On the structure of quasivariety lattices

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May 12, 2021

Quasivarieties

A *quasivariety* is a class of algebraic structures defined by *quasi-identities*, i.e., universal sentences of the form

$$\forall \overline{x}\ t_0(\overline{x})\ \&\ \ldots\ \&\ t_n(\overline{x}) \longrightarrow t(\overline{x}),$$

where $t_0(\overline{x}), \ldots, t_n(\overline{x}), t(\overline{x})$ are atomic formulas.

For a class $\mathbf{K}$, let $\mathbf{Q}(\mathbf{K})$ denote the least q-variety containing $\mathbf{K}$.

$\mathbf{K}(\sigma)$ denotes the class of all structures of type σ.

Quasivariety lattices

For a q-variety $\mathbf{K}$, a q-variety $\mathbf{K}' \subseteq \mathbf{K}$ is a *subquasivariety of* $\mathbf{K}$.

$\mathrm{Lq}(\mathbf{K})$ denotes the lattice of subquasivarieties of $\mathbf{K}$.

Historic remarks

- We consider quasivarieties of finite type.
- Cardinality 2^ω: V. P. Belkin, A. I. Budkin, V. A. Gorbunov (1970-ies).
- Elements without covers (and without independent q-bases): V. A. Gorbunov (1977), V. K. Kartashov (1980-ies), S. V. Sizyĭ (1995), A. I. Budkin.
- No nontrivial identities ($\mathrm{FL}(\omega)$ embeds): V. I. Tumanov; V. A. Gorbunov, W. Dziobiak, M. P. Tropin.
- $\mathcal{Q}$-universality (the ideal lattice of $\mathrm{FL}(\omega)$ embeds): M. V. Sapir (1985); M. E. Adams and W. Dziobiak (1994, AD-classes), V. A. Gorbunov (1995); M. E. Adams and W. Dziobiak, V. Koubek and J. Sichler (after 2000).

B-classes

Definition (A. V. Kravchenko, A. M. Nurakunov, MS, 2017)

Let $\mathbf{M} \subseteq \mathbf{K}(\sigma)$ be a q-variety. A class $\mathbf{A} = \{\mathcal{A}_F \mid F \in \mathcal{P}_{fin}(\omega)\} \subseteq \mathbf{M}$ is a B-*class with respect to* $\mathbf{M}$, if

(B$_0$) for any nonempty $F \in \mathcal{P}_{fin}(\omega)$, $\mathcal{A}_F$ is finitely presented in $\mathbf{M}$; $\mathcal{A}_\varnothing$ is a trivial structure;

(B$_1$) if $F = G \cup H$ then $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G, \mathcal{A}_H)$;

(B$_2$) if $F \neq \varnothing$ and $\mathcal{A}_F \in \mathbf{Q}(\mathcal{A}_G)$ then $F = G$;

(B$_3$) if $f \in \mathrm{Hom}(\mathcal{A}_F, \mathcal{A}_{\{i\}})$ then either $f(\mathcal{A}_F) \cong \mathcal{A}_\varnothing$ or $i \in F$;

(B$_4$) $\mathbf{H}(\mathcal{A}_F) \cap \mathbf{M} \subseteq \mathbf{A}$.

A B-class consisting of finite structures is a *finite* B-*class with respect to* $\mathbf{M}$.

If a q-variety $\mathbf{K}$ contains a [finite] B-class with respect to some q-variety $\mathbf{M} \subseteq \mathbf{K}$, then $\mathrm{Lq}(\mathbf{K})$ is "complex"—it contains continuum many elements $\mathbf{K}'$ such that:

1. $\mathbf{K}'$ is Q-universal;
2. $\mathbf{K}'$ has no independent but has an ω-independent q-equational basis relative to $\mathbf{K}$;
3. $\mathbf{K}'$ has an independent q-equational basis relative to $\mathbf{K}$; the q-theory of $\mathbf{K}'$ is undecidable; the finite membership problem for $\mathbf{K}'$ is undecidable;
4. all finite lattices are representable as $\mathbf{K}'$-congruence lattices of structures from $\mathbf{K}'$;
5. etc...

Q-varieties possessing a [finite] B-class are not too exotic. B-classes are to find in quasivarieties of

1. Abelian groups with a constant;
2. commutative rings with 1;
3. [un]directed graphs;
4. Cantor algebras;
5. modular $(0, 1)$-lattices;
6. unary algebras;
7. etc...

Boolean structures

A topology τ is *Boolean* if it is compact, Hausdorff, and has a basis consisting of clopen sets.

A structure $\mathcal{A} = \langle A; \sigma, \tau \rangle$ is *Boolean topological* if τ is a Boolean topology on A, each $f \in \sigma^F$ is continuous and each $p \in \sigma^P$ is closed with respect to the corresponding product topology defined by τ.

Standard quasivarieties

Let **K** be a quasivariety.

Profinite structures [*with respect to* **K**] are exactly those isomorphic to inverse limits of finite structures [from **K**].

Profinite structures are naturally equipped with Boolean topologies which are in this case product topologies.

A quasivariety **K** is *standard* if each Boolean topological structure $\mathcal{A} \in \mathbf{K}$ is profinite with respect to **K**.

Theorem (A. V. Kravchenko, A. M. Nurakunov, MS, 2021)

Let a q-variety $\mathbf{K}$ *contains a finite* B*-class relative to* $\mathbf{M} \subseteq \mathbf{K}$*. Then* $\mathbf{K}$ *contains continuum many non-standard subquasivarieties which*

1. *have an independent q-basis relative to* $\mathbf{K}$*;*
2. *do not have an independent q-basis relative to* $\mathbf{K}$*.*