On invariant surfaces in phase portraits of circular gene network models

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3D system with smooth functions

$$\begin{cases} \dfrac{dx_1}{dt} = k_1(f_1(x_3) - \gamma_1(x_1)); \\ \\ \dfrac{dx_2}{dt} = k_2(f_2(x_1) - \gamma_2(x_2)); \\ \\ \dfrac{dx_3}{dt} = k_3(f_3(x_2) - \gamma_3(x_3)), \end{cases} \tag{1}$$

where $f_i(x_{i-1})$ are smooth monotonous decreasing functions, $\gamma_i(x_{i-1})$ are smooth monotonous increasing functions, $i = 1, 2, 3$.

The system (1) has a unique equilibrium $S_0 = (x_1^0, x_2^0, x_3^0)$ in the invariant domain.

Lemma 1

The parallelepiped $\mathcal{Q}^3 = \prod_{j=1}^{3}[0, M_j]$, *where* $M_j = \min\{f_j(0), \max\gamma_j\}$, *is a positively invariant domain for the trajectories of the three-dimensional system* (1).

The hyperplanes $x_j = x_j^0$ through the equilibrium S_0 split $\mathcal{Q}^3$ into 8 blocks: $\{\varepsilon_1\varepsilon_2\varepsilon_3\}$, where $\varepsilon_j = 1$, if $x_j > x_j^0$, and $\varepsilon_j = 0$ otherwise.

The valence of the n-dimensional block B is a number of its $(n-1)$-dimensional faces, through which the trajectories of the dynamical system can pass from B to adjacent blocks.

$\mathcal{Q}^3 = W_1 \cup W_3$
W_1 — 6 one-valent blocks;
W_3 — three-valent blocks $\{000\}$, $\{111\}$.

Lemma 2

For any pair of adjacent blocks B_1, B_2 trajectories of points of their common $(n-1)$-dimensional face pass either from B_1 to B_2 , i.e. $B_1 \to B_2$, or from B_2 to B_1, $B_2 \to B_1$.

$$\begin{array}{ccccc} \{101\} & \xrightarrow{F_0} & \{001\} & \xrightarrow{F_1} & \{011\} \\ {\scriptstyle F_5}\uparrow & & & & \downarrow{\scriptstyle F_2} \\ \{100\} & \xleftarrow[F_4]{} & \{110\} & \xleftarrow[F_3]{} & \{010\} \end{array} \tag{2}$$

Theorem 1

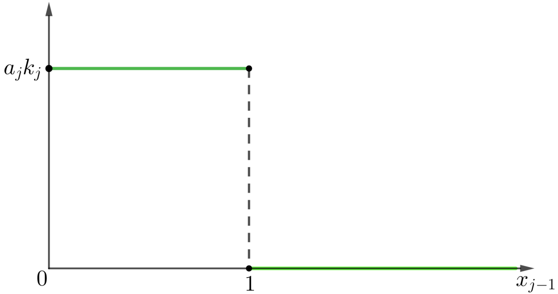
If S_0 is an unstable hyperbolic equilibrium point then the system (1) has a cycle passing through the blocks of the diagram (2).

We have shown that the linearization matrix has eigenvalues $\lambda_1 < 0$, with $\operatorname{Re}\lambda_{2,3} > 0$.

3D piecewise linear system

$$\begin{cases} \dot{x}_1 = L_1(x_3) - k_1 x_1; \\ \dot{x}_2 = L_2(x_1) - k_2 x_2; \\ \dot{x}_3 = L_3(x_2) - k_3 x_3, \end{cases} \tag{3}$$

$$L_j(x_{j-1}) = \begin{cases} a_j k_j > 0\,,\ 0 \leqslant x_{j-1} < 1; \\ 0\,,\ x_{j-1} \geqslant 1; \end{cases}$$



Lemma 3

The parallelepiped $\mathcal{Q}^3 = \prod_{j=1}^{3}[0, a_j]$ *is a positively invariant domain for the trajectories of the three-dimensional system* (3).

The system (3) has no equilibrium points, the hyperplanes $x_j = 1$ through the point $E = (1, 1, 1)$ split $\mathcal{Q}^3$ into 8 blocks.
After 6 steps each trajectory returns to the face F_0.

Theorem[1,2] 2

If $a_j > 1$, $j = 1, 2, 3$, *then the system* (3) *has exactly one cycle, which is stable.*

[1] Golubyatnikov V. P., Ivanov V. V., Minushkina L. S. On existence of a cycle in one non-symmetrical circular gene network model. //Sib. Journ. of P. and Appl. Math., 2018, V.18, N 3.

[2] Golubyatnikov V. P., Ivanov V. V. Uniqueness and stability of a cycle in three-dimensional block linear circular gene network models. // Sib. Journ. of P. and Appl. Math., 2018, V.18, N 4.

Monotonicity of the Poincaré map

Let $\Phi(y_0, z_0) := (y_6, z_6)$, where $\Phi = f_5 \dots f_0 : F_0 \to F_0$ is a Poincaré map, $x_0 = x_6 = 1$. A "normalized" Poincaré map $\Psi = \mathcal{L} \circ \Phi \circ \mathcal{L}^{-1}$, where $\mathcal{L} : K^2 \to F_0$ is a map of the form

$$\mathcal{L}(u_1, u_2) = (1, 1 - u_1, 1 + (a_3 - 1)u_2).$$

Lemma 4

Let (u_1^0, u_2^0) and (v_1^0, v_2^0) belong to K^2. Ψ is monotonous, i.e., if $u_1^0 \leqslant v_1^0$ and $u_2^0 \leqslant v_2^0$, then $u_1^6 \leqslant v_1^6$ and $u_2^6 \leqslant v_2^6$. If one of the first inequalities is strict, them both second inequalities are strict.

Lemma 5

If $P = (u_1^0, u_2^0) \in K^2$ is sufficiently close to the origin, and does not coincide with it, then $u_1^0 < u_1^6$ and $u_2^0 < u_2^6$.

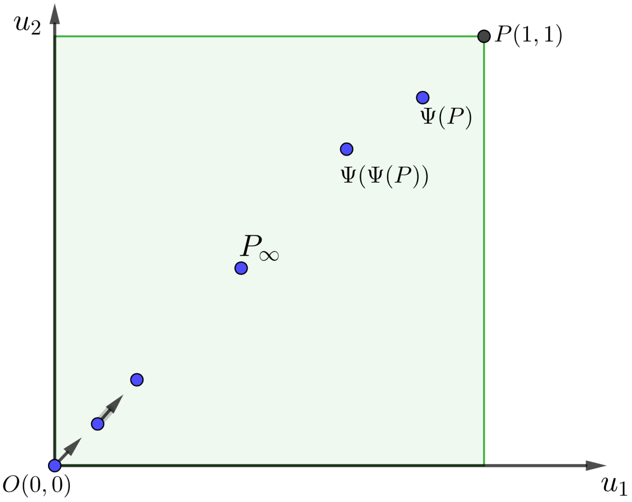


Figure 1: Monotonicity of the Poincaré map Ψ in $K^2 = [0,1] \times [0,1]$

Lemma (P. Hartman, Ordinary Differential Equations)

Let A,C *be non-singular constant matrices, where* A *is a* $d \times d$ *matrix,* C *is a* $e \times e$ *matrix,and*

$$a = ||A|| < 1 \text{ and } 1/c = ||C^{-1}|| < 1. \tag{4}$$

Let $\Pi : (y_0, z_0) \to (y_1, z_1)$ *be a map of the form*

$$\Pi : y_1 = Ay_0 + Y(y_0, z_0), \quad z_1 = Cz_0 + Z(y_0, z_0),$$

where Y, Z *are functions of class* C^1 *for small* $||y_0||$, $||z_0||$ *which vanish together with their Jacobian matrices at* $(y_0, z_0) = 0$. *Then there exists a continious one-to-one map*

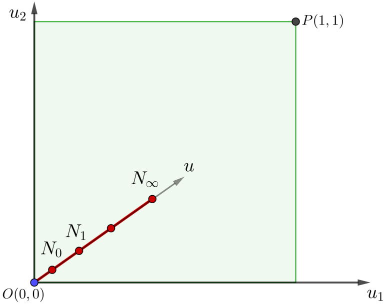
$$R : u = \Phi(y, z), \quad v = \Psi(y, z)$$

of a neighborhood of $(y, z) = 0$ *onto a neighborhood of* $(u, v) = 0$ *such that* R *transforms* Π *into the linear map*

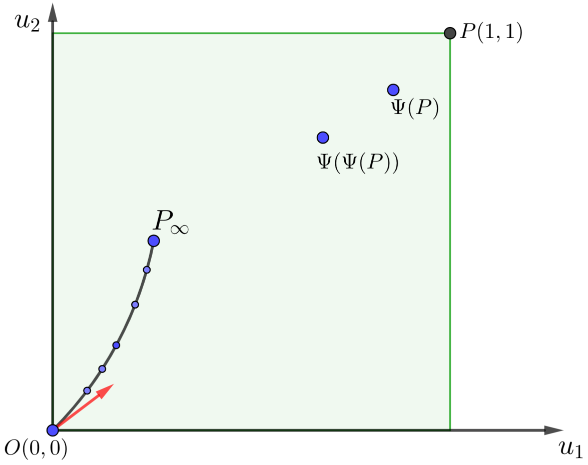
$$R\Pi R^{-1} = L : u_1 = Au_0, \quad v_1 = Cv_0.$$

It has been shown that $\det J_\Psi(0) = 1$ so the Jacobian matrix $J_\Psi(0)$ has eigenvalues $\lambda_1 > 1 > \lambda_2 > 0$. Under conditions of the Lemma $d = e = 1$ and $C = \lambda_1$, $A = \lambda_2$. In the coordinate system OYZ, where OY and OZ are parallel to eigenvectors corresponding to λ_1 and λ_2, the Poincaré map Π has a form

$$\Pi(Y_0, Z_0) = (Y_1, Z_1), \quad Y_1 = \lambda_1 Y_0 + G(Y_0, Z_0), \quad Z_1 = \lambda_2 Z_0 + H(Y_0, Z_0).$$



The linear map $L = R\Pi R^{-1}$ maps a point $N_0 = (\nu_0, 0)$ in the neighborhood of the origin to a point $N_1 = (\nu_1, 0)$ with $\nu_1 = \lambda_1 \nu_0$.



Theorem 4 (V. P. Golubyatnikov, N. B. Ayupova)

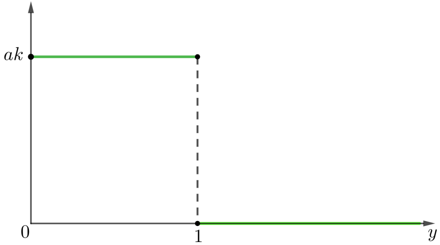
If $a_j > 1$, $j = 1, 2, 3$, then the domain W_1 contains an invariant surface of the system (3).

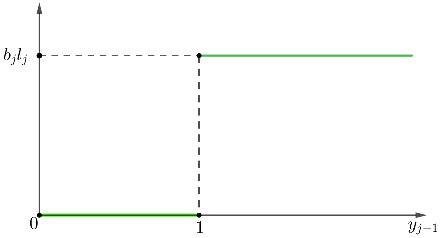
4D piecewise linear system[3]

$$\frac{dy_1}{dt} = L(y_4) - ky_1; \quad \frac{dy_j}{dt} = \Gamma_j(y_{j-1}) - l_j y_j, \tag{5}$$

$j = 2, 3, 4$, where

$$L(y) = \begin{cases} ak > 0\ ,\ 0 \leqslant y < 1; \\ 0\ ,\ y \geqslant 1; \end{cases} \qquad \mathit{\Gamma}_j(y_{j-1}) = \begin{cases} 0\ ,\ 0 \leqslant y_{j-1} \leqslant 1; \\ b_j l_j > 0\ ,\ y_{j-1} > 1. \end{cases}$$





[3] Glass L., Pasternack J.S. Stable oscillations in mathematical models of biological control systems. // Journal of Mathematical Biology. 1978. V. 6

Lemma 6

All trajectories of the system (5) *do not leave the domain* $Q^4 = [0,a] \times [0,b_2] \times [0,b_3] \times [0,b_4]$ *as time increases.*

The hyperplanes $y_i = 1$ through the point $E = (1,1,1,1)$ split Q^4 into 2^4 blocks: $\{\varepsilon_1\varepsilon_2\varepsilon_3\varepsilon_4\}$, where $\varepsilon_i = 1$ if $y_i > 1$ and $\varepsilon_i = 0$ otherwise.
W_1 contains 8 one-valent blocks:

$$
\begin{tikzcd}
\{1111\} \arrow[r, "F_0=\{y_1=1\}"] & \{0111\} \arrow[r, "F_1=\{y_2=1\}"] & \{0011\} \arrow[d, "F_2=\{y_3=1\}"] \\
\{1110\} \arrow[u, "F_7=\{y_4=1\}"] & & \{0001\} \arrow[d, "F_3=\{y_4=1\}"] \\
\{1100\} \arrow[u, "F_6=\{y_3=1\}"] & \{1000\} \arrow[l, "F_5=\{y_2=1\}"'] & \{0000\} \arrow[l, "F_4=\{y_1=1\}"']
\end{tikzcd}
\tag{6}
$$

Theorem[4] 5

If $a, b_j > 1$, $j = 2, 3, 4$, one-valent domain W_1 in the phase portrait of the system (5) contains exactly one cycle $\mathcal{C}$ that passes from block to block according to the arrows of the diagram (6); this cycle contains the unique fixed point $P_0 = \mathcal{L}^{-1}U_0$ of the Poincaré map $\Psi : F_0 \to F_0$.

Theorem[4] 6

The cycle $\mathcal{C}$ is stable.

[4] Golubyatnikov V. P., Minushkina L. S. On uniqueness and stability of a cycle in one gene network // Sib. Electr. Math. Reports, 2021, V. 18, N. 1.

The Jacobian determinant $\det J_\Psi(0)$ of the Poincaré map in the origin is equal to 1. It follows from Frobenius – Perron Theorem that Jacobian matrix $J_\Psi(0)$ has a positive eigenvalue $\lambda_1 > 1$ with the corresponding eigenvector having positive coordinates. The next four cases are possible:

1. $\lambda_1 > 1 > |\lambda_2| > |\lambda_3|$;
2. $\lambda_1 > |\lambda_2| > 1 > |\lambda_3|$;
3. $\lambda_1 > 1 > |\lambda_2| = |\lambda_3|$.

Even if the norm of $J_{\lambda_2} = \begin{pmatrix} \lambda_2 & 1 \\ 0 & \lambda_2 \end{pmatrix}$ can be more than 1, for a sufficiently large $m > 0$ the norm $J_{\lambda_2}^m$ becomes less than 1.

4. $\lambda_1 > |\lambda_2| = 1 > |\lambda_3|$. This case does not satisfy the condition (4) from the Hartman's Lemma.

Example. The matrix

$$\begin{pmatrix} 3 & 1 & 1 \\ 1/2 & 1 & 1/2 \\ 1 & 1 & 1 \end{pmatrix}$$

has eigenvalues $\lambda_1 = 2 + \sqrt{3} > \lambda_2 = 1 > \lambda_3 = 2 - \sqrt{3}$.

Theorem 7 (V. P. Golubyatnikov, L. S. Minushkina)

If $a, b_j > 1$, $j = 2, 3, 4$, and the Jacobian matrix $J_\Psi(0)$ of the Poincaré map does not have eigenvalues equal to 1 in absolute value, then the domain W_1 contains an invariant surface of the system (5) with the cycle $\mathcal{C}$ from Theorems 5,6 lying on it.

Thank you for attention!