COARFA FORMULA FOR FUNCTIONS ON SUB-LORENTZIAN STRUCTURES

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Main Result and Generalizations

- Coarea formula for functions:

$$\int\limits_{\Omega}\Big((X_1\varphi(v))^2-\sum_{j=2}^{n}(X_j\varphi(v))^2\Big)^{1/2}\,d\mathcal{H}^{\nu}(v)=\int\limits_{\mathbb{R}}dy\int\limits_{\varphi^{-1}(y)\cap\Omega}d\mathcal{H}^{\nu-1}_{\mathfrak{d}}(u)$$

- Coarea formula for $\mathbb{R}^{\widetilde{n}}$-valued mappings:

$$\int\limits_{\Omega}\Big(\det\Big(D_H^{\widetilde{n}}\varphi(D_H^{\widetilde{n}}\varphi)^*-\big(D_H\setminus D_H^{\widetilde{n}}\varphi\big)\big(D_H\setminus D_H^{\widetilde{n}}\varphi\big)^*\Big)\Big)^{1/2}d\mathcal{H}^{\nu}$$

$$=\int\limits_{\mathbb{R}^{\widetilde{n}}}d\mathcal{H}^{\widetilde{n}}(z)\int\limits_{\varphi^{-1}(z)}d\mathcal{H}^{\nu-\widetilde{n}}_{\mathfrak{d}}(y).$$

- The analog is valid also for mappings of Carnot groups

Outline of the Talk

- ◇ Preliminaries
- ◇ Coarea formula and applications
- ◇ Minkowski geometry
- ◇ Carnot groups

- ◇ Sub-Lorentzian structures

- ◇ Coarea formula for functions
- ◇ Main ideas of the proof

- ◇ Coarea formula for $\mathbb{R}^{\widetilde{n}}$-valued mappings
- ◇ Specifics

- ◇ Coarea formula for mappings of Carnot groups
- ◇ Specifics

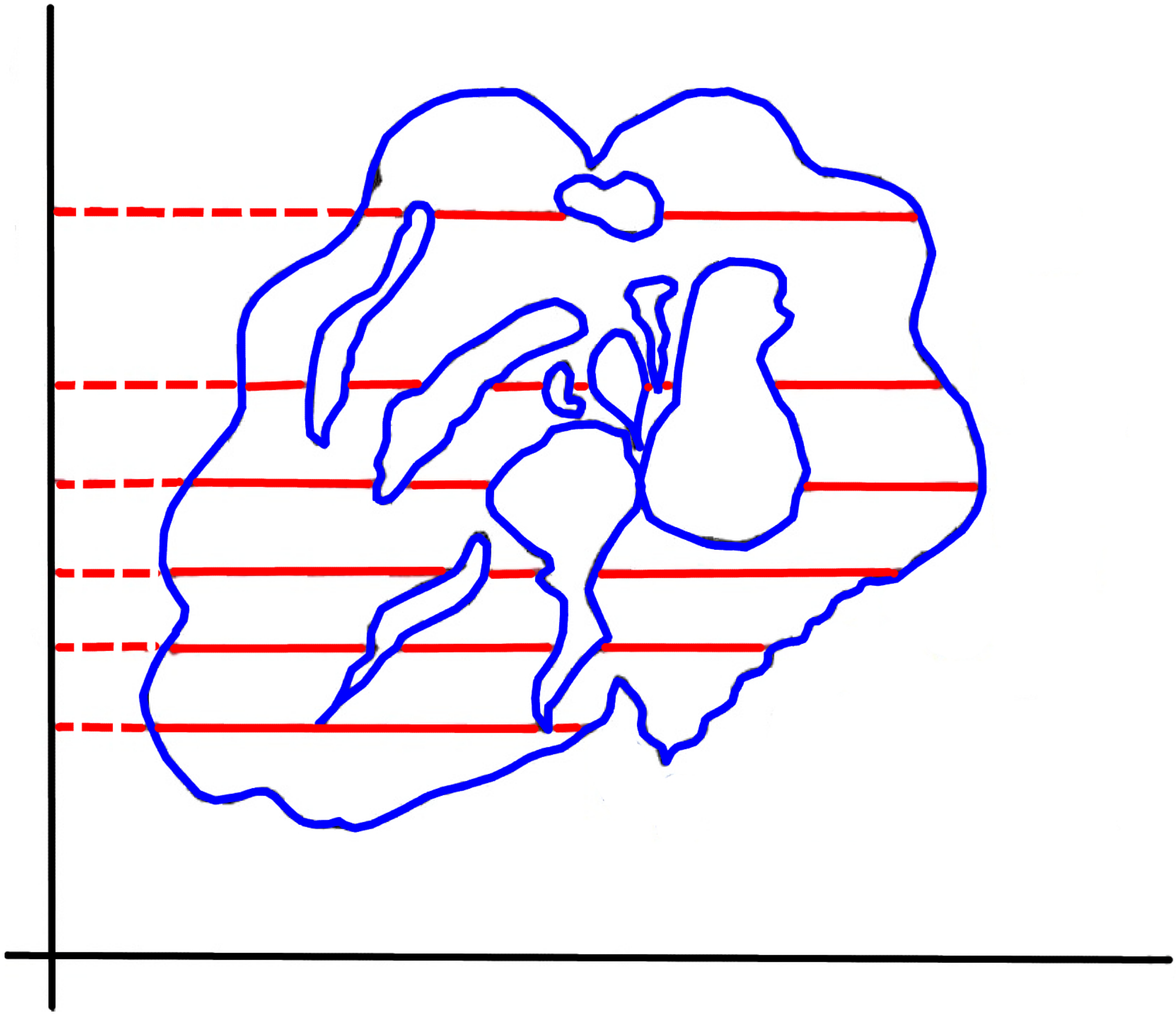
Coarea formula

- curvilinear version of Fubini Theorem:
- $(\mathbb{R}^n, |\cdot|_n)$; $(\mathbb{R}^m, |\cdot|_m)$; $(\mathbb{R}^{n+m}, |\cdot|_{n+m})$
- $E \subset \mathbb{R}^{n+m}$; $\pi_{\mathbb{R}^m}$ is a projection: $\mathbb{R}^{n+m} \ni (x,y) \overset{\pi_{\mathbb{R}^m}}{\mapsto} y \in \mathbb{R}^m$

◇ $\pi^{-1}_{\mathbb{R}^m}(y) \cap E = \{x \in \mathbb{R}^n : (x,y) \in E\} = \{(x,y) \in E : \pi_{\mathbb{R}^m}(x,y) = y\}$

⇒ Then, the value of $|E|_{n+m}$ equals

$$|E|_{n+m} = \int_{\mathbb{R}^m} |\pi^{-1}_{\mathbb{R}^m}(y) \cap E|_n \, dy$$



Coarea formula

- $(\mathbb{R}^n, |\cdot|_n)$; $(\mathbb{R}^m, |\cdot|_m)$; $(\mathbb{R}^{n+m}, |\cdot|_{n+m})$

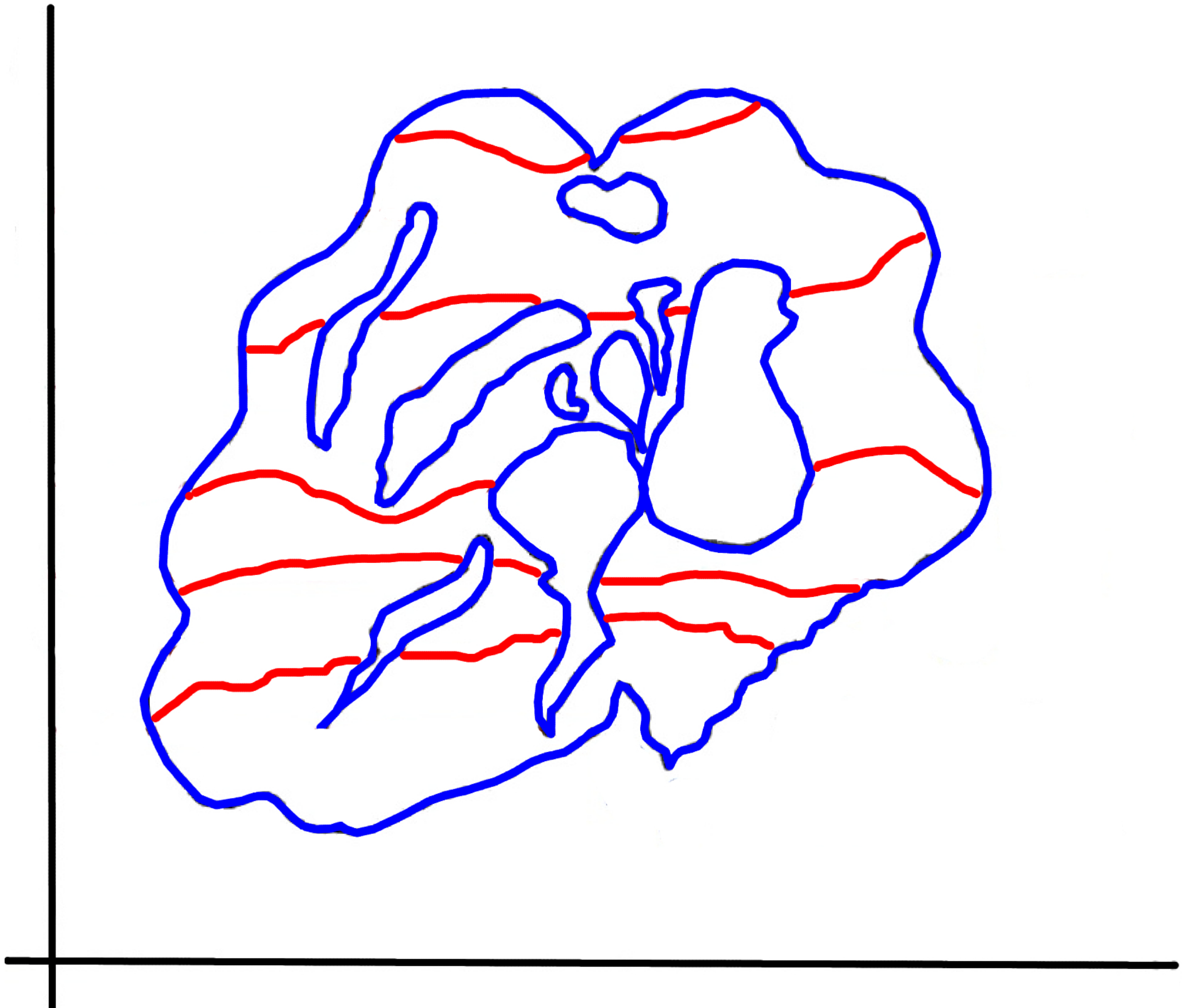
- $E \subset \mathbb{R}^{n+m}$; $\varphi \in C^1(E, \mathbb{R}^m)$

◇ $\varphi^{-1}(y) \cap E = \{z \in E : \varphi(z) = y\}$

- $\mathcal{H}^n(A) = \omega_n \lim\limits_{\delta \to 0} \inf \Big\{ \sum\limits_{i \in \mathbb{N}} r_i^n : \bigcup\limits_{i \in \mathbb{N}} B(x_i, r_i) \supset A, \ r_i < \delta \Big\}$

⇒ Then

$$\int\limits_E \mathcal{J}_m(\varphi, z)\, dz = \int\limits_E \sqrt{\det(D\varphi(z) D\varphi^*(z))}\, dz = \int\limits_{\mathbb{R}^m} \mathcal{H}^n(\varphi^{-1}(y) \cap E)\, dy$$



The Case $m = 1$

- $(\mathbb{R}^N, |\cdot|_N)$; $(\mathbb{R}, |\cdot|)$; $\mathbb{R}^N = \mathbb{R}^{N-1} \times \mathbb{R}$

- $E \subset \mathbb{R}^N$, $N > 1$

$\Rightarrow$ Then

$$\int_E \langle \nabla\varphi(z), \nabla\varphi(z)\rangle^{1/2}\, dz = \int_{\mathbb{R}} \mathcal{H}^{N-1}(\varphi^{-1}(y) \cap E)\, dy,$$

where $\varphi \in C^1(E, \mathbb{R})$

$\diamond$ $\nabla\varphi(z) = (\frac{\partial\varphi}{\partial x_1}(z), \ldots, \frac{\partial\varphi}{\partial x_N}(z))$ is a gradient of φ at z

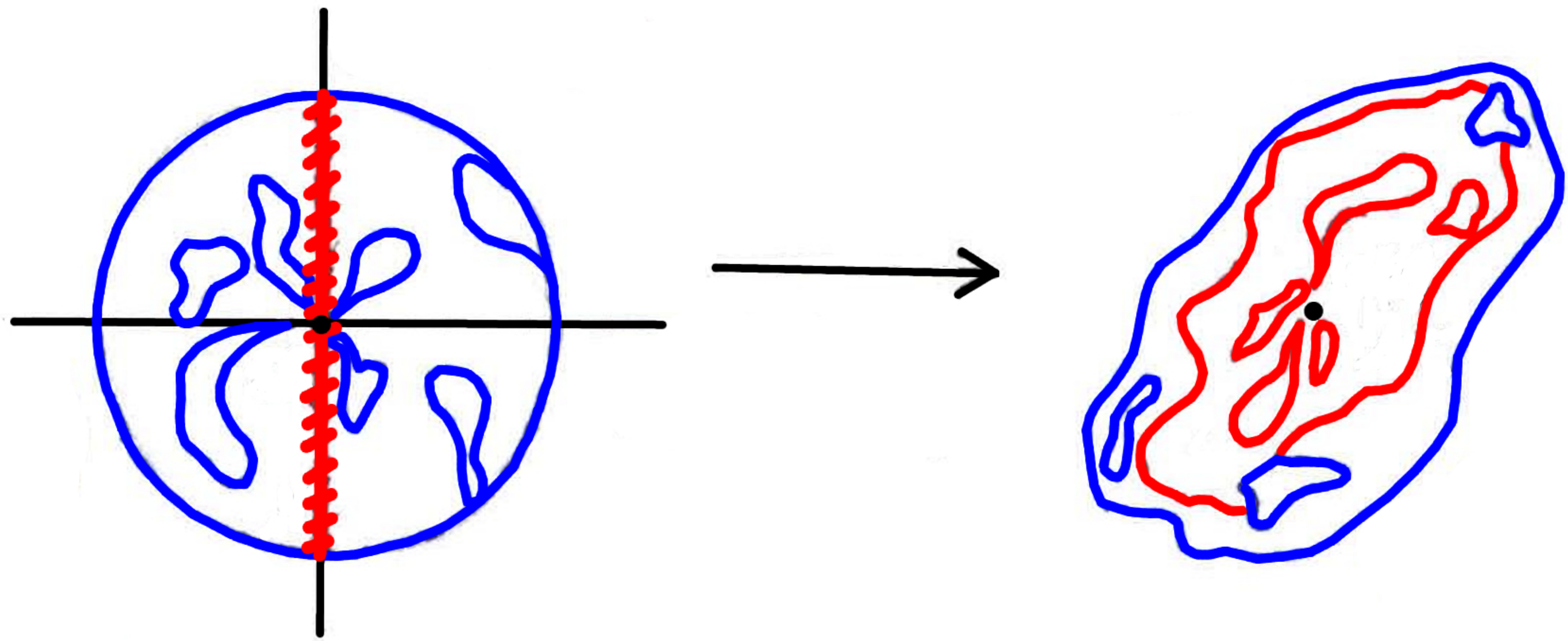
Geometric Sense of the Coarea Factor

$\Rightarrow$ Local measure distortion w. r. t. $\ker^{\perp}$

$\Rightarrow$ We have

$$\mathcal{J}_m(\varphi, z) = \sqrt{\det(D\varphi(z)D\varphi^*(z))}$$
$$= \lim_{r\to 0} \frac{\mathcal{H}^m\Big(\varphi\Big(B(z,r)\cap E\cap z + (\ker D\varphi(z))^{\perp}\Big)\Big)}{\omega_m r^m}$$

almost everywhere



Applications of the Coarea Formula

- Extremal surfaces theory

$\Rightarrow$ Its development to Minkowski geometry structures, non-holonomic geometry, new properties of solutions to differential equations, modeling of various processes, etc.

- Currents theory

- Algebraic geometry

- New proof of Stokes formula

$\Rightarrow$ Deduction of the analog of $\int\limits_{\mathcal{M}} d\omega = \int\limits_{\partial\mathcal{M}} \omega$ on general structures, definition of new objects, etc

(see lectures by S.K. Vodopyanov)

Minkowski Geometry

- Consider the following definition of a vector's length in $\mathbb{R}^4$:

$$|v|_L = \begin{cases} \sqrt{x^2+y^2+z^2-t^2}, & x^2+y^2+z^2-t^2 \geq 0 \\ i\sqrt{|x^2+y^2+z^2-t^2|}, & x^2+y^2+z^2-t^2 < 0, \end{cases}$$

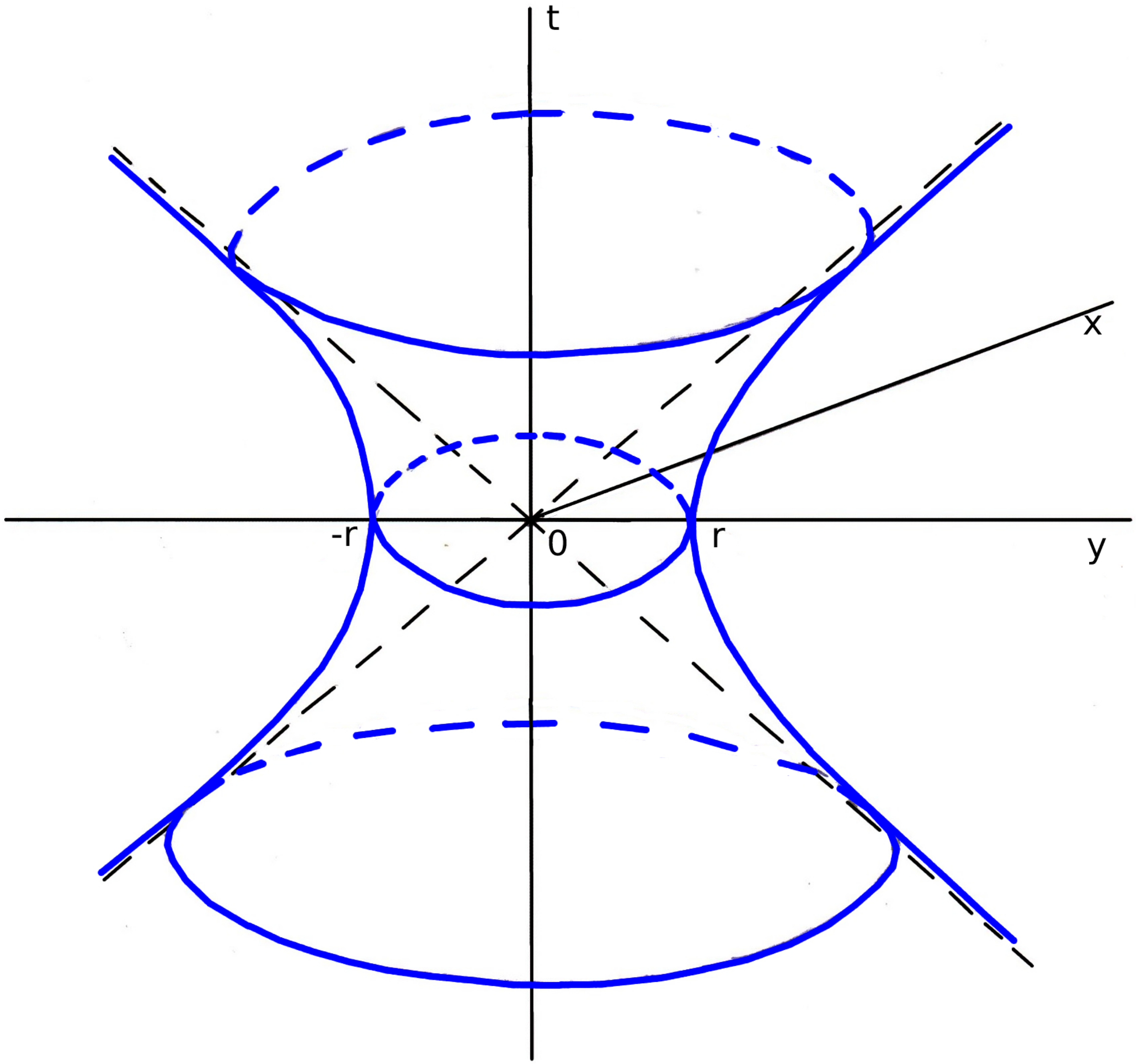
where $v = xe_1 + ye_2 + ze_3 + te_4$.

- If $a, b \in \mathbb{R}^4$ and $b = a + v$ then

$$d_L(a,b) = |v|_L.$$

See

- Miklyukov V. M., Klyachin A. A., Klyachin V. A. Maximal Surfaces in Minkowski Space-Time
http://www.uchimsya.info/maxsurf.pdf


t
x
-r
0
r
y

Vectors and Surfaces

- $v \in \mathbb{R}^4_1$

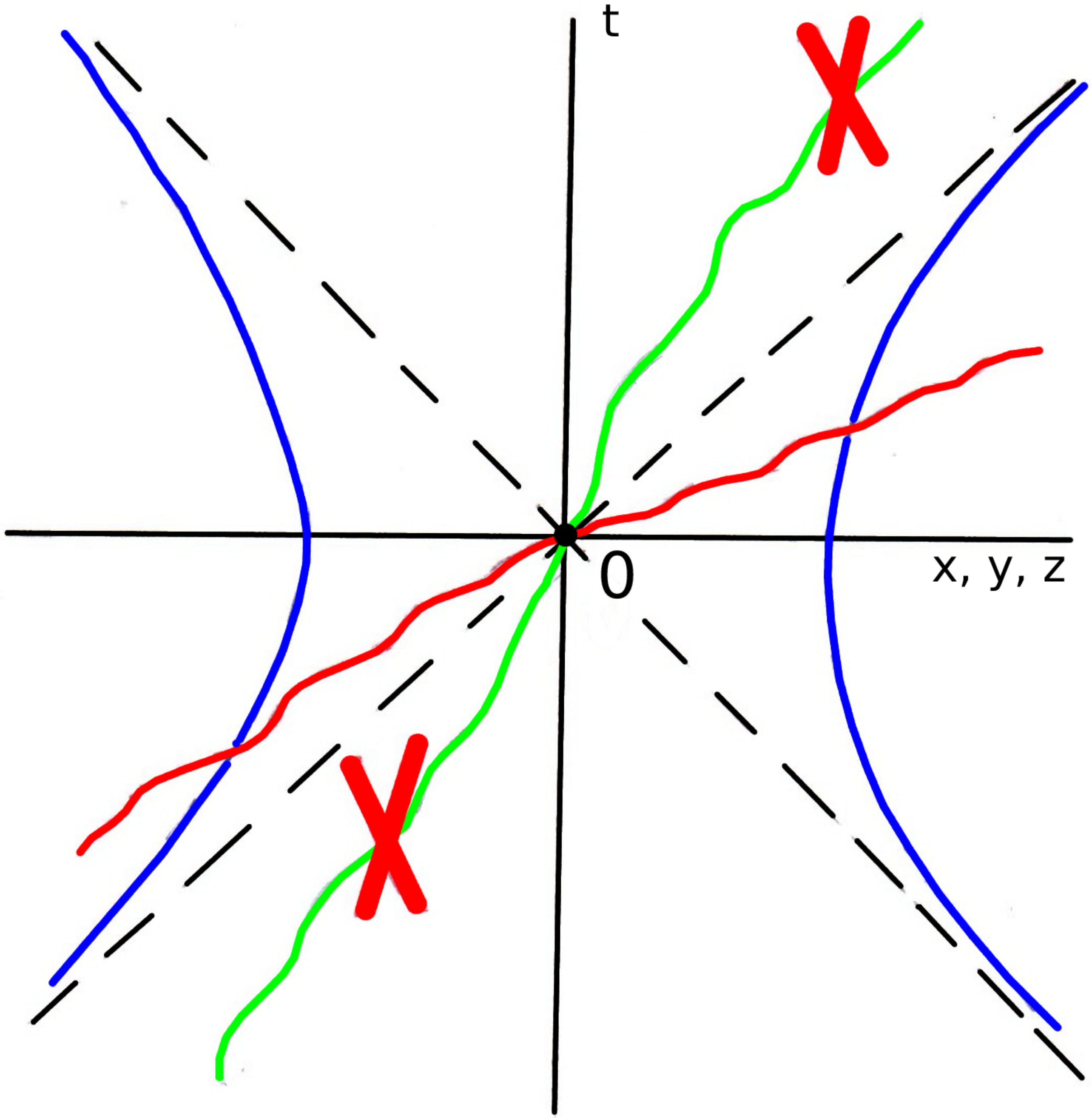
◇ $|v|^2_L > 0 \Rightarrow v$ is a spacelike vector

- Light cone: $\{\mathbf{x}' \in \mathbb{R}^4_1 : d^2_L(\mathbf{x}', 0) = 0\} = \partial B_L(0, 0)$

- Light cone centered at $\mathbf{x} \in \mathbb{R}^4_1$:

$$\{\mathbf{x}' \in \mathbb{R}^4_1 : d^2_L(\mathbf{x}', \mathbf{x}) = 0\} = \partial B_L(\mathbf{x}, 0)$$

◇ All tangent vectors are spacelike
$\Rightarrow$ the surface is spacelike
$\Leftrightarrow$ for each light cone centered at this surface, the surface lies locally outside it (except the center)


t
0
x, y, z

Carnot Groups

- Let $\mathbb{G}$ be a connected simply connected Lie group, where

◇ Lie algebra V can be represented as $V = \bigoplus_{k=1}^{M} V_k$, where

$$V_{k+1} = [V_1, V_k],\ k = 1, \ldots, M-1, \quad \text{and} \quad [V_1, V_M] = \{0\}$$

- This structure is called a Carnot group of the depth M

- Put $V_1 = \text{span}\{X_1, \ldots, X_n\}$, $n < N$

- $X_l \in V_k \Leftrightarrow \deg X_l = k$

Applications of Non-Holonomic Structures

- subelliptic equations
- non-holonomic mechanics
- contact geometry
- physics
- amoeba theory in thermodynamics
- robotechnics (J.-P. Laumond, А. Аграчев, Ю. Сачков, etc.)
- neurobiology (J. Petitot, G. Citti, A. Sarti, etc.)
- astrodynamics (J.K. Whiting)

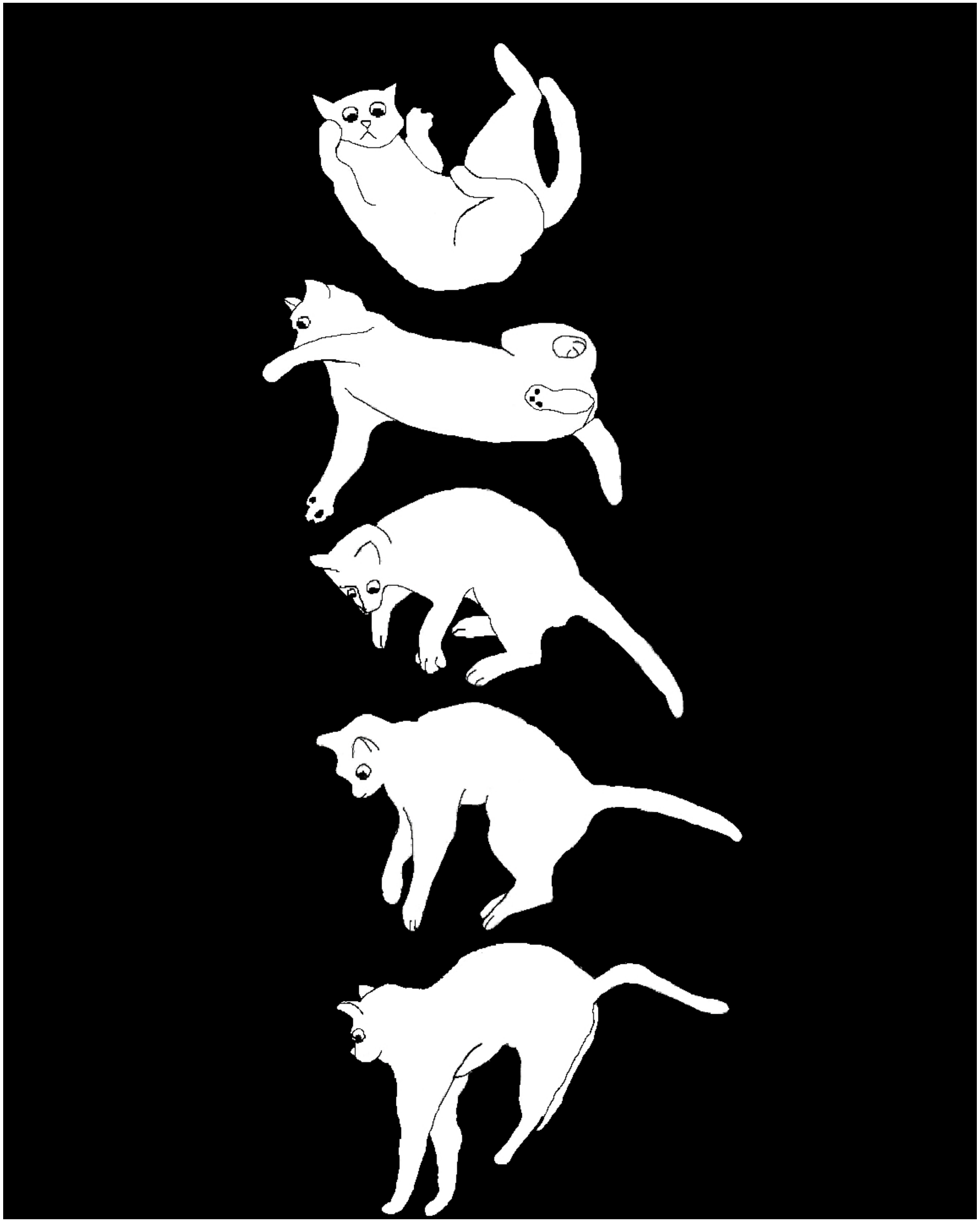
An Example: a Falling Cat Problem

- Model: a pair of cylinders that can change relative orientation

- Description in terms of connection of configuration space containing relative motions of these two parts admissible by physics

- Solution: horizontal (admissible w. r. t. physics) curve γ with given endpoints. I. e., $\dot{\gamma} \in V_1$

- Montgomery R. Gauge Theory of the Falling Cat // In: Enos, M.J. (ed.), Dynamics and Control of Mechanical Systems, American Mathematical Society, 1993. P. 193–218.



Group Operation

◇ <u>Homogeneous degree</u> $\lambda = (\lambda_1, \ldots, \lambda_N)$: $|\lambda|_h = \sum_{k=1}^{N} \lambda_k \deg X_k$

• Let $x = \exp\left(\sum_{i=1}^{N} x_i X_i\right)(0)$, $y = \exp\left(\sum_{j=1}^{N} y_j X_j\right)(0)$. Then

$$x \cdot y = \exp\left(\sum_{j=1}^{N} y_j X_j\right)(x) = \exp\left(\sum_{k=1}^{N} z_k X_k\right)(0),$$

where

$$z_k = \begin{cases} x_k + y_k, & k \leq n \\ x_k + y_k + \sum\limits_{\substack{|\alpha+\beta|_h = \deg X_k \\ \alpha > 0,\, \beta > 0}} F^k_{\alpha\beta} x^\alpha y^\beta, & k > n \end{cases}$$

The Distance d_2

- Let $x = \exp\left(\sum_{j=1}^{N} x_j X_j\right)(y)$. Then $d_2(x,y)$ equals

$$\max\left\{\left(\sum_{j:\,X_j\in V_1} x_j^2\right)^{\frac{1}{2}},\ \left(\sum_{j:\,X_j\in V_2} x_j^2\right)^{\frac{1}{2\cdot 2}},\ldots,\left(\sum_{j:\,X_j\in V_M} x_j^2\right)^{\frac{1}{2\cdot M}}\right\}$$

- $\exists C < \infty : \ d_2(v,w) \leq C(d_2(v,u) + d_2(u,w)) \quad \forall w,v,u \in \mathbb{G}$

- In normal coordinates θ_x^{-1}, the ball $\mathrm{Box}_2(x,r)$ in d_2 equals the Cartesian product of Euclidean balls

$$B(0,r) \times B(0,r^2) \times \ldots \times B(0,r^M)$$

Sub-Riemannian Differentiability

Definition [Pansu, Vodopyanov]. Let $\Omega \subset \mathbb{R}^m$. The mapping $\varphi : (\Omega, d_2) \to (\widetilde{\mathbb{G}}, \widetilde{d}_2)$ is *hc-differentiable* at $u \in \Omega$ if there exists a horizontal homomorphism

$$L_u : (\mathbb{G}, d_2) \to (\widetilde{\mathbb{G}}, \widetilde{d}_2)$$

such that

$$\widetilde{d}_2(\varphi(w), L_u(w)) = o(d_2(u, w)), \ \Omega \cap \mathbb{G} \ni w \to u.$$

- Denote hc-differential of φ at u by $\widehat{D}\varphi(u)$

Theorem [Vodopyanov]. *Let $\varphi : (\Omega, d_2) \to (\widetilde{\mathbb{G}}, \widetilde{d}_2)$ be a C^1_H-smooth and contact: that is, $V_1\varphi \subset \widetilde{V}_1$, and horizontal derivatives are continuous.*

Then φ is continuously hc-differentiable everywhere.

Sub-Riemannian Differential of a Function

- $(\widetilde{\mathbb{G}}, \widetilde{d}_2) = (\mathbb{R}, |\cdot|) \Rightarrow \widetilde{V}_1 = \mathbb{R}$

- $\varphi : (\mathbb{G}, d_2) \to (\mathbb{R}, |\cdot|)$

- $L_u = \widehat{D}\varphi(u) = (X_1\varphi(u), \ldots, X_n\varphi(u), 0, \ldots, 0)$

- $w = \exp\Big(\sum_{i=1}^{N} w_i X_i\Big)(u)$ is from a neighborhood of u

$\Rightarrow$ $\widehat{D}\varphi(u)(w) = \Big\langle (X_1\varphi(u), \ldots, X_n\varphi(u), 0, \ldots, 0), (w_1, \ldots, w_N) \Big\rangle$

Measures on Carnot Groups

- Riemannian measure:

$$\mathcal{H}^N(A) = \omega_N \lim_{\delta\to 0} \inf\Big\{\sum_{i\in\mathbb{N}} r_i^N : \bigcup_{i\in\mathbb{N}} \mathrm{B}(x_i, r_i) \supset A,\ x_i \in A,\ r_i < \delta\Big\},$$

where $\mathrm{B}(x_i, r_i)$ are balls in Riemannain metric d_R

(i. e., the length of a curve connection x and $y = \exp\Big(\sum_{j=1}^{N} y_j X_j\Big)(x)$)

- Sub-Riemannian measure; $\nu := \dim_{\mathcal{H}}(\mathbb{G}) = \sum_{k=1}^{M} k \cdot \dim V_k$:

$$\mathcal{H}^\nu(A) = \omega_N \lim_{\delta\to 0} \inf\Big\{\sum_{i\in\mathbb{N}} r_i^\nu : \bigcup_{i\in\mathbb{N}} \mathrm{Box}_2(x_i, r_i) \supset A,\ x_i \in A,\ r_i < \delta\Big\},$$

where $\mathrm{Box}_2(x_i, r_i)$ are balls in sub-Riemannian quasimetric d_2

Sub-Lorentzian Structure on Carnot Groups

- Let X_1 be a timelike direction; $V_1 = \text{span}\{X_1, \dots, X_n\}$

- $d_L^2(x,y) = (x_2)^2 + (x_3)^2 + (x_4)^2 - (x_1)^2$ for $x = \exp\left(\sum_{j=1}^{4} x_j \frac{\partial}{\partial x_j}\right)(y)$

- $$d_2^2(x,y) \quad = \quad \max\left\{\sum_{j=1}^{n} x_j^2, \left(\sum_{j:\, X_j \in V_2} x_j^2\right)^{\frac{1}{2}}, \dots, \left(\sum_{j:\, X_j \in V_M} x_j^2\right)^{\frac{1}{M}}\right\}$$

for $x = \exp\left(\sum_{j=1}^{N} x_j X_j\right)(y)$

$$\Rightarrow \mathfrak{d}_2^2(x,y) = \max\left\{\sum_{j=2}^{n} x_j^2 - x_1^2,\ \left(\sum_{j:\, X_j \in V_2} x_j^2\right)^{\frac{1}{2}}, \dots, \left(\sum_{j:\, X_j \in V_M} x_j^2\right)^{\frac{1}{M}}\right\}$$

- $\mathfrak{d}_2^2$ is a squared sub-Lorentzian distance

References and Main Results

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Grochowski M. Reachable sets for the Heisenberg sub-Lorentzian structure on $\mathbb{R}^3$. An estimate for the distance function // J. Dyn. Control Syst. 2006. V. 12, no. 2. P. 145–160.

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Applications

- Equations of motion of a charged particle in a five-dimensional model of general relativity with a nonholonomic four-dimensional velocity space

- The solution of the variational problem for the length functional satisfies the equations of motion of a charged particle of the general theory of relativity

Krym V. R., Petrov N. N. Equations of motion of a charged particle in a five-dimensional model of the general theory of relativity with a nonholonomic four-dimensional velocity space // Vestn. St. Petersburg Univ. Math. 2007. V. 40, no. 1. P. 52–60.

Krym V. R., Petrov N. N. The curvature tensor and the Einstein equations for a four-dimensional nonholonomic distribution // Vestn. St. Petersburg Univ. Math. 2008. V. 41, no. 3. P. 256–265.

Sub-Lorentzian Balls on Carnot Groups

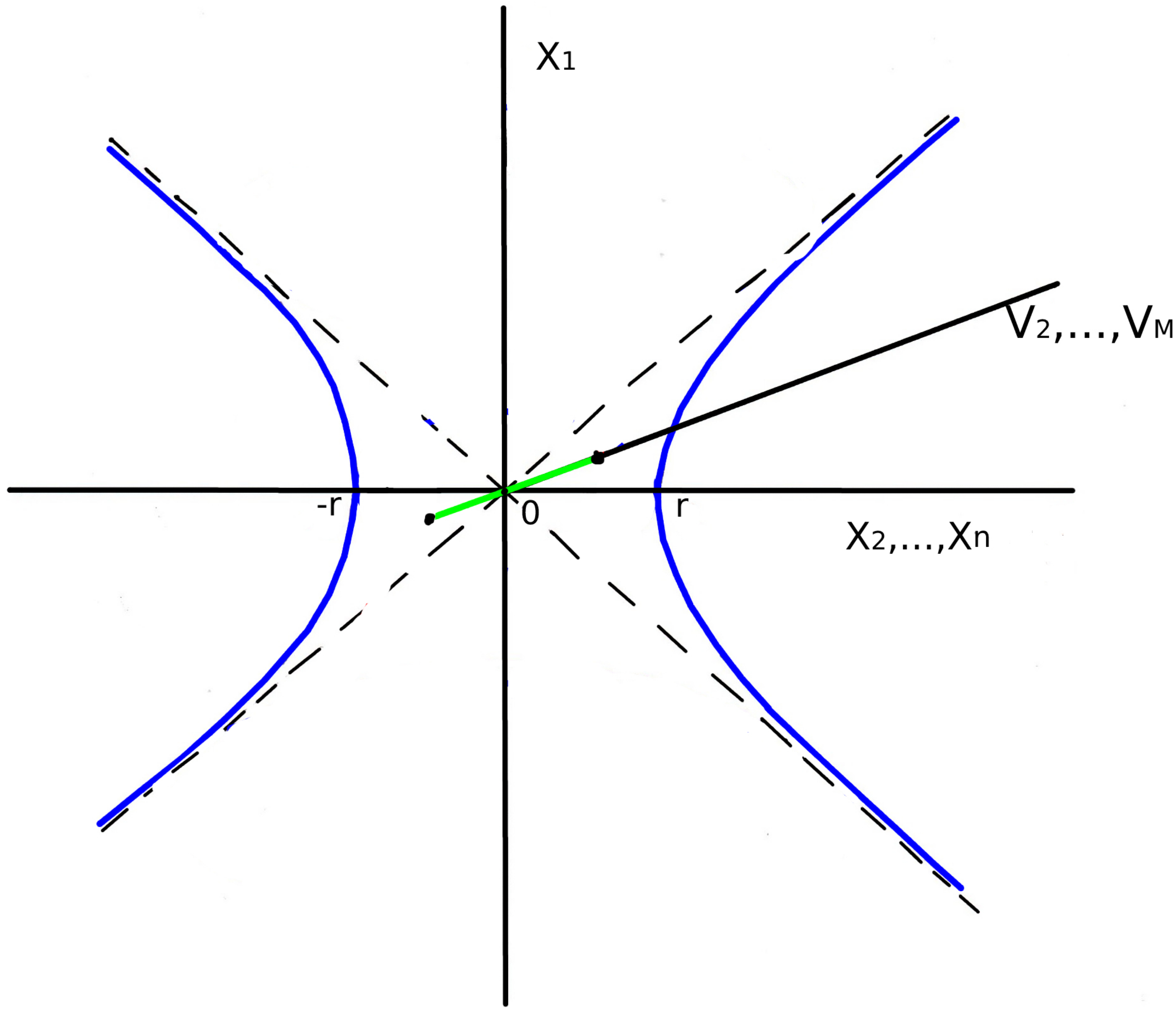
- $x = \exp\left(\sum_{j=1}^{N} x_j X_j\right)(y)$

$\Rightarrow$ $\mathfrak{d}_2^2(x,y) = \max\left\{\sum_{j=2}^{n} x_j^2 - x_1^2,\ \left(\sum_{j:\,X_j \in V_2} x_j^2\right)^{\frac{1}{2}}, \dots, \left(\sum_{j:\,X_j \in V_M} x_j^2\right)^{\frac{1}{M}}\right\}$

- Sub-Lorentzian ball centered at x of the radius $r \geq 0$:

$$\mathrm{Box}_{\mathfrak{d}}(x,r) = \{y : \mathfrak{d}_2^2(x,y)) < r^2\}$$

$\Rightarrow$ In normal coordinates: Cartesian product of $B_L(0,r)$ and $B(0,r^2) \times \ldots \times B(0,r^M)$



X1
V2,...,VM
-r
0
r
X2,...,Xn

Light Cone

- Light cone: let $y = \exp\left(\sum_{j=1}^{N} y_j X_j\right)(0)$

$$\{y : \mathfrak{d}_2^2(0, y)) = 0\}$$

$$= \left\{y : \sum_{j=2}^{n} y_j^2 - y_1^2 = 0,\ \max\left\{\left(\sum_{j:\, X_j \in V_2} x_j^2\right)^{\frac{1}{2}}, \ldots, \left(\sum_{j:\, X_j \in V_M} x_j^2\right)^{\frac{1}{M}}\right\} = 0\right\}$$

$$= \left\{y : \sum_{j=2}^{n} y_j^2 - y_1^2 = 0,\ y_{n+1} = \ldots = y_N = 0\right\} = \partial\, \mathrm{Box}_{\mathfrak{d}}(0, 0)$$

$\Rightarrow$ Light cone lies in $\exp(V_1)(0)$ **!**

- Light cone centered at x: $\{y : \mathfrak{d}_2^2(x, y) = 0\}$

$\Rightarrow$ It lies in $\exp(V_1)(x)$

Spacelike Surfaces

◇ All tangent vectors are spacelike

$\Rightarrow$ the surface is spacelike

$\Leftrightarrow$ for each light cone centered at this surface, the surface lies locally outside it (except the center)

- Spacelike surface S (a local property):

$$y \in S \Rightarrow \mathrm{Box}_{\mathfrak{d}}(y, 0) \cap S = \{y\}$$

- This property must be checked only for horizontal coordinates**!**

Hausdorff Measure

- $\mathcal{H}^l(A) = \omega_l \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^l : \bigcup_{i \in \mathbb{N}} \mathrm{Box}_{\mathfrak{d}}(x_i, r_i) \supset A,\ x_i \in A,\ r_i < \delta \right\}$

◇ This definition makes sense for spacelike surfaces only

- A light cone is a set of a <u>measure zero</u> for all $l > 0$**!**

◇ It is sufficient to consider $\mathrm{Box}_{\mathfrak{d}}(x, r)$, where x is a cone's center

From Classical to Sub-Lorentzian

- Classical type:

$$\int_{\Omega} \langle \nabla\varphi(z), \nabla\varphi(z)\rangle^{1/2}\, d\mathcal{H}^{N}(z) = \int_{\mathbb{R}} dy \int_{\varphi^{-1}(y)\cap\Omega} d\mathcal{H}^{N-1}$$

- Riemannian case:

$$\mathcal{J}^{R}(\varphi, z) = \frac{\langle \nabla\varphi(z), \nabla\varphi(z)\rangle_{g}^{1/2}}{\sqrt{\det(g|_{\nabla\varphi(z)}(z))}} = \langle \nabla\varphi(z), \nabla\varphi(z)\rangle^{1/2}\frac{\sqrt{\det(g|_{\ker\nabla\varphi(z)}(z))}}{\sqrt{\det(g(z))}}$$

- We should prove <u>sub-Lorentzian type</u>:

$$\int_{\Omega} \mathcal{J}^{SL}(\varphi, z)\, d\mathcal{H}^{\nu}(z) = \int_{\mathbb{R}} dy \int_{\varphi^{-1}(y)\cap\Omega} d\mathcal{H}_{\mathfrak{d}}^{\nu-1}$$

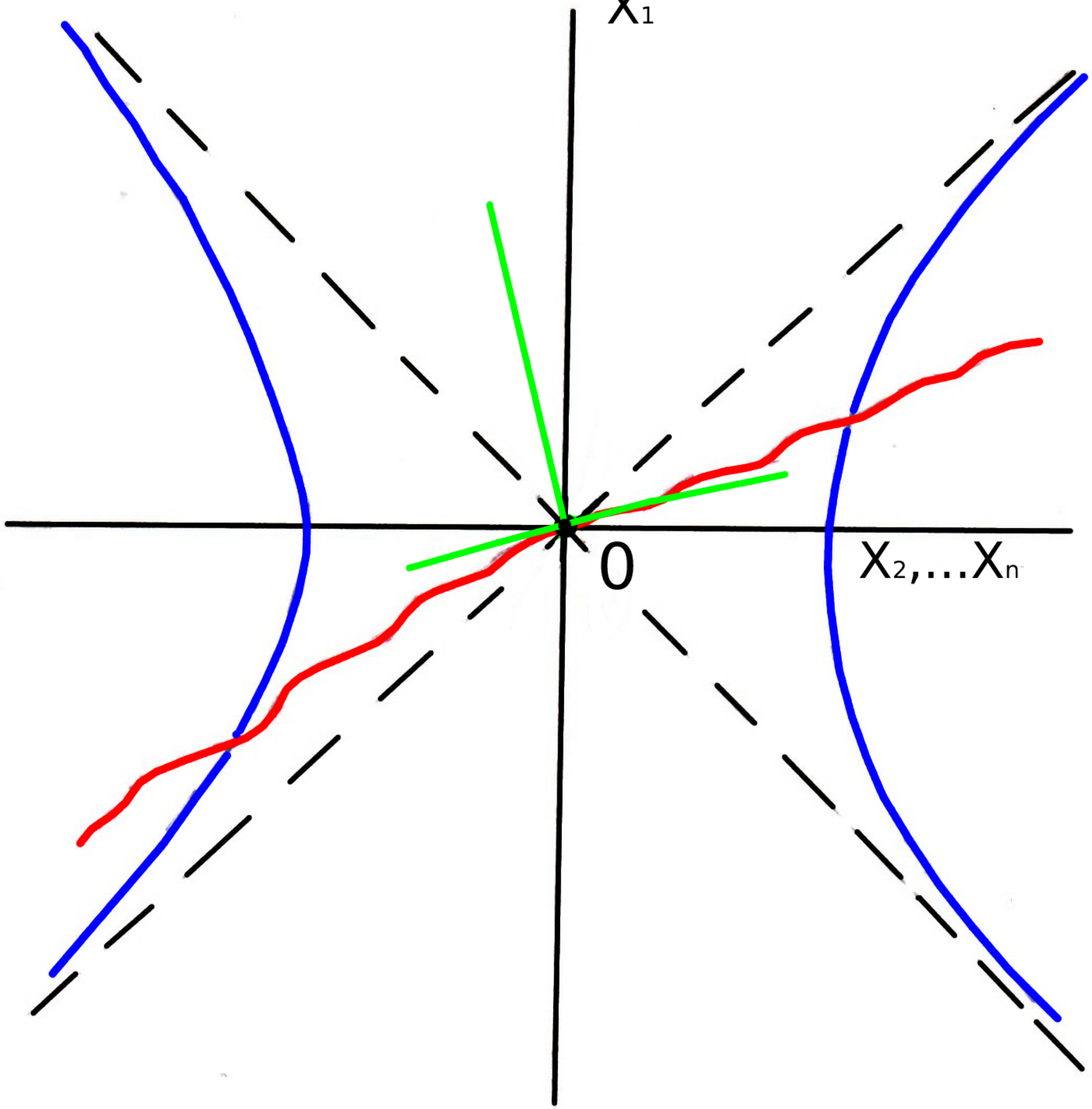
Main Steps of the Proof

- Riemannian measure: $\mathcal{H}^N(\mathrm{Box}_2(x,r)) = \sqrt{\det(g(x))} \cdot \omega_N r^\nu \cdot (1 + o(1))$

- Sub-Riemannian Measure: $\mathcal{H}^\nu(\mathrm{Box}_2(x,r)) = \omega_N r^\nu \cdot (1 + o(1))$

- Spacelike condition for surfaces: the following should hold

$$(X_1\varphi(x))^2 - \sum_{j=2}^{n} (X_j\varphi(x))^2 > 0$$


X1
0
X2,...Xn

Hausdorff Measure

- For $A \subset \varphi^{-1}(z)$, put

$$\mathcal{H}_{\mathfrak{d}}^{\nu-1}(A) =$$

$$= \omega_{n-1} \prod_{k=2}^{M} \omega_k \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^{\nu-1} : \bigcup_{i \in \mathbb{N}} \mathrm{Box}_{\mathfrak{d}}(x_i, r_i) \supset A,\ x_i \in A,\ r_i < \delta \right\}$$

◇ Level sets are spacelike

◇ Put $\nabla_H \varphi = \widehat{D}\varphi$

Main Steps of Proof

- Linearization:

$$\mathcal{H}^{N-1}\Big(\varphi^{-1}(\varphi(v))\cap \mathrm{Box}_{\mathfrak{d}}(v,r)\Big) = (1+o(1))\mathcal{H}^{N-1}\Big(\ker \nabla\varphi(v) \cap \mathrm{Box}_{\mathfrak{d}}(v,r)\Big)$$

- Compute $\mathcal{H}^{N-1}\Big(\ker \nabla\varphi(v) \cap \mathrm{Box}_{\mathfrak{d}}(v,r)\Big)$ for $\ker \nabla\varphi$:

$$\mathcal{H}^{N-1}\Big(\ker \nabla_H\varphi(v) \cap \mathrm{Box}_{\mathfrak{d}}(v,r)\Big) \cdot \frac{\langle \nabla\varphi(v), \nabla\varphi(v)\rangle^{1/2}}{\langle \nabla_H\varphi(v), \nabla_H\varphi(v)\rangle^{1/2}}$$

- Compute $\mathcal{H}^{N-1}\Big(\ker \nabla_H\varphi(v) \cap \mathrm{Box}_{\mathfrak{d}}(v,r)\Big)$ for $\ker \nabla_H\varphi$:

$$\frac{\langle \nabla_H\varphi(v), \nabla_H\varphi(v)\rangle^{1/2}}{\Big((X_1\varphi(v))^2 - \sum_{j=2}^{n}(X_j\varphi(v))^2\Big)^{1/2}} \cdot \omega_{n-1}\omega_{N-n} r^{\nu-1}\sqrt{\det(g|_{\ker \nabla\varphi(v)}(v))}$$

- Compute $\mathcal{H}_{\mathfrak{d}}^{\nu-1}\Big(\varphi^{-1}(\varphi(v)) \cap \mathrm{Box}_{\mathfrak{d}}(v,r)\Big) = \omega_{n-1}\prod_{k=2}^{M}\omega_k r^{\nu-1}$

Coarea Formula

- Define

$$\mathcal{J}^{SL}(\varphi, z) = \left((X_1\varphi(v))^2 - \sum_{j=2}^{n} (X_j\varphi(v))^2 \right)^{1/2}$$

- We infer from the previous slide that

$$\int_{\varphi^{-1}(y)} \frac{\mathcal{J}^{SL}(\varphi, v)}{\langle \nabla\varphi(v), \nabla\varphi(v)\rangle^{1/2} \sqrt{\det(g|_{\ker \nabla\varphi(v)}(v))}} d\mathcal{H}^{N-1} = \int_{\varphi^{-1}(y)} d\mathcal{H}_{\mathfrak{d}}^{\nu-1}(u)$$

- We obtain

$$\int_{\Omega} \left((X_1\varphi(v))^2 - \sum_{j=2}^{n} (X_j\varphi(v))^2 \right)^{1/2} d\mathcal{H}^{\nu}(v) = \int_{\mathbb{R}} dy \int_{\varphi^{-1}(y)\cap\Omega} d\mathcal{H}_{\mathfrak{d}}^{\nu-1}(u)$$

Minkowski Geometry

- The result is new even for Minkowski geometry!

- We have

$$\int_{\Omega} \sqrt{\Big(\frac{\partial \varphi}{\partial t}(v)\Big)^2 - \sum_{j=1}^{m} \Big(\frac{\partial \varphi}{\partial x_j}(v)\Big)^2 \Big)} \, d\mathcal{H}^m(v) = \int_{\mathbb{R}} dy \int_{\varphi^{-1}(y)\cap\Omega} d\mathcal{H}^{m-1}_{d_L}$$

- The measure $\mathcal{H}^{m-1}_{d_L}$ is constructed w. r. t.

$$d_L^2(\mathbf{x}, \mathbf{x}') = \sum_{j=1}^{m} (x_j - x_j')^2 - (t - t')^2.$$

Multidimensional Timelike Coordinates

Craig W., Weinstein S. On Determinism and Well-Posedness in Multiple Time Dimensions // Proc. R. Soc. A. 2008. V. 465, no. 2110. P. 3023–3046.

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The Case $\varphi : \mathbb{G} \to \mathbb{R}^{\widetilde{n}}$

- $X_1, \dots, X_{n^-}$ are timelike fields, $n^- \le \tilde{n} < n$:

$$\mathfrak{d}_2^2(x,y) = \max\Big\{ \sum_{j=n^-+1}^{n} x_j^2 - \sum_{k=1}^{n^-} x_k^2,\ \Big(\sum_{j\colon X_j \in V_2} x_j^2\Big)^{\frac{1}{2}}, \dots, \Big(\sum_{j\colon X_j \in V_M} x_j^2\Big)^{\frac{1}{M}} \Big\}$$

- $D_H^{\widetilde{n}}\varphi(x)$: a part of $D_H\varphi(x)$ consisting of the first $\tilde{n}$ columns

- $D_H \setminus D_H^{\widetilde{n}}\varphi$: $(\tilde{n} \times (n - \tilde{n}))$-matrix obtained by deletion of columns $D_H^{\widetilde{n}}\varphi$ and columns numbered $n+1, \dots, N$ from $D_H\varphi$

Coarea Formula for $\varphi : \mathbb{G} \to \mathbb{R}^{\widetilde{n}}$

- Coarea Factor $\mathcal{J}_{\widetilde{n}}^{n^-}(\varphi, x)$:

$$\left(\det\left(D_H^{\widetilde{n}}\varphi(D_H^{\widetilde{n}}\varphi)^* + \left(D_H^{\widetilde{n}}\varphi\right)E_{n^-}^{-}\left(D_H^{\widetilde{n}}\varphi\right)^{-1}\left(D_H \backslash D_H^{\widetilde{n}}\varphi\right)\left(D_H \backslash D_H^{\widetilde{n}}\varphi\right)^*\right)\right)^{1/2},$$

- Coarea formula:

$$\int_{\Omega} \mathcal{J}_{\widetilde{n}}^{n^-}(\varphi, x)\, d\mathcal{H}^{\nu}(x) = \int_{\mathbb{R}^{\widetilde{n}}} d\mathcal{H}^{\widetilde{n}}(z) \int_{\varphi^{-1}(z)} d\mathcal{H}_{\mathfrak{d}}^{\nu - \widetilde{n}}(y)$$

- Coarea formula for $n^- = \widetilde{n}$:

$$\int_{\Omega} \left(\det\left(D_H^{\widetilde{n}}\varphi(D_H^{\widetilde{n}}\varphi)^* - \left(D_H \setminus D_H^{\widetilde{n}}\varphi\right)\left(D_H \setminus D_H^{\widetilde{n}}\varphi\right)^*\right)\right)^{1/2} d\mathcal{H}^{\nu}$$

$$= \int_{\mathbb{R}^{\widetilde{n}}} d\mathcal{H}^{\widetilde{n}}(z) \int_{\varphi^{-1}(z)} d\mathcal{H}_{\mathfrak{d}}^{\nu - \widetilde{n}}(y)$$

The Case $\varphi : \mathbb{G} \to \widetilde{\mathbb{G}}$

- $\varphi : (\mathbb{G}, \bigoplus_{k=1}^{M} V_k) \to (\widetilde{\mathbb{G}}, \bigoplus_{k=1}^{\widetilde{M}} \widetilde{V}_k)$
- $M \geq \widetilde{M}$, $\dim V_k > \dim \widetilde{V}_k$; put $\dim H_k = \sum_{l=1}^{k} \dim V_k$, $\dim H_0 = 0$
- $X_1, \ldots, X_{n_1^-}, X_{\dim H_1+1}, \ldots, X_{\dim H_1+n_2^-}, X_{\dim H_2+1}, \ldots, X_{\dim H_{M-1}+1}, \ldots, X_{\dim H_{M-1}+n_M^-}$ are timelike fields, $n_k^- \leq \dim \widetilde{V}_k < \dim V_k$, $k = 1, \ldots, \widetilde{M}$:

$$\mathfrak{d}_2^2(x, y) = \max_{k=1,\ldots,M} \Big\{ \Big(\sum_{j=\dim H_{k-1}+n_k^-+1}^{\dim H_k} x_j^2 - \sum_{j=\dim H_{k-1}+1}^{\dim H_{k-1}+n_k^-} x_j^2 \Big)^{\frac{1}{k}} \Big\}$$

- In each block of $\widehat{D}\varphi$, there exist $D_k^- \varphi(x)$, the analog of $D_H^{\widetilde{n}} \varphi(x)$

Coarea Formula for $\varphi : \mathbb{G} \to \widetilde{\mathbb{G}}$

- Coarea factor $\mathcal{J}^{-}(\varphi, x)$:

$$\prod_{k=1}^{M} \Big(\det \Big(D_k^{-}\varphi (D_k^{-}\varphi)^* + \big(D_k^{-}\varphi\big) E_{n_k^{-}}^{-} \big(D_k^{-}\varphi\big)^{-1} \big(\widehat{D}^k \backslash D_k^{-}\varphi\big) \big(\widehat{D}^k \backslash D_k^{-}\varphi\big)^* \Big) \Big)^{1/2},$$

- Coarea formula:

$$\int_{\Omega} \mathcal{J}^{-}(\varphi, x)\, d\mathcal{H}^{\nu}(x) = \int_{\widetilde{\mathbb{G}}} d\mathcal{H}^{\widetilde{\nu}}(z) \int_{\varphi^{-1}(z)} d\mathcal{H}_{\mathfrak{d}}^{\nu - \widetilde{\nu}}(y)$$

Coarea formula for $\varphi : \mathbb{G} \to \widetilde{\mathbb{G}}$

- Coarea formula for $n_k^- \le \dim \widetilde{V}_k$, $k = 1, \ldots, M$:

$$\int\limits_{\Omega} \prod_{k=1}^{M} \Big(\det\Big(D_k^- \varphi (D_k^- \varphi)^* - \big(\widehat{D}^k \setminus D_k^- \varphi\big)\big(\widehat{D}^k \setminus D_k^- \varphi\big)^*\Big)\Big)^{1/2} d\mathcal{H}^{\nu}$$

$$= \int\limits_{\widetilde{\mathbb{G}}} d\mathcal{H}^{\widetilde{\nu}}(z) \int\limits_{\varphi^{-1}(z)} d\mathcal{H}_{\mathfrak{d}}^{\nu - \widetilde{\nu}}(y)$$

$\ominus$ Essential difficulty: approximation of C^1-mapping by order higher than 1

Publications

Karmanova M. B. Coarea formula for functions on 2-step Carnot groups with sub-Lorentzian structure // Dokl. Math. 2020. V. 101. P. 129–131.

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THANK YOU!