Classification of graphs of diameter 2

Tatiana Fedoryaeva

Novosibirsk, Russia
Sobolev Institute of Mathematics SB RAS
Novosibirsk State University
e-mail: fti@math.nsc.ru

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Diametral vertices

We study finite labeled ordinary graphs. For a connected graph G

- distance $\rho_G(u, v)$ between the vertices $u, v \in V$ is defined as the length of the shortest path connecting these vertices and
- $d(G) = \max_{u,\, v \in V} \rho_G(u, v)$ is the diameter of graph G.
- A shortest path of length $d(G)$ is called the diametral path of the graph G and
- its endpoints are diametral vertices.

Earlier, the author has asymptotically investigated n-vertex graphs of given diameter k and the properties of their typical graphs. Let us formulate the necessary definitions.

Almost all graphs

In the asymptotic analysis of graphs with a certain property, the notion of almost all is often used to estimate the measure of the number of graphs with this property; with this approach, the studied property is considered for graphs with a large number of vertices. Let $\mathcal{J}_n$ be the class of labeled n-vertex graphs with a fixed set of vertices $V = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Consider some property $\mathcal{P}$, which each graph may or may not possess. By $\mathcal{J}_n^{\mathcal{P}}$ we denote the set of all graphs from $\mathcal{J}_n$ that possess the property $\mathcal{P}$.

- Almost all graphs possess the property $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 1$.
- Almost no graphs with the property $\mathcal{P}$ if $\lim_{n\to\infty} \frac{|\mathcal{J}_n^{\mathcal{P}}|}{|\mathcal{J}_n|} = 0$.

Class of typical graphs

When studying and selecting almost all graphs for the considered class of graphs, it is often useful to determine not characteristic properties themselves for the concept of almost all, but directly distinguish a subclass of typical graphs itself. Let Ω be an arbitrary class of graphs such that $\Omega_n \neq \varnothing$ for all large enough n, where $\Omega_n = \Omega \cap \mathcal{J}_n$.

- A subclass $\Omega^* \subseteq \Omega$ is a class of typical graphs of the class Ω if

$$\lim_{n\to\infty} \frac{|\Omega_n^*|}{|\Omega_n|} = 1.$$

In this definition, we consider graphs as combinatorial objects and use the more general concept of a class of typical combinatorial objects for a given class of objects that admit the concept of dimension. In our case the dimension of a graph is understood as the number of its vertices.

Graphs of diameter k, $k \geq 3$

In previous papers of the author [1-3] with the aim of finding asymptotically the exact value of the number of n-vertex graphs of fixed diameter typical graphs and their properties in the class $\mathcal{J}_{n,\, d=k}$ of all n-vertex labeled graphs of diameter k are investigated. It turned out that almost all n-vertex graphs of a given diameter $k \geq 3$ have a unique pair of diametral vertices.

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[1] T.I.Fedoryaeva, The diversity vector of balls of a typical graph of small diameter, Diskret. Anal. Issled. Oper., 6(2015), 43–54.
[2] T.I.Fedoryaeva, Structure of the diversity vector of balls of a typical graph with given diameter, Siber. Electr. Math. Reports, 13(2016), 375–387.
[3] T.I.Fedoryaeva, Asymptotic approximation for the number of n-vertex graphs of given diameter, J.Appl.Ind.Math., 2(2017), 204–214.

Graphs of diameter k, $k \geq 3$

In addition, some properties of typical graphs of the class $\mathcal{J}_{n,\, d=k}$ related to the variety of metric balls in the graph are satisfied. This unexpected result is valid even for more wide classes of graphs and, in particular, for graphs that are not necessarily connected, but contain a shortest path of the given length $k \geq 3$.

Formulate this previously obtained result.
Denote by

- $\mathcal{J}_{n,\,d=k}$ — the class of n-vertex graphs of diameter k,
- $\mathcal{J}_{n,\,d\geq k}$ — the class of n-vertex connected graphs of diameter at least k,
- $\mathcal{J}^*_{n,\,d\geq k}$ — the class of n-vertex (not necessarily connected) graphs with a shortest path of length at least k.

The following theorem is valid.

Theorem 1 [3].

Let $k \geq 3$. Then for every $n \in \mathbb{N}$ the following inequalities hold:

$$2^{\binom{n}{2}}\xi_{n,k}(1-o(1)) \leq |\mathcal{T}_{n,d=k}| \leq |\mathcal{J}_{n,\,d=k}| \leq |\mathcal{J}_{n,\,d\geq k}| \leq |\mathcal{J}^*_{n,\,d\geq k}|$$

$$\leq 2^{\binom{n}{2}}\xi_{n,k}(1+o(1)),$$

$$\text{where } \xi_{n,k} = q_k\,(n)_{k-1}\left(\frac{3}{2^{k-1}}\right)^{n-k+1}, \; q_k = \frac{1}{2}(k-2)\,2^{-\binom{k-1}{2}}.$$

Here $\mathcal{T}_{n,d=k}$ is the class of constructed typical graphs of diameter k.

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[3] T.I.Fedoryaeva, Asymptotic approximation for the number of n-vertex graphs of given diameter, J.Appl.Ind.Math., 2(2017), 204–214.

Graphs of diameter 2

All constructed typical graphs of diameter $k \geq 3$ (the class $\mathcal{T}_{n,d=k}$) have a single pair of diametral vertices. However, for graphs of diameter 2 it was found that this property is not valid, i.e., the number of n-vertex graphs of diameter 2 with a single pair of diametral vertices is asymptotically small. This prompted a detailed investigation of more general properties of graphs of diameter 2 related to the number of pairs* of diametral vertices contained in the graph.

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*By a pair of vertices we mean an unordered sample of two elements.

In particular, one of the questions that arises here is whether it is possible to limit the number of pairs of diametral vertices to receive almost all graphs of diameter 2.

In addition, the class $\mathcal{J}_{n,\,d=2}$ has always been of particular interest in view of, on the one hand, the apparent simplicity of its objects, on the other hand, the breadth of its "coverage" of all graphs: it is well known that almost all graphs have a diameter 2 (J.W. Moon, L. Moser).

Theorem 2 [4].

Almost all graphs have diameter 2.

In our paper one of the simplest justifications of this fact is given on the basis of asymptotic properties in terms of metric balls contained in a graph.

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[4] J.W. Moon, L. Moser, *Almost all* $(0{,}1)$ *matrices are primitive*, Stud. Sci. Math. Hung., **1** (1966), 153-156.

In this connection, a more subtle classification of graphs of diameter 2 is interesting, when subclasses are distinguished that form a partition of the entire class $\mathcal{J}_{n,\, d=2}$. Moreover, for meaningful classification it is required that none of the considered subclasses would not be poor and too rich, i.e. asymptotically did not coincide with the whole class.

At the same time, considering the "wealth" of the whole class $\mathcal{J}_{n,\, d=2}$, apparently we should not expect a good characterization of the selected subclasses; however, there is a natural problem of describing or constructing a class of typical graphs inside each studied subclass in order to clarify the structure of such graphs with a large number of vertices.

Thus, we have the following tasks

- classification of graphs of diameter 2 and
- constructing a class of typical graphs for each distinguished subclass of graphs of diameter 2.

In the present paper the classification of graphs of diameter 2 by the number of pairs of diametral vertices contained in the graph is designed.

Definition (class $\mathcal{J}_{n,\,q=k}$).

For any nonnegative integer k by $\mathcal{J}_{n,\,q=k}$ we denote the class of all n-vertex labeled graphs of diameter 2 containing exactly k pairs of diametral vertices.

Partition of class $\mathcal{J}_{n,\, d=2}$ of graphs of diameter 2

It's obvious that

$$\mathcal{J}_{n,\, d=2} = \bigcup_{k \geq 0} \mathcal{J}_{n,\, q=k}.$$

Moreover, $\mathcal{J}_{n,\, q=0} = \varnothing$ and nonempty subclasses $\mathcal{J}_{n,\, q=k}$ form a partition of the class of all n-vertex graphs of diameter 2.

Firstly, we find out when in the class of n-vertex graphs of diameter 2 there exists a graph containing exactly k pairs of diametral vertices.

Example: class $\mathcal{J}_{n,\,q=1}$

For k = 1 graphs of class $\mathcal{J}_{n,\,q=k}$ are arranged quite simply:

Case $k = 1$

(i) For $n \geq 3$, $\mathcal{J}_{n,\,q=1}$ consists of graphs isomorphic $\overline{K}_2 + K_{n-2}$;
(ii) $\mathcal{J}_{n,\,q=1} = \varnothing$ if $n < 3$.

The generally accepted notations of graph theory are used here:

- $\overline{G}$ — complement of the graph G,
- $G + H$ — the graph obtained by the join operation from graphs G and H,
- K_n — complete n-vertex graph.

The general case is solved in the next theorem, where the necessary and sufficient condition under which $\mathcal{J}_{n,\,q=k} \neq \varnothing$ is established.

Condition of $\mathcal{J}_{n,\,q=k} \neq \varnothing$

The following theorem establishes all possible values of parameters n and k for which there exists an n-vertex graph of diameter 2 containing exactly k pairs of diametral vertices.

Theorem 3 (condition of $\mathcal{J}_{n,\,q=k} \neq \varnothing$)

Let $k \geq 1$. Then there exists a graph containing exactly k pairs of diametral vertices in the class of n-vertex graphs of diameter 2 iff $n \geq \lceil 0.5(3+\sqrt{1+8k}) \rceil$.

Under proving this theorem, we also establish a number of properties of such graphs and methods of constructing them.

The previous theorem gives the smallest order of graphs of the class $\mathcal{J}_{n,\,q=k}$. Further we also investigate such graphs with a large number of vertices; they are counted in the following theorem.

Theorem 4 (number of graphs of class $\mathcal{J}_{n,\,q=k}$).

Let $k \geq 1$ be a given integer. Then the number of labeled n-vertex graphs of diameter 2 containing exactly k pairs diametral vertices is equal to $\binom{\binom{n}{2}}{k}$ for every $n > k+1$.

Note that the estimate $n > k+1$ in Theorem 4 is unimprovable in the sense of the above equality $|\mathcal{J}_{n,\,q=k}| = \binom{\binom{n}{2}}{k}$. And this fact deserves special attention in view of its possible further applications.

Typical graphs of class $\mathcal{J}_{n,\,q=k}$

After the partition of the graphs class of diameter 2 into subclasses and counting the number of graphs with a large number of vertices in them, we turn to the problem of constructing a class of typical graphs for each defined subclass of graphs of diameter 2. Distinguished typical graphs of class $\mathcal{J}_{n,\,q=k}$ are given in the following theorem.

Theorem 5 (typical graphs of class $\mathcal{J}_{n,\,q=k}$).

Let $k \geq 1$ be any fixed integer. Then graphs isomorphic $\overline{kK_2 + K_{n-2k}}$ for $n \neq 2$ form a class of typical graphs of the class $\mathcal{J}_{n,\,q=k}$.

The class of all such typical n-vertex graphs denote by $\mathcal{J}^{\top}_{n,\,q=k}$.

Atypical graphs of class $\mathcal{J}_{n,\,q=k}$

After constructing the subclass $\mathcal{J}^{\top}_{n,\,q=k}$ of typical graphs for the class $\mathcal{J}_{n,\,q=k}$ naturally we want to understand what is left into the class of graphs outside of typical graphs, i.e. how much we "deepened" into the class itself.
Note that graphs isomorphic to $\overline{K}_{1,k}+K_{n-(k+1)}$ form a broad subclass $\mathcal{J}^{\perp}_{n,\,q=k}$ of the whole class $\mathcal{J}_{n,\,q=k}$. Moreover, they are atypical for $k \geq 2$:

$$\mathcal{J}^{\perp}_{n,\,q=k} \subseteq \mathcal{J}_{n,\,q=k} \setminus \mathcal{J}^{\top}_{n,\,q=k}.$$

This means, that for $k \geq 2$ the constructed subclass $\mathcal{J}^{\top}_{n,\,q=k}$ of typical graphs does not exhaust the entire class $\mathcal{J}_{n,\,q=k}$ (and for $k = 1$ it coincides with the whole class $\mathcal{J}_{n,\,q=1}$ of graphs that are explicitly described).

Atypical graphs of class $\mathcal{J}_{n,\, q=k}$

Thus,

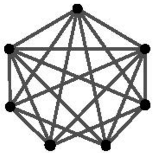
Example of atypical graphs of class $\mathcal{J}_{n,\, q=k}$

For any fixed $k \geq 2$ in class $\mathcal{J}_{n,\, q=k}$ there is almost no graphs are isomorphic $\overline{K}_{1,k} + K_{n-(k+1)}$.

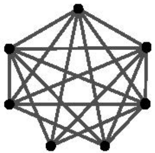
Note also that any subclass of the class $\mathcal{J}_{n,\, q=k}$ that has an asymptotically "small" intersection with the constructed class of typical graphs possesses the similar atypical property.

Examples of typical and atypical graphs of the class $\mathcal{J}_{n=7,\, q=2}$

The following figure shows examples of a typical graph $\overline{2K}_2 + K_3 \in \mathcal{J}^{\top}_{7,\, q=2}$ and atypical graph $\overline{K}_{1,2} + K_4 \in \mathcal{J}^{\perp}_{7,\, q=2}$ for the class $\mathcal{J}_{7,\, q=2}$.



$\overline{2K_2} + K_3 \in \mathcal{J}^{\top}_{7,q=2}$



$\overline{K}_{1,2} + K_4 \in \mathcal{J}^{\perp}_{7,q=2}$

Рис. 1: Graphs of the class $\mathcal{J}_{7,\, q=2}$

Class $\mathcal{J}_{n,\,q \geq k}$

Now let's turn to the question of the possibility of limiting the number of pairs of diametral vertices sufficient to obtain almost all graphs diameter 2. Define the following class.

Definition (the class $\mathcal{J}_{n,\,q \geq k}$.)

For an arbitrary non-negative integer k by $\mathcal{J}_{n,\,q \geq k}$ we denote the class of all n-vertex labeled graphs of diameter 2 containing at least k pairs of diametral vertices.

Consider the more general case when $k = k(n)$ is a function depending on n and taking nonnegative integer values with the growth restriction under consideration, covering the case of fixed integer $k \geq 1$.

Theorem 6 (the class $\mathcal{J}_{n,\, q \geq k}$).

Let $0 < k \leq \epsilon \binom{n}{2}$ as $n \to \infty$, where ϵ is a fixed constant and $0 < \epsilon < \frac{1}{2}$. Then almost all n-vertex graphs of diameter 2 have at least k pairs of diametral vertices.

In the proof of Theorem 6, the well-known entropy inequality for the sum of binomial coefficients was used:

Lemma (entropy inequality).

If $0 < k \leq \frac{n}{2}$, *then the following inequality holds*

$$\sum_{i=0}^{k} \binom{n}{i} \leq 2^{nH(\frac{k}{n})}.$$

Here $H(\epsilon) = -\epsilon \log_2 \epsilon - (1-\epsilon)\log_2(1-\epsilon)$ is the function of entropy.

As a consequence of Theorem 6, in particular it follows that

Corollary.

For any fixed integer $k \geq 1$, almost all graphs of diameter 2 have at least k pairs of diametral vertices.

In other words, this means that it is impossible to limit the number of pairs of diametral vertices by given fixed integer k in order to obtain almost all graphs of diameter 2.

Summary

The classification of graphs of diameter 2 by the number of pairs of diametral vertices contained in the graph is designed.

- All possible values of the parameters n and k are established for which there exists an n-vertex graph of diameter 2 that has exactly k pairs of diametral vertices. As a corollary, the smallest order of these graphs is found.
- Such graphs with a large number of vertices are also described and counted.
- In addition, for any fixed integer $k \geq 1$ inside each distinguished class of n-vertex graphs of diameter 2 containing exactly k pairs of diametral vertices, a class of typical graphs is constructed.
- For the introduced classes, the almost all property is studied for any $k = k(n)$ with the growth restriction under consideration, covering the case of fixed integer $k \geq 1$.

Thank you for your attention!