Element orders of finite almost simple groups

Maria Grechkoseeva

Sobolev Institute of Mathematics

12 May, 2021

Orders of elements

If G is a group and $g \in G$, then the order of g is the smallest positive integer k such that $g^k = e$.

$\omega(G)$ is the set of all numbers that are the orders of elements of G.

Example

If G is the symmetry group of the regular triangle, then $\omega(G) = \{1, 2, 3\}$ with 1, 2 and 3 be the orders of the identity, a reflection and a nontrivial rotation respectively.

Example

If $G = PGL_2(q)$, the projective general linear group of dimension 2 over the field of order $q = p^m$, then $k \in \omega(G)$ iff k divides $q + 1$, or $q - 1$, or p.

Finite almost simple groups

A finite group S is a simple group if $S \neq 1$ and its only normal subgroups are 1 and S itself. Every finite group G can be "constructed" from simple groups via extensions.

A finite group G is almost simple if $S \leqslant G \leqslant \mathrm{Aut}(S)$ for some nonabelian simple group S. This group S is the socle of G.

Finite almost simple groups

A finite group S is a simple group if $S \neq 1$ and its only normal subgroups are 1 and S itself. Every finite group G can be "constructed" from simple groups via extensions.

A finite group G is almost simple if $S \leqslant G \leqslant \mathrm{Aut}(S)$ for some nonabelian simple group S. This group S is the socle of G.

Classification of Finite Simple Groups

- cyclic groups C_p of prime order p (abelian)
- alternating permutation groups Alt_n, $n \geqslant 5$ (nonabelian)
- simple groups of Lie type (nonabelian)
- 26 sporadic groups (nonabelian)

The main problem

Problem

Given a nonabelian simple group S and G with $S \leqslant G \leqslant \mathrm{Aut}(S)$, decribe $\omega(G)$.

The problem is easy if S is an alternating or sporadic because

- $\mathrm{Aut}(Alt_n) = Sym_n$ for $n \neq 6$
- Alt_6 and sporadic groups are in "The Atlas of Finite Groups"

So we may assume that S is a group of Lie type.

The main problem

Problem

Given a nonabelian simple group S and G with $S \leqslant G \leqslant \mathrm{Aut}(S)$, decribe $\omega(G)$.

The problem is easy if S is an alternating or sporadic because

- $\mathrm{Aut}(Alt_n) = Sym_n$ for $n \neq 6$
- Alt_6 and sporadic groups are in "The Atlas of Finite Groups"

So we may assume that S is a group of Lie type.

Also it should be noted that the sets $\omega(S)$ are known (Suzuki, Deriziotis, ..., Buturlakin; the final case is $S = E_8(q)$ due to Buturlakin, 2018).

Automorphisms of groups of Lie type

The easiest but still illustrartive example of a group of Lie type is $PSL_n(q)$, the projective special linear group of dimension n over the field of order q. Its authomorphism group is generated by inner automorphisms and the following:

- diagonal automorphisms:
  $A \mapsto D^{-1}AD$ with D a fixed diagonal matrix in $GL_n(q)$
- graph automorphisms:
  relating to the inverse-transpose map $A \mapsto A^{-\top}$
- field automorphisms:
  $(a_{ij}) \mapsto (a_{ij}^{\varphi})$ with φ an automorphism of the underlying field

Steinberg's Theorem

Every automorphism of a simple group of Lie type is a product of inner, diagonal, graph and field automorphisms.

Field automorphisms

Theorem (Zavarnitsine, 2006)

Let $S = PSL_n(q)$, where $q = q_0^m$, and φ be a field automorphism of S of order m. Then

$$\omega(\langle S, \varphi \rangle) = \bigcup_{k|m} \frac{m}{k} \cdot \omega(PSL_n(q_0^k)).$$

Similar results hold for other groups of Lie type and for some other automorphisms relating to field authomorphisms (Zavarnitsine, 2006; Grechkoseeva, 2017).

In some sense, this reduce the whole problem to graph and diagonal automorphisms.

Graph automorphisms

Let $S = PSL_n(q)$ and γ be the inverse-transpose automorphism. To find $\omega(\langle S, \gamma \rangle)$, one needs to know which matrices can be written as

$$AA^{-\top} \text{ for some } A \in SL_n(q).$$

A similar problem with $A \in GL_n(q)$ was solved by Wall in 1962 as a part of the classification of bilinear forms on $GF(q)^n$. Wall's results provided a basis for calculating $\omega(\langle S, \gamma \rangle)$.

Theorem (Grechkoseeva, 2017)

Let $S = PSL_n(q)$, where q and n are odd, and γ the graph automorphism of S or order 2. Then

$$\omega(\langle S, \gamma \rangle) = \omega(S) \cup 2 \cdot \omega(Sp_{2n-1}(q)).$$

Similar results were obtained for involutary graph automorphisms of orthogonal groups of even dimensions (Grechkoseeva, 2018).

Ongoing projects

- G is the extension of an exceptional group of Lie type by diagonal automorphisms (with A. Buturlakin)

  Tools: Carter's description of centralizers of semisimple elements in the corresponding adjoint groups

- G is the extension of an exceptional group of Lie type by graph authomorphism

  Tools: using Cartan subgroups instead of maximal tori and the corresponding modification of the above Carter's description