,n}^{\sin}\right)_{L_2(Z,(1-s^2)^{-1/2})} \mathbf{T}_{k,n}^{\sin} \Bigg). \qquad (20)$$

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$\partial B = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ — unit sphere.

Basic definitions, 3D case

We use the following differential operators:
1) gradient operator $\mathrm{d} : H^k(B) \to H^{k-1}(S^1(B))$, which acts on the potential ψ by formula:

$$\mathrm{d}\psi = \left(\frac{\partial\psi}{\partial x}, \quad \frac{\partial\psi}{\partial y}, \quad \frac{\partial\psi}{\partial z}\right); \tag{21}$$

2) rotor operator $\mathrm{rot} : H^k(S^1(B)) \to H^{k-1}(S^1(B))$, which acts on a vector field $\mathbf{w}$ by next way:

$$\mathrm{rot}\mathbf{w} = \left(\frac{\partial w_3}{\partial y} - \frac{\partial w_2}{\partial z}, \frac{\partial w_1}{\partial z} - \frac{\partial w_3}{\partial x}, \frac{\partial w_2}{\partial x} - \frac{\partial w_1}{\partial y}\right); \tag{22}$$

3) divergence operator $\delta : H^k(S^1(B)) \to H^{k-1}(B)$, which acts on a vector field $\mathbf{w}$ by rule:

$$\delta\mathbf{w} = \frac{\partial w_1}{\partial x} + \frac{\partial w_2}{\partial y} + \frac{\partial w_3}{\partial z}. \tag{23}$$

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Basic definitions, 3D case

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A vector field $\mathbf{v} \in H^k(S^1(B))$ is a solenoidal vector field, if $\delta\mathbf{v} \in H^{k-1}(B) = 0$.

It is obvious that field $\mathbf{u} = \mathrm{rot}\mathbf{v}$ is solenoidal.

It is well known that every vector field $\mathbf{w} \in L_2(S^1(B))$ in R^3 can be decomposed uniquely in a sum of potential and solenoidal parts

$$\mathbf{w} = \mathrm{d}\,\phi + \mathrm{rot}\mathbf{v}. \tag{24}$$

where $\phi \in H^1_0(B)$ and $\mathbf{v} \in H^1(S^1(B))$.

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Basic definitions, 3D case

Let $\mathbb{Z} = \{(\xi, s) \big| |\xi| = 1, s \in \mathbb{R}\}$.
The Radon transform $\mathcal{R}f : L_2(R^3) \to L_2(Z, \rho)$ of function $f(\mathbf{x})$ is given by formula

$$\mathcal{R}f(s, \xi) = \int\limits_{P_{\xi,s}} f(u\mathbf{e_1} + v\mathbf{e_2} + s\xi)\, du\, dv. \tag{25}$$

Integral in the right-hand side does not depend on the choice of the basis $\mathbf{e_1}$, $\mathbf{e_2}$ on the plane of integration.

Basic definitions, 3D case

Let $T = \{(u, v, \xi) \big| u \in [-\sqrt{1-v^2}, \sqrt{1-v^2}], v \in [-1, 1], |\xi| = 1\}$.

The ray transform $\mathcal{P} : L_2(S^1(B)) \to L_2(T)$ of a vector field $\mathbf{w}$ is given by formula

$$(\mathcal{P}\mathbf{w})(u, v, \xi) = \int_{-\infty}^{\infty} \langle \mathbf{w}, \xi \rangle \, dt. \qquad (26)$$

Easy to show that the kernel of the operator consist of potential vector fields $\mathrm{d}\phi \in L_2(S^1(B))$ with potential $\phi \in H_0^1(B)$. That is if we know the ray transform of a vector field we can reconstruct only its solenoidal part.

Basic definitions, 3D case

The normal Radon transform $\mathcal{R}^{\perp} : L_2(S^1(B)) \to L_2(Z, (1-s^2)^{-1})$ of a vector field $\mathbf{w} = \mathbf{w}(x, y, z) = (w_1, w_2, w_3)$ is given by formula

$$\mathcal{R}^{\perp}\mathbf{w} = \iint\limits_{P_{\xi,s}} \left(w_1\xi^1 + w_2\xi^2 + w_3\xi^3\right) du\ dv. \tag{27}$$

Basic definitions, 3D case

Lemma.
The kernel of the normal Radon transform consists of solenoidal vector fields, that is, the following equation holds

$$\mathcal{R}^{\perp}(\mathrm{rot}\mathbf{w}) = 0 \text{ with } \mathbf{w}\big|_{\partial B} = 0. \tag{28}$$

In other words if we know the normal Radon transform of a vector field, we can reconstruct only its potential part.

A connection between the normal Radon transform of a vector field and the Radon transform of a potential $f \in H_0^1(B)$:

$$\left(\mathcal{R}^{\perp}(\mathrm{d}f)\right)(s,\xi) = \frac{\partial}{\partial s}((\mathcal{R}f)(s,\xi)). \tag{29}$$

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Statement of the problem

Let we have some potential vector field

$$\mathrm{d}\phi \in L_2(S^1(B)), \phi \in H_0^1(B), \tag{30}$$

which is given in a unit ball B.

One has to recover this field by its known the normal Radon transform.

A singular value decomposition of the normal Radon transform operator

We choose the following system of polynomials as the potentials

$$\Phi_{k,n}(x,y,z) = (1-x^2-y^2-z^2)\, H_k(x,y,z)\, P_n^{(k+2{,}5,k+1{,}5)}(x^2+y^2+z^2), \tag{31}$$

$k, n = 0, 1, 2, \ldots$

or in spherical system of coordinates

$$\Phi_{k,n}(r,\theta,\varphi) = (1-r^2)\, r^k\, P_n^{(k+2{,}5,k+1{,}5)}(r^2)\, Y_k(\omega). \tag{32}$$

A singular value decomposition of the normal Radon transform operator

System of functions (??) is not orthogonal in the space $H_0^1(B)$, but this is not required.

An application of the operator d leads to a set of potential vector fields

$$\mathbf{T}_{k,n}(x,y,z) \overset{def}{=} \mathrm{d}\Phi_{k,n}(x,y,z). \tag{33}$$

A singular value decomposition of the normal Radon transform operator

Theorem.
System of potentials (in spherical coordinates)

$$F_{k,n}(r,\theta,\phi) = a_{n,k}\,(1-r^2)\,r^k\,P_n^{(k+2{,}5,k+1{,}5)}(r^2)\,Y_k(\omega) \qquad (34)$$

with

$$a_{n,k} = \frac{\Gamma(n+k+1.5)}{(n+1)!\Gamma(k+1.5)\|Y_k(\omega)\|}\sqrt{\frac{2n+k+2.5}{2}}$$

forms a system of potential vector fields

$$(\mathbf{F}_{k,n})(x,y,z) = \mathrm{d}F_{k,n}(x,y,z), \qquad (35)$$

which is orthonormal in space $L_2(S^1(B))$.

A singular value decomposition of the normal Radon transform operator

Proposition. (Louis, 1984)
Let $\nu > 0.5$, k, $n \geqslant 0$,

$$\Psi(\omega,\ s) = (1 - s^2)^{\nu - 0.5}\ C_{2n+k}^{(\nu)}(s)\ Y_k(\omega), \tag{36}$$

with $C_{2n+k}^{(\nu)}(s)$ – Gegenbauer polynomials. Then

$$\Phi = \mathcal{R}^{-1}\Psi = c(n,k,\nu)\left(1 - r^2\right)^{\nu - 1.5} r^k\ P_n^{(k+\nu, k+1.5)}(r^2)\ Y_k(\omega), \tag{37}$$

with $c(n,k,\nu) = \dfrac{(-1)^n\ 2^{1-2\nu}\Gamma(2n+k+2\nu)\Gamma(k+n+1.5)}{\sqrt{\pi}\Gamma(\nu)\ \Gamma(n+\nu-0.5)\Gamma(k+1.5)}$
and $P_n^{(p,q)}(r^2)$— Yakobi polynomials of degree n with indices p, q.

A singular value decomposition of the normal Radon transform operator

Theorem.
A system of function

$$G_{k,n} = b_{n,k}(1-s^2)C^{(1.5)}_{2n+k+1}(s)Y_k(\omega) \tag{38}$$

with

$$b_{n,k} = \frac{(-1)^{n-1}\sqrt{2n+k+2.5}}{\sqrt{(2n+k+3)(2n+k+2)}\|Y_k(\omega)\|}$$

forms an orthonormal system in space $L_2(Z,\, (1-s^2)^{-1})$.

A singular value decomposition of the normal Radon transform operator

We have the following relation:

$$(\mathcal{R}^{\perp}\mathbf{F}_{k,n})(s,\,\theta,\,\phi) = \sigma_{k,n}\cdot G_{k,n}(s,\,\theta,\,\phi),\quad k,n = 0,1,2\ldots, \tag{39}$$

where

$$\sigma_{k,n} = \frac{2\sqrt{2}}{\sqrt{(2n+k+2)(2n+k+3)}} \tag{40}$$

– singular value of operator $\mathcal{R}^{\perp}$.

A singular value decomposition of the normal Radon transform operator

A singular value decomposition of the normal Radon transform operator $\mathcal{R}^{\perp}$ has the form

$$\mathcal{R}^{\perp} v = \sum_{k,n=0}^{\infty} \sigma_{k,n} \left(v, \mathbf{F}_{k,n}\right)_{L_2(S^1(B))} G_{k,n}, \tag{41}$$

the inverse operator can be calculated by the formula

$$v = \left(\mathcal{R}^{\perp}\right)^{-1} g = \sum_{k,n=0}^{\infty} \sigma_{k,n}^{-1} \left(g, G_{k,n}\right)_{L_2(Z,(1-s^2)^{-1})} \mathbf{F}_{k,n}. \tag{42}$$

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Publications

1. I.E. Svetov, A.P. Polyakova, *Comparison of two algorithms for the numerical solution of two-dimensional vector tomography* , Siberian Electronic Mathematical Reports, **10** (2013), 90–108. (in Russian)
2. I.E. Svetov, A.P. Polyakova, *Approximate solution of two-dimensional 2-tensor tomography problem using truncated singular value decomposition*, Siberian Electronic Mathematical Reports, **12** (2015), 480–499. (in Russian)
3. A.P. Polyakova, I.E. Svetov, *Numerical Solution of the Problem of Reconstructing a Potential Vector Field in the Unit Ball from Its Normal Radon Transform*, Journal of Applied and Industrial Mathematics, **9**:4 (2015), 547-558.
4. A.P. Polyakova, I.E. Svetov, *Numerical solution of reconstruction problem of a potential symmetric 2-tensor field in a ball from its normal Radon transform*, Siberian Electronic Mathematical Reports, **13** (2016), 154–174. (in Russian)

THANK YOU FOR YOUR ATTENTION!