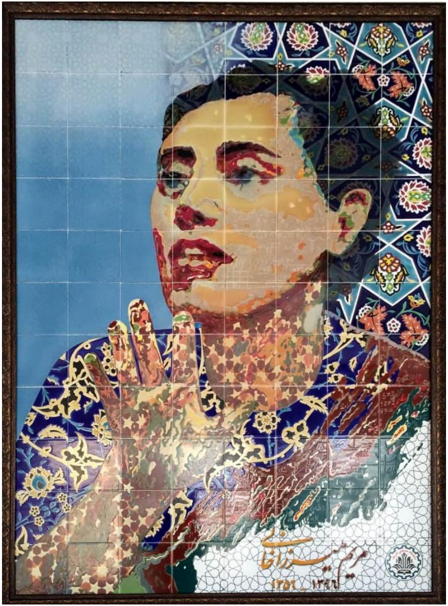
Maryam Mirzakhani - 12 May 1977 – 14 July 2017



Academic Council Hall, Sharif University of Technology, Tehran, Iran

Avicenna Building (the main classroom building)

View fom Avicenna Building (24/10/2018)

A general view of the main campus (from Wikipedia)

Maryam Mirzakhani: career

• Growing up in Tehran during the Iranian Revolution and the Iran-Iraq war, Maryam won two gold medals at the International Mathematical Olympiad: in Hong Kong in 1994 (41/42) and in Canada in 1995 (42/42).

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• In 1999, after graduating from Sharif University of Technology in Tehran, she moved to US and received her PhD. from Harvard University under the supervision of Curtis McMullen (Fields Medalist 1998).

Maryam Mirzakhani: career

Stanford University, 2009

• In 2004, she was awarded a Research Fellowship from the Clay Mathematics Institute and became an assistant professor at Princeton University.

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• In 2004, she was awarded a Research Fellowship from the Clay Mathematics Institute and became an assistant professor at Princeton University.

• In 2008, she became a full professor at Stanford University. She received the most prestigious distinctions for her work, among these the Clay Research Award in 2014 and the Fields Medal in 2014 for *'her outstanding contributions to the dynamics and geometry of Riemann surfaces'*

Maryam Mirzakhani: style to work

1983

• Maryam described herself as a *'slow'* mathematician, saying that *'you have to spend some energy and effort to see the beauty of math.'*

• To solve problems, she would draw doodles on sheets of paper and write mathematical formulas around the drawings.

• *'I don't have any particular recipe [for developing new proofs] ... It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck, you might find a way out.'*

Maryam Mirzakhani: mathematical polyglot

Clay Mathematics Institute, 2004

• Maryam was mathematically a true polyglot, able to combine techniques coming from very different areas.

• Maryam citation for the Fields Medal observes that *'her insights have integrated methods from algebraic geometry, topology and probability theory.'*

• There was a good deal of graph-theoretic intuition behind her work as well.

Maryam's papers in graph theory

Mahmoodian, Ebadollah, and Mirzakhani, Maryam, Decomposition of complete tripartite graphs into 5-cycles, *Combinatorics advances* (Tehran, 1994), 235-–241, Math. Appl. 329, Kluwer Acad. Publ., Dordrecht, 1995.

DECOMPOSITION OF COMPLETE TRIPARTITE GRAPHS INTO 5-CYCLES

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ABSTRACT

A result of Sotteau on the necessary and sufficient conditions for decomposing the complete bipartite graphs into even cycles has been shown in many occasions, that it is a very important tool in the theory of graph decomposition into *even* cycles. In order to have similar tools in the case of odd cycle decomposition, obviously bipartite graphs are not suitable to be considered. Searching for such tools, we have considered decomposition of complete tripartite graphs, $K_{r,s,t}$, into 5-cycles. There are some necessary conditions that we have shown their sufficiency in the case of $r = t$, and some other cases. Our conjecture is that these conditions are always sufficient.

Maryam's papers in graph theory

Mirzakhani, Maryam, A small non-4-choosable planar graph, *Bull. Ins Combin. Appl.* 17 (1996) 15—18.

Result: there exist planar 3-colorable graphs which are not 4-choosable.

Tommy Jensen, co-author of the book on Graph Coloring Problems, praised the construction, emphasizing the delicacy of finding counterexamples to list coloring conjectures.

'*It is hard to do this in a short and elegant way, and it seems that Maryam was able to find a solution that is possibly the most beautiful of its kind so far'*, - Jensen wrote.

Maryam's mentors: Saieed Akbari

The problem of finding such a counterexample was posed to Maryam by Prof. Saieed Akbari, Sharif University of Technology in Tehran.

Akbari offered 10 US dollars for the construction of a non-4-choosable planar graph. He notes '*If I had offered* 100 US *dollars for the construction of such a graph, nobody would dare to attempt the problem.*'

After a few days Maryam handed him her one-page proof that there exist planar 3-colorable graphs which are not 4-choosable.

In spite of initial incredulity at the brevity of the solution, Akbari checked the construction and awarded her the ten-dollar prize the next day. Of her solution, he writes '*I found it to be very creative and beautiful!!*'

William J. Martin, On an early paper of Maryam Mirzakhani, `https://arxiv.org/abs/1709.07540`

Maryam's mentors: Saieed Akbari (1 March, 2020)

Chromatic properties of the Pancake graphs

Definition

The Pancake graph P_n is the Cayley graph over the symmetric group Sym_n with generating set $\{r_i \in \mathrm{Sym}_n, 1 \leqslant i < n\}$, where any r_i reverses the order of any substring $[1, i]$, $1 < i \leqslant n$, of π when multiplied on the right, i.e.,

$$[\underline{\pi_1 \dots \pi_i}\pi_{i+1} \dots \pi_n]r_i = [\underline{\pi_i \dots \pi_1}\pi_{i+1} \dots \pi_n].$$

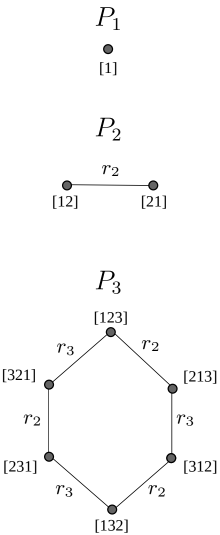
Properties

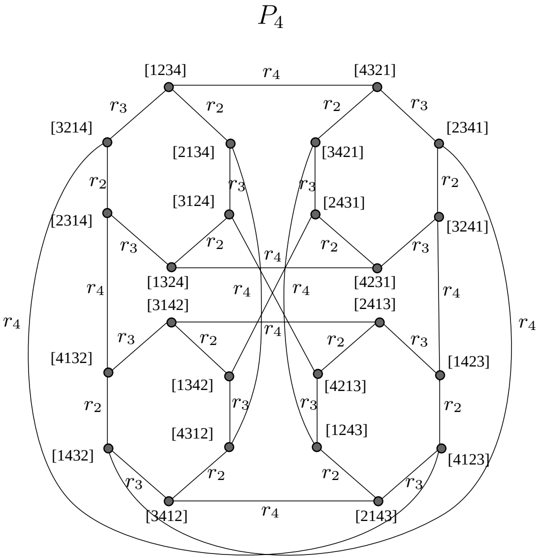
- connected, $(n-1)$–regular, vertex–transitive
- has a hierarchical structure: P_n has n copies of P_{n-1}
- is hamiltonian; contains all cycles C_l of length l, $6 \leqslant l \leqslant n!$

Original paper by Jacob E. Goodman, 1975

This graph is well known because of the open combinatorial Pancake problem of finding its diameter (the Pancake problem)

Pancake graphs P_2, P_3, P_4





Chromatic characteristics of graphs

A mapping $c : V(\Gamma) \to \{1, 2, \ldots, k\}$ is a *proper k–coloring* of a graph $\Gamma = (V, E)$ if $c(u) \neq c(v)$ whenever the vertices u and v are adjacent.

Chromatic number

The chromatic number $\chi(\Gamma)$ of a graph Γ is the least number of colors needed to color vertices of Γ.

A k–coloring is the same as a partition of $V(\Gamma)$ into k independent sets.

Chromatic index

The chromatic index $\chi'(\Gamma)$ is the least number of colors needed to color edges of Γ such that no two adjacent edges share the same color.

In the *total coloring* no adjacent vertices, edges, and no edge and its endvertices are assigned the same color.

Total chromatic number

The total chromatic number $\chi''(\Gamma)$ of a graph Γ is the least number of colors needed in any total coloring of Γ.

Chromatic properties of the Pancake graph [K17]

Total chromatic number

$\chi''(P_n) = n$ for any $n \geqslant 3$.

Total n-coloring is based on efficient dominating sets in the graph.

Chromatic index: the 1st class!

$\chi'(P_n) = n - 1$ for any $n \geqslant 3$.

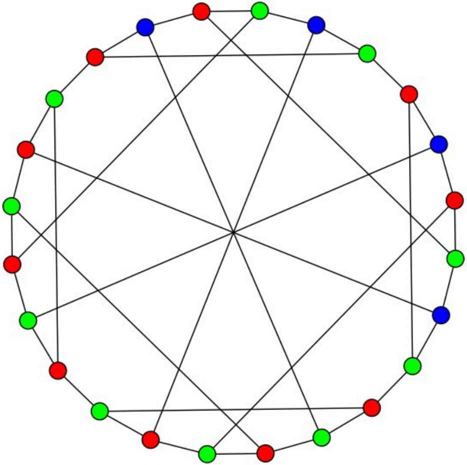
It is obtained from Vizing's bound $\chi' \geqslant \Delta$ taking into account the edge coloring in which the color $(i-1)$ is assigned to r_i, $2 \leqslant i \leqslant n$.

Chromatic number: trivial bounds

$$3 \leqslant \chi(P_n) \leqslant n - 1 \text{ for any } n \geqslant 4.$$

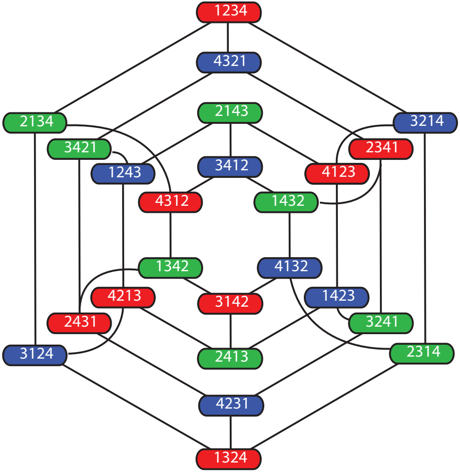
[K17] E. V. Konstantinova, Chromatic properties of the Pancake graphs, Discussiones Mathematicae Graph Theory, **37** (2017) 777–787.

3-coloring of P_4: hamiltonian drawing



Picture: Tomo Pisanski

3-coloring of P_4: hierarchical drawing



Picture: K. Rogalskaya

Improved chromatic numbers

The Catlin's bound [1978] for C_4–free graphs, that is $\chi \leqslant \frac{2}{3}(\Delta + 3)$, gives one more bound for any $n \geqslant 8$:

$$\chi(P_n) \leqslant \frac{2}{3}(n+2). \tag{1}$$

The bound (1) was improved in [K17]:

$$\chi(P_n) \leqslant \begin{cases} n-k, & \text{if } n \equiv k \pmod 4 \text{ for } k = 1,3; \\ n-2, & \text{if } n \text{ is even}; \end{cases} \qquad \textit{for } 5 \leqslant n \leqslant 8;$$

$$\chi(P_n) \leqslant \begin{cases} n-(k+2), & \text{if } n \equiv k \pmod 4 \text{ for } k = 1,3; \\ n-4, & \text{if } n \text{ is even}; \end{cases} \qquad \textit{for } 9 \leqslant n \leqslant 16;$$

$$\chi(P_n) \leqslant \begin{cases} n-(k+4), & \text{if } n \equiv k \pmod 4 \text{ for } k = 1,2,3; \\ n-8, & \text{if } n \equiv 0 \pmod 4; \end{cases} \qquad \textit{for } n \geqslant 17.$$

Chromatic numbers for small n:

n	3	4	5	6	7	8	9
$\chi(P_n)$	2	3	3	4	4	4	4
$\|V(P_n)\|$	6	24	120	720	5040	40320	362880

$\chi(P_3) = 2$ since $P_3 \cong C_6$

$\chi(P_4) = 3$ since there are 7–cycles in P_n, $n \geqslant 4$

Optimal 3–coloring for P_5 was given in [K17]

Optimal 4-coloring for P_6 was computed by J. Azarija, University of Ljubljana, Slovenia

Since P_{n-1} is an induced subgraph of P_n, then $\chi(P_7)$ is at least 4, and by improved bounds we have $\chi(P_7) = 4$

Optimal 4-colorings for P_8, P_9 were computed by A. H. Ghodrati, Sharif University of Technology, Iran

New bound: joint work with Leen Droogendijk, https://arxiv.org/abs/2103.11092

New bound [DK-2021+]

$$\chi(P_n) \leqslant 4\left\lfloor \frac{n}{9} \right\rfloor + \chi\left(P_{n \pmod 9}\right)$$

with $\chi(P_0) = 0$, $\chi(P_1) = 1$, and $\chi(P_2) = 2$.
For $n = 3, \ldots, 9$, $\chi(P_n)$ can be taken from the table above.

The proof is an consequence of the following result.

Subbadditive property

$\chi(P_{n+m}) \leqslant \chi(P_n) + \chi(P_m)$ for all positive integers n and m.

With $\chi(P_9) = 4$ this immediately gives the new general upper bound.

Chromatic number: open problems

Problem 1

It is unknown if $\chi(P_n)$ ever exceeds 4 for $n \geqslant 10$.

If it turns out that $\chi(P_n) \leqslant 4$ for all n, then efforts on finding upper bounds are pointless.

One of the ways to ruling out the possibility $\chi(P_n) \leqslant 4$ for all n would be

Problem 2

To show that the Pancake graph P_n does not have an independent set of size $\frac{n!}{4}$ if n is large enough.

Equitable Chromatic number: conjecture

A graph Γ is said to be *equitably k-colorable* if Γ has a proper k-coloring such that the sizes of any two color classes differ by at most one.

Equitable chromatic number

The *equitable chromatic number* $\chi_=(\Gamma)$ is the smallest integer k such that Γ is equitably k-colorable.

Equitable coloring was introduced due to scheduling problems.

Fact

For any $n = 3, 4, 5, 6, 7$, we have $\chi(P_n) = \chi_=(P_n)$.

Conjecture

For any $n \geqslant 8$, $\chi(P_n) = \chi_=(P_n)$.

Thanks for attention!