On a model of population dynamics

Skvortsova Maria

Predator – prey model

Gourley S.A., Kuang Y. (2004)

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\ \\ \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\ \\ \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

Boundary value problem

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \quad t > 0, \\ \\ \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \quad t > 0, \\ \\ \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t), \quad t > 0, \\ \\ \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \quad x(+0) = \varphi(0), \quad \varphi \in C([-\tau, 0]), \\ \\ y(t) = \psi(t), \quad t \in [-\tau, 0], \quad y(+0) = \psi(0), \quad \psi \in C([-\tau, 0]), \\ \\ z(0) = z^0 \end{cases}$$

A solution is **exists and unique**.

Nonnegativity of solutions

$$
\begin{cases}
\dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\[2em]
\dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\[1em]
\dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t)
\end{cases}
$$

If $x(t) \geq 0$, $y(t) \geq 0$ for $t \in [-\tau, 0]$, then $x(t) \geq 0$, $y(t) \geq 0$ for all $t > 0$. If

$$
z(0) \geq \int_{-\tau}^{0} bpe^{c\xi}x(\xi)y(\xi)d\xi,
$$

then $z(t) \geq 0$ for all $t > 0$.

Boundness of solutions

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\ \\ \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\ \\ \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

Solutions are limited:

$$x(t) \leq M_1, \quad y(t) \leq M_2, \quad z(t) \leq M_3.$$

Equilibrium points

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\ \\ \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\ \\ \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

Case $bpe^{-c\tau}K \leq d$

2 equilibrium points: $(0,0,0)$, $(K,0,0)$

Case $bpe^{-c\tau}K > d$

3 equilibrium points: $(0,0,0)$, $(K,0,0)$,

$$(x_0, y_0, z_0) = \left(\frac{d}{bpe^{-c\tau}},\ \frac{r}{p}\left[1 - \frac{d}{bpe^{-c\tau}K}\right],\ \frac{d}{c}\frac{r}{p}\left[1 - \frac{d}{bpe^{-c\tau}K}\right]\left[e^{c\tau} - 1\right]\right)$$

Predator – prey model

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\[2ex] \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\[2ex] \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

Aim:

study of stability of equilibrium points of the system,

obtaining estimates of solutions characterizing
the rate of convergence to equilibrium points,

finding attraction domains.

Linear system

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + B\vec{y}(t-\tau)$$

Criterion for asymptotic stability.
The zero solution to linear system is asymptotically stable $\Longleftrightarrow$ all root of

$$\det\left(\lambda I - A - e^{-\lambda\tau}B\right)$$

are in the left-half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$.

Sufficient condition for instability.
If there exists a root λ_0 such that $\operatorname{Re}\lambda_0 > 0$, then the zero solution is unstable.

Nonlinear system

$$\frac{d}{dt}\vec{y}(t) = A\vec{y}(t) + B\vec{y}(t-\tau) + F(\vec{y}(t), \vec{y}(t-\tau)),$$

$$F(\vec{y}_1, \vec{y}_2) \in C^1(\mathbb{R}^{2n}), \qquad \frac{\|F(\vec{y}_1, \vec{y}_2)\|}{\|(\vec{y}_1, \vec{y}_2)\|} \to 0 \quad \text{as} \quad (\vec{y}_1, \vec{y}_2) \to 0$$

Stability in the first approximation.

I) If all roots of

$$\det\left(\lambda I - A - e^{-\lambda\tau}B\right)$$

are in the left-half plane, then the zero solution is asymptotically stable.

II) If there exists a root λ_0 such that $\mathrm{Re}\,\lambda_0 > 0$, then the zero solution to nonlinear system is unstable.

Instability of equilibrium point $(0, 0, 0)$

$$
\begin{cases}
\dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\[2ex]
\dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\[2ex]
\dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t)
\end{cases}
$$

Linearized system

$$
\begin{cases}
\dfrac{d}{dt}x(t) = rx(t), \\[2ex]
\dfrac{d}{dt}y(t) = -dy(t), \\[2ex]
\dfrac{d}{dt}z(t) = -cz(t)
\end{cases}
$$

Equilibrium point $(0, 0, 0)$ is unstable

Stability of equilibrium point $(K, 0, 0)$

Change of variables $x(t) = K + \widetilde{x}(t)$. Linearized system

$$\frac{d}{dt}\begin{pmatrix}\widetilde{x}(t)\\ y(t)\\ z(t)\end{pmatrix} = \begin{pmatrix}-r & -pK & 0\\ 0 & -d & 0\\ 0 & bpK & -c\end{pmatrix}\begin{pmatrix}\widetilde{x}(t)\\ y(t)\\ z(t)\end{pmatrix} + \begin{pmatrix}0 & 0 & 0\\ 0 & bpe^{-c\tau}K & 0\\ 0 & -bpe^{-c\tau}K & 0\end{pmatrix}\begin{pmatrix}\widetilde{x}(t-\tau)\\ y(t-\tau)\\ z(t-\tau)\end{pmatrix}$$

$$\det\left(\lambda I - A - e^{-\lambda\tau}B\right) = (\lambda + r)\left(\lambda + d - bpe^{-c\tau}Ke^{-\lambda\tau}\right)(\lambda + c)$$

$bpe^{-c\tau}K < d \implies (K, 0, 0)$ is asymptotically stable
(system has 2 equilibrium points)

$bpe^{-c\tau}K > d \implies (K, 0, 0)$ is unstable
(system has 3 equilibrium points)

Stability of equilibrium point (x_0, y_0, z_0)

Change of variables $x(t) = x_0 + u(t)$, $y(t) = y_0 + v(t)$, $z(t) = z_0 + w(t)$.

Linearized system

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} -\frac{rx_0}{K} & -px_0 & 0 \\ 0 & -d & 0 \\ bpy_0 & bpx_0 & -c \end{pmatrix}\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ \frac{y_0}{x_0}d & d & 0 \\ -\frac{y_0}{x_0}d & -d & 0 \end{pmatrix}\begin{pmatrix} u(t-\tau) \\ v(t-\tau) \\ w(t-\tau) \end{pmatrix}$$

$$\det\left(\lambda I - A - e^{-\lambda\tau}B\right) = \left[\left(\lambda + \frac{rx_0}{K}\right)(\lambda + d) - de^{-\lambda\tau}\left(\lambda + \frac{rx_0}{K} - py_0\right)\right](\lambda + c)$$

$d < bpe^{-c\tau}K \leq 3d \implies$ (x_0, y_0, z_0) is asymptotically stable

$bpe^{-c\tau}K > 3d \implies$ there exists $\tau_0 > 0$ such that:
if $0 \leq \tau < \tau_0$, then (x_0, y_0, z_0) is asymptotically stable
if $\tau > \tau_0$, then (x_0, y_0, z_0) is unstable

Predator – prey model

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\[2ex] \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\[2ex] \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

Aim:

obtaining estimates of solutions characterizing the rate of convergence to equilibrium points,

finding attraction domains.

System of ordinary differential equations

$$\frac{d}{dt}y = Ay$$

Theorem (A.M. Lyapunov).
The zero solution to linear system is asymptotically stable $\iff$ there exists a solution to matrix Lyapunov equation $H = H^* > 0$:

$$HA + A^*H = -I.$$

Theorem (M.G. Krein).
Let there exist a solution to matrix Lyapunov equation $H = H^* > 0$. Then

$$\|y(t)\| \le \sqrt{2\|A\|\|H\|}\,\mathrm{exp}\left(-\frac{t}{2\|H\|}\right)\|y(0)\|.$$

Linear system with delay

$$\frac{d}{dt}y(t) = Ay(t) + By(t-\tau)$$

Theorem (N.N. Krasovskii).
Let there exist matrices $H = H^* > 0$ and $K = K^* > 0$ such that

$$\begin{pmatrix} HA + A^*H + K & HB \\ B^*H & -K \end{pmatrix} < 0.$$

Then the zero solution is asymptotically stable.

Lyapunov – Krasovskii functional

$$V(t,y) = \langle Hy(t), y(t)\rangle + \int_{t-\tau}^{t} \langle Ky(s), y(s)\rangle ds$$

Using Lyapunov – Krasovskii functional, it is unable to obtain estimates that are analogues of the Krein's estimate.

Linear system with delay

$$\begin{cases} \dfrac{d}{dt}y(t) = Ay(t) + By(t-\tau), \quad t > 0, \\ \\ y(t) = \varphi(t), \quad t \in [-\tau, 0], \quad y(+0) = \varphi(0), \end{cases} \qquad \varphi(t) \in C([-\tau, 0])$$

Modified Lyapunov – Krasovskii functional

$$V(t,y) = \langle Hy(t), y(t)\rangle + \int_{t-\tau}^{t} \langle K(t-s)y(s), y(s)\rangle ds$$

Theorem (G.V. Demidenko, I.I. Matveeva).
Let there exist matrices $H = H^* > 0$ and $K(s) \in C^1([0,\tau])$ such that $K(s) = K^*(s) > 0$, $\frac{d}{ds}K(s) < 0$, and

$$\mathbf{C} = -\begin{pmatrix} HA + A^*H + K(0) & HB \\ B^*H & -K(\tau) \end{pmatrix} > 0.$$

Then

$$\|y(t)\| \leq \sqrt{h_{\min}^{-1} V(0,\varphi)}\, e^{-\varepsilon t/2},$$

$$\varepsilon = \min\left\{\frac{c_{\min}}{\|H\|}, k\right\}, \qquad \frac{d}{ds}K(s) + kK(s) \leq 0.$$

Estimates of the rate of convergence to $(K, 0, 0)$

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\ \\ \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\ \\ \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

$bpe^{-c\tau}K < d \implies (K, 0, 0)$ is asymptotically stable

Notation. Let $\theta > 0$ and æ > 0 such that

$$\tilde{d} = d - bpe^{-c\tau}(K + \theta)e^{æ\tau} > 0,$$

$$\delta_0 = \min\left\{\tilde{d}, æ\right\}, \quad \delta_1 = \min\left\{\tilde{d}, æ, r\right\}, \quad \delta_2 = \min\left\{\tilde{d}, æ, c\right\}.$$

Modified Lyapunov – Krasovskii functional

$$v(t, y) = y^2(t) + bpe^{-c\tau}(K + \theta)e^{æ\tau}\int\limits_{t-\tau}^{t} e^{-2æ(t-s)}y^2(s)ds$$

Estimates of the rate of convergence to $(K, 0, 0)$

Theorem 1. For solutions with initial conditions

$$0 \leq x(t) \leq K + \theta, \quad t \in [-\tau, 0], \quad x(0) > 0,$$

$$y(t) \geq 0, \quad t \in [-\tau, 0], \quad z(0) \geq \int_{-\tau}^{0} bpe^{c\xi} x(\xi) y(\xi) d\xi,$$

the estimates hold

$$|x(t) - K| \leq \frac{K \exp\left(p\sqrt{v(0,y)}/\delta_0\right)}{\min\{K, x(0)\}} \left[|x(0) - K| + x(0)\, p\sqrt{v(0,y)}\, t\right] e^{-\delta_1 t},$$

$$y(t) \leq \sqrt{v(0,y)}\, e^{-\delta_0 t},$$

$$z(t) \leq \left[y(0) + z(0) + \max\{c - d + bp(K + \theta), 0\} \sqrt{v(0,y)}\, t\right] e^{-\delta_2 t}.$$

Estimates of the rate of convergence to (x_0, y_0, z_0)

$$\begin{cases} \dfrac{d}{dt}x(t) = rx(t)\left(1 - \dfrac{x(t)}{K}\right) - px(t)y(t), \\[2ex] \dfrac{d}{dt}y(t) = bpe^{-c\tau}x(t-\tau)y(t-\tau) - dy(t), \\[2ex] \dfrac{d}{dt}z(t) = bpx(t)y(t) - bpe^{-c\tau}x(t-\tau)y(t-\tau) - cz(t) \end{cases}$$

$d < bpe^{-c\tau}K < 3d \implies (x_0, y_0, z_0)$ is asymptotically stable

Change of variables $x(t) = x_0 + u(t)$, $y(t) = y_0 + v(t)$, $z(t) = z_0 + w(t)$**:**

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = A\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + B\begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix} + \begin{pmatrix} u(t)\left(-\dfrac{r}{K}u(t) - pv(t)\right) \\ bpe^{-c\tau}u(t-\tau)v(t-\tau) \end{pmatrix}, \\[2ex] \dfrac{d}{dt}(v(t) + w(t)) = bp(y_0 + v(t))u(t) + (c - d + bpx_0)v(t) - c(v(t) + w(t)), \end{cases}$$

$$A = \begin{pmatrix} -\dfrac{rx_0}{K} & -px_0 \\ 0 & -d \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \dfrac{y_0}{x_0}d & d \end{pmatrix}.$$

Estimates of the rate of convergence to (x_0, y_0, z_0)

Modified Lyapunov – Krasovskii functional

$$V(t,u,v) = \left\langle H \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}, \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} \right\rangle + \int\limits_{t-\tau}^{t} \left\langle K(t-s) \begin{pmatrix} u(s) \\ v(s) \end{pmatrix}, \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} \right\rangle ds$$

Matrix H

$$H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix} > 0, \quad -HA - A^*H - B^*B - \widetilde{H}^*\widetilde{H} > 0, \quad \widetilde{H} = \begin{pmatrix} 0 & 0 \\ h_{12} & h_{22} \end{pmatrix} :$$

$$h_{11} = \frac{d^2}{py_0}\left(\frac{y_0}{x_0}\right)^2, \; h_{22} = d, \; 0 < h_{12}px_0 < \sqrt{d^2 py_0 \left(\frac{2rx_0}{K} - py_0\right) + \left(\frac{rx_0}{K}\right)^4} - \left(\frac{rx_0}{K}\right)^2.$$

Matrix $K(s)$

$$K(s) = e^{-ks}\left[B^*B + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}\right] > 0 :$$

$$k > 0 : \quad C = -HA - A^*H - B^*B - e^{k\tau}\widetilde{H}^*\widetilde{H} > 0, \quad 0 < \mu < \frac{d^2}{\|\widetilde{H}^*\widetilde{H}\|} c_{\min}.$$

Estimates of the rate of convergence to (x_0, y_0, z_0)

Theorem 2a. Let

$$\max_{t\in[-\tau,0]} |u(t)| \leq \frac{\mu x_0}{d^2} e^{-k\tau/2}, \qquad \sqrt{V(0,u,v)} < \frac{\varepsilon}{q},$$

$$\frac{1}{\sqrt{h_{\min}}} \frac{\sqrt{V(0,u,v)}}{\left(1 - \frac{q}{\varepsilon}\sqrt{V(0,u,v)}\right)} \leq \frac{\mu x_0}{d^2} e^{-k\tau/2},$$

where

$$\varepsilon = \min\left\{\frac{1}{\|H\|}\left(c_{\min} - \frac{\mu}{d^2}\|\widetilde{H}^*\widetilde{H}\|\right), k\right\}, \qquad q = \frac{\sqrt{h_{11}}}{h_{\min}}\left(\frac{r}{K} + \sqrt{\left(\frac{r}{K}\right)^2 + p^2}\right).$$

Then the estimate holds

$$u^2(t) + v^2(t) \leq \frac{1}{h_{\min}} \frac{V(0,u,v)}{\left(1 - \frac{q}{\varepsilon}\sqrt{V(0,u,v)}\right)^2} e^{-\varepsilon t}.$$

Estimates of the rate of convergence to (x_0, y_0, z_0)

Theorem 2b. Let the conditions of theorem 2a be fulfilled. then the estimate holds

$$|w(t)| \leq \left[|v(0) + w(0)| + \frac{1}{\sqrt{h_{\min}}} \frac{\sqrt{V(0,u,v)}}{\left(1 - \frac{q}{\varepsilon}\sqrt{V(0,u,v)}\right)} (1 + \beta t) \right] e^{-\gamma t},$$

where

$$\gamma = \min\left\{\frac{\varepsilon}{2}, c\right\}, \quad \beta = bp\left(y_0 + \frac{\mu x_0}{d^2} e^{-k\tau/2}\right) + |c - d + bpx_0|\,.$$

Thank you for your attention!