$\Sigma-$relations and generalized computability on structures

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Classical computability

- Recursive functions
- Turing machines

Computable structures

A structure $\mathcal{M}$ of the signature $\sigma = \{P_i^{n_i}, i \in \omega\}$

consists of:

- the domain set M,
- $P_i^{\mathcal{M}} \subseteq M^{n_i}$ the interpretation of the symbol P_i in $\mathcal{M}$;

$\mathcal{M}$ is computable if its domain is a computable set and $P_i^{\mathcal{M}}$ are uniformly computable in i.

Hereditarily finite and list superstructures

- Barwise: Admissible sets.
- Ershov: $\Sigma-$definability in hereditarily finite superstructures.
- Goncharov and Sviridenko: lists over the elements of a given abstract data type.

Hereditarily finite superstructure

Let $\mathcal{M}$ be a structure of σ

- $HF_0(M) = M$
- $HF_{n+1}(M) = HF_n(M) \cup \mathcal{P}_\omega(HF_n(M))$
- $HF(M) = \bigcup_{n<\omega} HF_n(M)$

Then HF superstructure $\mathbb{HF}(\mathcal{M}) = \langle M, HF(M), \sigma \cup \{\varnothing, \in^2, U^1\}\rangle$

Hereditarily finite list superstructure

Let $\mathcal{M}$ be a structure of σ

- elements of $S^0(M)$ are finite lists of M,
- elements of $S^{n+1}(M)$ are finite lists of $S^n(M) \cup M$.
- $S(M) = \bigcup_{n \in \omega} S^n(M)$

Then $HW(\mathcal{M}) = \langle M, S(M), \sigma \cup \{head, tail, cons, nil, \in\} \rangle$

- $head(< x_1, x_2, \ \ldots, x_n >) = x_n$,
  $head(nil) = nil$
- $tail(< x_1, x_2, \ \ldots, x_{n+1} >) = (< x_1, x_2, \ \ldots, x_n >)$,
  $tail(< y >) = tail(nil) = nil$
- $cons(< x_1, x_2, \ \ldots, x_n >, y) = (< x_1, x_2, \ \ldots, x_n, y >)$,
- $y \in < x_1, x_2, \ \ldots, x_n > \iff y = x_i$, for some $1 \leq i \leq n$.
- $< y_1, y_2, \ \ldots, y_m > \sqsubseteq < x_1, x_2, \ \ldots, x_n > \iff m \leq n$ and $y_i = x_i$, for all $1 \leq i \leq m$.

Δ_0–formulas and Σ–formulas:

- Δ_0–formulas:
  closure of the set of all quantifier–free formulas under
  $\wedge, \vee, \neg, \rightarrow, \exists x \in y, \forall x \in y$,
- Σ–formulas:
  closure of the set of all Δ_0–formulas under
  $\wedge, \vee, \exists x \in y, \forall x \in y, \exists x$,

- $\Sigma-$relation $P(x_1, x_2, \dots, x_n)$ is a relation defined by a $\Sigma-$formula
- Function $f(x)$ is Σ-definable, if its graph is a Σ-relation.

Classical computability

- Σ-definable sets in $\mathbb{HF}(\varnothing)$ are c.e.
- Δ–definable sets (Σ-definable together with complements) in $\mathbb{HF}(\varnothing)$ are computable.

Computability over reals

$\mathbb{R} = \langle R, +, \cdot, 0, 1, < \rangle$

- In $HW(\mathbb{R})$ only algebraic functions are Σ-definable.
  M.V. Korovina (1990)

$\mathbb{R}_{\text{exp}} = \langle R, +, \cdot, 0, 1, <, exp(x) \rangle$

- Exponential polinomials

$\mathbb{C}_{\text{exp}} = \langle C, +, \cdot, 0, 1, <, exp(x) \rangle$

- sin, cos, etc.

$\Sigma-$definability in admissible sets

Ershov

$\mathcal{A} = \langle A; P_0^{n_0}, \ldots, P_k^{n_k} \rangle$ is $\Sigma-$definable in $\mathbb{A}$, if there are $\Sigma-$formulas (with parameters in $\mathbb{A}$) $S(x)$, $E^+(x,y)$, $E^-(x,y)$, $\Psi_i^+(x_1,\ldots,x_{n_i})$, $\Psi_i^-(x_1,\ldots,x_{n_i})$, $i = 1,\ldots,k$, such that

1. $S(x)$ defines a nonempty subset S^* of $\mathbb{A}$,
2. Ψ_i^+ и Ψ_i^+ define new predicates P_i^* on S^* and their complements,
3. E^+ и E^- define an equivalence η on S^* and its complement,
4. η is a congruence on $\mathcal{A}^* = \langle S^*; P_1^*, \ldots, P_k^* \rangle$,
5. $\mathcal{A}^*/\eta \cong \mathcal{A}$.

Classical computability for structures

- Structures Σ-definable over $\mathbb{HF}(\varnothing)$ are structures isomorphic to computable ones.
- If $\mathcal{M}$ has a computable copy then structures Σ-definable over $\mathbb{HF}(\mathcal{M})$ are structures isomorphic to computable ones.

$\Sigma-$definability of superstructures

Theorem (A)

$\mathbb{HF}(\mathcal{M})$ is $\Sigma-$definable in $HW(\mathcal{M})$ and $HW(\mathcal{M})$ is $\Sigma-$definable in $\mathbb{HF}(\mathcal{M})$.

- The isomorphism $\mathcal{A}^*/\eta \cong \mathcal{A}$ in both cases is identical on $\mathcal{M}$.

Corollary

$X \subseteq M$ is $\Sigma-$definable in $\mathbb{HF}(\mathcal{M})$ iff X is $\Sigma-$definable in $HW(\mathcal{M})$

Uniformization

Universal function

- $(n+1)$-ary function F is an universal function for some class of n-ary functions G, if for any y $F(y, \overline{x}) \in G$ and for any $f \in G$ there is y, such that $F(y, \overline{x}) = f(\overline{x})$.

- There is a partial recursive universal function for all p.r.f.
- There is no recursive universal function for all r.f.

Uniformization property

- Let P be a subset of a cartesian product $X \times Y$. P' uniformizes P, if P' is a graph of a function with domain $\{x : \exists y P(x, y)\}, x \in X, y \in Y$ and $P' \subseteq P$.
- If any P of class G can be uniformized by some P' of G, G has the uniformization property.
- Σ-definable $(n+1)$-ary function F is an universal function for some class of n-ary functions G, if for any y $F(y, \overline{x}) \in G$ and for any $f \in G$ there is y, such that $F(y, \overline{x}) = f(\overline{x})$.

Results for $\mathbb{HF}(\mathcal{M})$ and $HW(\mathcal{M})$

- In any admissible set there is an universal Σ-predicate.
  Yu.L. Ershov (1996)
- There is an algebraic system $\mathcal{M}$ such that in $\mathbb{HF}(\mathcal{M})$ an universal Σ-function does not exist.
  V. Rudnev (1986)
- $HW(\mathbb{R})$ has universal Σ-function.
  M.V. Korovina (1990)
- There is a $\Sigma-$definable universal partial function for $\Sigma-$definable in $HW(\mathbb{R}_{exp})$ functions $\mathbb{R}^n \times \mathbb{R}$.

Computable analysis vs. $\Sigma-$definability

Definition

The Cauchy representation $\rho_C :\subseteq \Sigma^\omega \to \mathbb{R}$ of the real numbers is a representation where a real number x is represented by a one-way infinite stream of symbols if this one-way infinite stream encodes a sequence of rational numbers converging rapidly to x:
$\rho_C(w_1\_w_2\_\dots) = x ;\Leftrightarrow\ |x - \nu_{\mathbb{Q}}(w_i)| < 2^{-i}$ for all $i \in \mathbb{N}$.

Computable analysis

Definition

A function $f : R^n \to R$ is computable if there is an oracle Turing machine that, given any $k \in N$, may ask for arbitrarily good rational approximations of the input $x \in dom(f)$; i.e., it may ask finitely many questions of the kind "Give me a vector $p \in Q_n$ of rational numbers with $d(x, p) < 2^{-i}$," where the exponent i may depend on the answers to the previous questions, and after finitely many steps, it writes a rational number q on the output tape with $|f(x) - q| < 2^{-k}$.

Definition

A representation of a set X is a surjective function $\delta :\subseteq \Sigma^\omega \to X$, where Σ is some alphabet. Then for any $x \in X$ and any $p \in \Sigma^\omega$ with $\delta(p) = x$, the sequence p is called a $\delta-$name of x.

Theorem (A)

Let $\mathbb{R}^*$ *be an expansion of the ordered real field with a decidable theory. Then there exists a computable real function, which is not* $\Sigma-$*definable in* $\mathbb{HF}(\mathbb{R}^*)$.

Schanuel Conjecture:

- if n is a natural number, $a_1, \ldots, a_n \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$. Then the transcendence degree of expansion $\mathbb{Q}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) \geq n$.

- If the Schanuel Conjecture is true, then the theory of $\mathbb{R}_{exp}$ is decidable.

A. Macintyre, A. Wilkie "On the decidability of the real exponential field."(1996)

Corollary

There exists a computable real function, which is not $\Sigma-$definable in hereditarily finite superstructure over the real exponential field.

Remark

The $sign$ function

$$sign(x) = \begin{cases} 0, \ if \ x < 0, \\ 1, \ if \ x \geq 0. \end{cases}$$

is $\Sigma-$definable in $\mathbb{HF}(\mathbb{R})$, but not computable.

Theorem (Bazhenov, A)

There exists a continuous $\Sigma-$definable in $\mathbb{HF}(\mathbb{R})$ real function, which is not computable.

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Thank you!