Fredholm Properties for Special Classes of Hypoelliptic Operators

Tumanyan Ani

Ph.D., Russian-Armenian University
Lead R&D Engineer at Siemens

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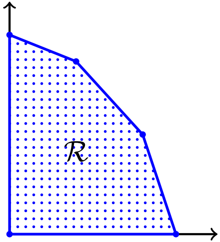
Introduction: Fredholm theory of regular hypoelliptic operators

The class of regular hypoelliptic operators is an important subclass of Hörmander's hypoelliptic operators. These operators were introduced in 60s-70s and have been studied by many authors: Nikolsky 1962, Mikhailov 1967, Friberg 1967, Cattabriga 1970, Volevich, Gindikin 1983 and others. The corresponding characteristic polynomials of regular hypoelliptic operators are «multi-quasi-elliptic», so they are natural generalization of elliptic, parabolic, $2b$–parabolic and quasielliptic polynomials with many applications in various anisotropic models.

The analysis of normal solvability and the Fredholm properties of regular hypoelliptic operators has certain difficulties - characteristic polynomials of such operators are not homogeneous as in the elliptic case and Fredholm theorems for compact manifolds cannot always be used in this case.



Basic concepts and definitions (1)

Definition 1

A bounded linear operator A, acting from a Banach space X to a Banach space Y, is called **Fredholm operator**, if the following conditions hold:

1. $Im(A) = \overline{Im(A)}$ (**normally solvable**);
2. $\dim Ker(A) < \infty$;
3. $\dim coker(A) = \dim Y/Im(A) < \infty$.

Definition 2

The difference between the dimensions of the kernel and the cokernel is called **index** of the operator:

$$ind\,(A) = \dim Ker(A) - \dim coker(A).$$

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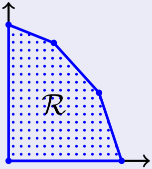
Basic concepts and definitions (2)

Let $\mathcal{N} \subset \mathbb{Z}_+^n$ be a finite set of multi-indices, $\mathcal{R} = \mathcal{R}(\mathcal{N})$ be a minimum convex polyhedron containing all the points $\mathcal{N}$. In this case $\mathcal{R}$ is called Newton polyhedron for the set $\mathcal{N}$.

Definition 3

A polyhedron $\mathcal{R}$ is called **completely regular** if the following holds:

1. $\mathcal{R}$ is a complete polyhedron: $\mathcal{R}$ has a vertex at the origin and further vertices on each coordinate axes in $\mathbb{R}^n$;
2. all components of the outer normals of $(n-1)$–dimensional non-coordinate faces of $\mathcal{R}$ are positive.



Let $\mathcal{R}$ be a completely regular polyhedron.
Denote by $\mathcal{R}_j^{n-1}$ $(j = 1, \ldots, I_{n-1})$ $(n-1)$–dimensional non-coordinate faces of $\mathcal{R}$ with corresponding outer normal μ^j such that all multi-indices $\alpha \in \mathcal{R}_j^{n-1}$ satisfy $(\alpha : \mu^j) = \frac{\alpha_1}{\mu_1^j} + ... + \frac{\alpha_n}{\mu_n^j} = 1$, $\partial\mathcal{R} = \bigcup_{j=1}^{I_{n-1}} \mathcal{R}_j^{n-1}$.
For $k \in \mathbb{R}_+$ denote $k\mathcal{R} := \{k\alpha = (k\alpha_1, k\alpha_2 \ldots, k\alpha_n) : \alpha \in \mathcal{R}\}$.

Let

$$P(x, \mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x) D^\alpha, \tag{1}$$

where $D^\alpha = D_1^{\alpha_1} ... D_n^{\alpha_n}$, $D_j = i^{-1}\frac{\partial}{\partial x_j}, x = (x_1, ..., x_n) \in \mathbb{R}^n, a_\alpha(x)$ are defined in $\mathbb{R}^n$.

Denote

$$P(x, \xi) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x)\, \xi^\alpha. \tag{2}$$

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Denote

$$P(x, \xi) = \sum_{\alpha \in \mathcal{R}} a_\alpha(x) \xi^\alpha. \tag{2}$$

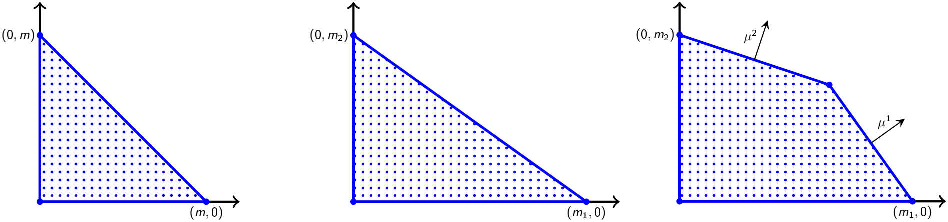
Definition 4

A differential operator $P(x, \mathbb{D})$ is called **regular at a point** x_0, if there exists a constant $\delta > 0$ such that:

$$1 + |P(x_0, \xi)| \geq \delta |\xi|_{\mathcal{R}}, \forall \xi \in \mathbb{R}^n, \text{ where } |\xi|_{\mathcal{R}} = \sum_{\alpha \in \mathcal{R}} |\xi^\alpha|.$$

A differential operator $P(x, \mathbb{D})$ is called **regular in** $\mathbb{R}^n$, if $P(x, \mathbb{D})$ is regular at each point $x \in \mathbb{R}^n$.

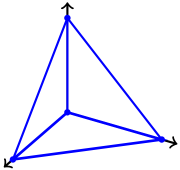
Examples of Newton polyhedrons for $n = 2$ case:



Isotropic Anisotropic Multianisotropic

Regular hypoelliptic operators

Elliptic operators:



$$P(x,\mathbb{D}) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha,$$

where $m \in \mathbb{N}, \alpha \in \mathbb{Z}_+^n, |\alpha| = \alpha_1 + \cdots + \alpha_n$.

Quasielliptic operators (parabolic, $2b$-parabolic, etc):

$$P(x,\mathbb{D}) = \sum_{(\alpha:\nu)\le m} a_\alpha(x) D^\alpha,$$

where

$m \in \mathbb{N}, \nu \in \mathbb{N}^n, \alpha \in \mathbb{Z}_+^n, (\alpha:\nu) = \frac{\alpha_1}{\nu_1} + \cdots + \frac{\alpha_n}{\nu_n}$.

Operators with completely regular Newton polyhedron:

$$P(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x) D^\alpha.$$

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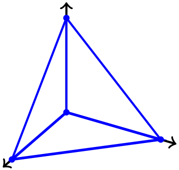

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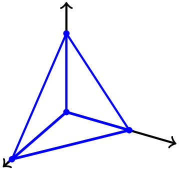

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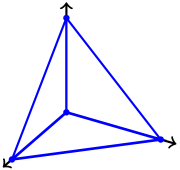

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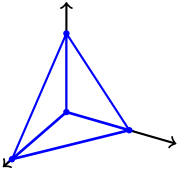

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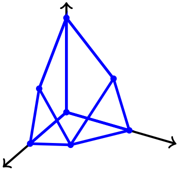
Operators with completely regular Newton polyhedron:



$$P(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x) D^\alpha.$$

Examples: regular hypoelliptic operators

1. Let $m \in \mathbb{N}$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0,\dots,0),(m,0,\dots,0),\dots,(0,0,\dots,m)$. In this case regularity conditions coincide with **ellipticity**.
2. Let $\nu \in \mathbb{N}^n$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0,\dots,0),(\nu_1,0,\dots,0),\dots,(0,0,\dots,\nu_n)$. In this case regularity conditions coincide with **quasiellipticity**.
3. Let $n=2$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0)$, $(8,0)$, $(0,8)$ and $(6,4)$. Then
$$P(x,\mathbb{D}) = a_1 D_1^8 + a_2 D_1^6 D_2^4 + a_3 D_2^8 + q(x)$$
is a **regular** differential operator in $\mathbb{R}^2$ with some $a_1,a_2,a_3>0$ and $q \in C(\mathbb{R}^2)$.
4. Let $n=3$ and $\mathcal{R}$ be a Newton polyhedron for the points $(0,0,0)$, $(8,0,0)$, $(0,8,0)$, $(6,4,0)$, $(6,0,6)$, $(0,6,6)$ and $(0,0,12)$. Then
$$P(x,\mathbb{D}) = D_1^8 + D_1^6 D_2^4 + D_2^8 + D_1^6 D_3^6 + D_2^6 D_3^6 + D_3^{12} + q(x)$$
is a **regular** differential operator in $\mathbb{R}^3$ with $q \in C(\mathbb{R}^3)$.

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Fredholm theory of hypoelliptic operators

Quasielliptic/Semielliptic operators:

- Fredholm property of the special classes of quasielliptic operators in anisotropic Sobolev spaces with certain weights.
  (Rabinovich V. S. 1971, Bagirov L. A. 1985, Karapetyan G. A., Darbinyan A. A. 2008)
- Isomorphism properties in special weighted Sobolev spaces for quasielliptic operators with constant coefficients.
  (Demidenko G. V. 1998–2016)
- Necessary and sufficient conditions for the Fredholm property of quasielliptic operators with special variable coefficients in $\mathbb{R}^n$ and for constant coefficients in bounded domains.
  (Karapetyan G. A., Darbinyan A. A., Tumanyan A. G. 2008–2020)
- Research on index stability on the scale of anisotropic spaces in $\mathbb{R}^n$.
  (Darbinyan A. A., Tumanyan A. G. 2016–2017)

Fredholm theory of hypoelliptic operators

Regular hypoelliptic operators:

- Fredholm criteria for an algebra of ΨDOs generated by smooth and suitably restricted at infinity functions and regular hypoelliptic operator with constant coefficients. (Cordes H. O., Taylor M. E. 1971)
- Fredholm properties and spectrum are studied for special ΨDOs in multianisotropic spaces with polynomial weights. (Rodino L., Boggiatto P., Schrohe E., Buzano E. 2000-2008)
- Fredholm and correct solvability of regular hypoelliptic operators with constant and special variable coefficients in multianisotropic Sobolev spaces in $\mathbb{R}^n$ and $\mathbb{R}^n_+$. (Karapetyan G., Darbinyan A., Tumanyan A., Petrosyan H., Khachaturyan M. 2008–2023)

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Covering of $\mathbb{R}^n$

Let $\{a_i\}_{i=0}^{\infty} \subset \mathbb{R}_+$ be such that $\sum_{i=0}^{\infty} a_i$ diverges and $a_{i+1} < \gamma a_i$, where $\gamma > 0$, $i = 0, 1, \ldots$, $\{b_i\}_{i=0}^{\infty}$ be such that $b_0 = 0$, $b_i = \sum_{j=0}^{i} a_j, i = 1, 2, \ldots$.

Denote

$$V_0 = \left\{ r : |r - b_0| < \frac{2\gamma + 1}{\gamma + 1} a_0 \right\},$$

$$V_i = \left\{ r : |r - b_i| < \frac{\gamma}{\gamma + 1} a_i \right\}, \ i = 1, 2, \ldots .$$

The system $\{V_i\}_{i=0}^{\infty}$ is an open covering for $\mathbb{R}_+$. Denote by U_j $(j = 1, \ldots, s)$ a system which covers a unit sphere $|x| = 1$. Denote

$$W_m = V_{\left[\frac{m-1}{s}\right]} \times U_{m - \left[\frac{m-1}{s}\right]s}, \ m = 1, 2, \ldots .$$

The system $\{W_m\}_{m=1}^{\infty}$ is an open covering for $\mathbb{R}^n$ and $\min\limits_{x \in \overline{W}_m} |x| \to \infty$ when $m \to \infty$.

Weight functions (1)

Denote

$$Q := \{g \in C(\mathbb{R}^n) : g(x) > 0, \forall x \in \mathbb{R}^n\}.$$

For $k \in \mathbb{Z}_+$, completely regular polyhedron $\mathcal{R}$, denote $Q^{k,\mathcal{R}}$:

1. $g \in Q, \frac{1}{g(x)} \rightrightarrows 0$ when $|x| \to \infty$;
2. for $\beta \in k\mathcal{R}, \beta \neq 0$ $D^\beta g(x) \in C(\mathbb{R}^n)$ and there exists $C_\beta > 0$ s.t. $\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\mu^j)}} \leq C_\beta$ for all $x \in \mathbb{R}^n, j = 1, \ldots, I_{n-1}$;
3. for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ and $m_0 = m_0(\varepsilon) > 0$ s.t. for all $m > m_0$ and $\max_{j=1,\ldots,s} \operatorname{diam} U_j < \delta$ the following holds:

$$\max_{x,y\in\overline{W}_m} \frac{|g(x) - g(y)|}{g(y)} < \varepsilon, \max_{x,y\in\overline{W}_m} \frac{1}{g(x)^{\frac{1}{\mu_{max}}} a_{\left[\frac{m-1}{s}\right]}} < \varepsilon,$$

where $\mu_{max} = \max_{1\leq i\leq I_{n-1}} \max_{1\leq j\leq n} \{\mu_j^i\}$.

Weight functions (2)

For $k \in \mathbb{Z}_+$, $\nu \in \mathbb{N}^n$, denote $\widetilde{Q}^{k,\nu}$:

1. there exists a constant $C > 0$ such that $0 < g(x) \leq C$ for all $x \in \mathbb{R}^n$;
2. for $\beta \in \mathbb{Z}^n_+, (\beta : \nu) \leq k, \beta \neq 0$ $D^\beta g \in C(\mathbb{R}^n)$ and there exists $C_\beta > 0$ such that $\frac{|D^\beta g(x)|}{g(x)^{1+(\beta:\nu)}} \leq C_\beta$ for all $x \in \mathbb{R}^n$;
3. for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ and $p_0 = p_0(\varepsilon) > 0$ such that for all $p > p_0$ when $\max\limits_{j=1,\dots,l} \operatorname{diam} U_j < \delta$ the following holds:
$$\max_{x,y \in \overline{W}_p} \frac{|g(x) - g(y)|}{g(y)} < \varepsilon, \quad \max_{x,y \in \overline{W}_p} \frac{1}{g(x)^{\frac{1}{\nu_{min}}} a_{\left[\frac{p-1}{l}\right]}} < \varepsilon,$$
where $\nu_{min} = \min\limits_{1 \leq i \leq n} \{\nu_i\}$.

$Q^{k,\mathcal{R}}$ examples: $(1 + |x|_\mathcal{R})^s, \exp{(1 + |x|_\mathcal{R})^r}$, where $s, r > 0$ and $|x|_\mathcal{R} = \sum_{\alpha \in \mathcal{R}} |x^\alpha|$. $\widetilde{Q}^{k,\nu}$ examples: $(1 + |x|_\nu)^l$, where $-\frac{\nu_{min}}{\nu_{max}} < l \leq 0$.

Weighted multianisotropic spaces

For $k \in \mathbb{Z}_+$, $1 < p < \infty$, a completely regular polyhedron $\mathcal{R}$ and $q \in Q$ denote

$$H_q^{k,\mathcal{R},p}(\mathbb{R}^n) := \left\{ u : \|u\|_{k,\mathcal{R},p,q} := \sum_{\alpha \in k\mathcal{R}} \left\| D^\alpha u \cdot q^{k - \max\limits_{1 \le j \le I_{n-1}} (\alpha:\mu^j)} \right\|_{L_p(\mathbb{R}^n)} < \infty \right\}$$

For $p = 2$ denote $H_q^{k,\mathcal{R}}(\mathbb{R}^n) := H_q^{k,\mathcal{R},2}(\mathbb{R}^n)$ and for $q \equiv 1$ denote by $H^{k,\mathcal{R},p}(\mathbb{R}^n)$.

$H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ spaces are generalization of multianisotropic Sobolev spaces. The functions from $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ have different regularity in different directions.

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Let $k \in \mathbb{Z}_+$. Consider the differential operator $P(x, \mathbb{D})$:

$$P(x,\mathbb{D}) = \sum_{\alpha\in\mathcal{R}} a_\alpha(x) D^\alpha = \sum_{\alpha\in\mathcal{R}} \left(a_\alpha^0(x) q(x)^{1-\max\limits_i(\alpha:\mu^i)} + a_\alpha^1(x) \right) D^\alpha, \quad (3)$$

where $D^\beta(a_\alpha^0(x)) = O\left(q(x)^{\min\limits_i(\beta:\mu^i)}\right)$,
$D^\beta(a_\alpha^1(x)) = o\left(q(x)^{1-\max\limits_i(\alpha-\beta:\mu^i)}\right)$ as $|x| \to \infty$ for all $\alpha \in \mathcal{R}, \beta \in k\mathcal{R}$.

$P(x, \mathbb{D})$ generates a bounded linear operator from $H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$ to $H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$.
This class generalizes so-called Schrödinger-type operators, associated with multi-quasi-elliptic symbols.

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A priori estimates in multianisotropic spaces (1)

Theorem 1

Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x, \mathbb{D})$ be the differential operator (3) with the coefficients that satisfy $\lim\limits_{m\to\infty} \max\limits_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.

Let there exist a constant $\kappa > 0$ such that:

$$\|u\|_{k+1,\mathcal{R},p,q} \leq \kappa \left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(\mathbb{R}^n)}\right), \forall u \in H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n).$$

Then $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max\limits_i(\alpha:\mu^i)}\xi^\alpha\right| \geq \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Tumanyan A.G. A priori estimates and Fredholm criteria for a class of regular hypoelliptic operators. *Siberian Advances in Mathematics*, 33:2 (2023), pp. 87–100

A priori estimates in multianisotropic spaces (2)

Theorem 2

Let $k \in \mathbb{Z}_+, q \in \widetilde{Q}^{k,\nu}$ and $P(x, \mathbb{D})$ be the differential operator (3) with the coefficients that satisfy $\lim\limits_{m\to\infty} \max\limits_{x,y\in\overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathbb{Z}^n_+, (\alpha : \nu) \leq 1$. Let there exist constants $\kappa > 0$ and $N > 0$ such that:

$$\|u\|_{k+1,\nu,p,q} \leq \kappa \left(\|Pu\|_{k,\nu,p,q} + \|u\|_{L_p(K_N)}\right), \forall u \in H^{k+1,\nu,p}_q(\mathbb{R}^n).$$

Then $P(x, \mathbb{D})$ is regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq 1} a^0_\alpha(x)\lambda^{1-(\alpha:\nu)}\xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Tumanyan A.G. Normal Solvability and Fredholm Properties for Special Classes of Hypoelliptic Operators. *Electronic Journal of Differential Equations* (in press).

A priori estimates in multianisotropic spaces (3)

Theorem 3

Let conditions from Theorem 1 hold, $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{\alpha \in \mathcal{R}} a_\alpha^0(x) \lambda^{1 - \max\limits_i (\alpha : \mu^i)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_{\partial \mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \quad (4)$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that:

$$\|u\|_{k+1,\mathcal{R},q,p} \leq \kappa \left(\|Pu\|_{k,\mathcal{R},q,p} + \|u\|_{L_p(K_N)} \right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \quad (5)$$

Corollary 1 (normal solvability)

Let operator $P(x, \mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ satisfy the conditions from Theorem 1. Then it is an $n-$normal operator: normally solvable ($Im(P) = \overline{Im(P)}$) and kernel is finite dimensional.

A priori estimates in multianisotropic spaces (3)

Theorem 3

Let conditions from Theorem 1 hold, $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left|\sum_{\alpha \in \mathcal{R}} a^0_\alpha(x) \lambda^{1-\max_i(\alpha:\mu^i)} \xi^\alpha\right| \geq \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M. \quad (4)$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that:

$$\|u\|_{k+1,\mathcal{R},q,p} \leq \kappa \left(\|Pu\|_{k,\mathcal{R},q,p} + \|u\|_{L_p(K_N)}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n). \quad (5)$$

Corollary 1 (normal solvability)

Let operator $P(x, \mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ satisfy the conditions from Theorem 1. Then it is an $n-$normal operator: normally solvable ($Im(P) = \overline{Im(P)}$) and kernel is finite dimensional.

Theorem 4

Let conditions from Theorem 2 hold, $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq 1} a_\alpha^0(x)\lambda^{1-(\alpha:\nu)}\xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that:

$$\|u\|_{k+1,\nu,p,q} \leq \kappa \left(\|Pu\|_{k,\nu,p,q} + \|u\|_{L_p(K_N)} \right), \forall u \in H_q^{k+1,\nu,p}(\mathbb{R}^n).$$

Corollary 2 (regularity of solutions)

Let operator $P(x, \mathbb{D}) : H_q^{k+1,\nu,p}(\mathbb{R}^n) \to H_q^{k,\nu,p}(\mathbb{R}^n)$ satisfy the conditions from Theorem 4. Then, if $u \in H_q^{k,\nu,p}(\mathbb{R}^n)$, $P(x, \mathbb{D})u \in H_q^{k,\nu,p}(\mathbb{R}^n)$, then $u \in H_q^{k+1,\nu,p}(\mathbb{R}^n)$.

A priori estimates in multianisotropic spaces (4)

Theorem 4

Let conditions from Theorem 2 hold, $P(x, \mathbb{D})$ be regular in $\mathbb{R}^n$ and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq 1} a_\alpha^0(x)\lambda^{1-(\alpha:\nu)}\xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Then there exist constants $\kappa > 0$ and $N > 0$ such that:

$$\|u\|_{k+1,\nu,p,q} \leq \kappa \left(\|Pu\|_{k,\nu,p,q} + \|u\|_{L_p(K_N)} \right), \forall u \in H_q^{k+1,\nu,p}(\mathbb{R}^n).$$

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Contents

Fredholm property in weighted multianisotropic spaces

Theorem 5

Let $k \in \mathbb{Z}_+, q \in Q^{k,\mathcal{R}}$ and $P(x, \mathbb{D})$ be the differential operator with the coefficients $a_\alpha(x) = a^0_\alpha(x) q(x)^{1-\max\limits_{1\le j\le l_{n-1}}(\alpha:\mu^j)} + a^1_\alpha(x)$ that satisfy:

1. $D^\beta \left(a^1_\alpha(x)\right) = o\left(q(x)^{1-\max\limits_{1\le j\le l_{n-1}}(\alpha-\beta:\mu^j)}\right)$ as $|x| \to \infty$ for all $\alpha \in \mathcal{R}$, $\beta \in k\mathcal{R}$;
2. $\lim\limits_{m\to\infty} \max\limits_{x,y\in \overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $\alpha \in \mathcal{R}$.

Then $P(x, \mathbb{D}) : H^{k+1,\mathcal{R},p}_q(\mathbb{R}^n) \to H^{k,\mathcal{R},p}_q(\mathbb{R}^n)$ is a Fredholm operator iff $P(x, \mathbb{D})$ is regular and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left|\sum_{\alpha\in\mathcal{R}} a^0_\alpha(x)\lambda^{1-\max\limits_{1\le j\le l_{n-1}}(\alpha:\mu^j)}\xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall\xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Fredholm property in weighted anisotropic spaces

Theorem 6

Let $k \in \mathbb{Z}_+, q \in \widetilde{Q}^{k,\nu}$ and $P(x, \mathbb{D})$ be the differential operator with the coefficients $a_\alpha(x) = a^0_\alpha(x) q(x)^{1-(\alpha:\nu)} + a^1_\alpha(x)$ that satisfy:

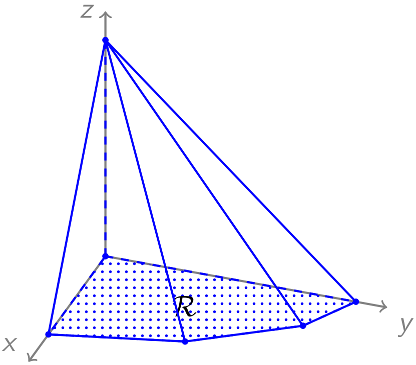
1. $D^\beta \left(a^1_\alpha(x)\right) = o\left(q(x)^{1-(\alpha-\beta:\nu)}\right)$ as $|x| \to \infty$ for all $(\alpha : \nu) \leq 1$, $(\beta : \nu) \leq k$;
2. $\lim\limits_{m\to\infty} \max\limits_{x,y \in \overline{W_m}} |a^0_\alpha(x) - a^0_\alpha(y)| = 0$ for all $(\alpha : \nu) \leq 1$.

Then $P(x, \mathbb{D}) : H^{k+1,\nu,p}_q(\mathbb{R}^n) \to H^{k,\nu,p}_q(\mathbb{R}^n)$ is a Fredholm operator iff $P(x, \mathbb{D})$ is regular and there exist constants $\delta > 0$ and $M > 0$ such that

$$\left| \sum_{(\alpha:\nu)\leq 1} a^0_\alpha(x) \lambda^{1-(\alpha:\nu)} \xi^\alpha \right| \geq \delta(\lambda + |\xi|_\nu), \forall \xi \in \mathbb{R}^n, \lambda > 0, |x| > M.$$

Notes on proof

We consider the symbol of the operator in $(n+1)-$dimensional space with the weight function as an additional variable. Corresponding characteristic polyhedron will be a hyperpyramid:



The proof is based on the special construction of regularizer/parametrix and a priori estimates from Theorem 1 and 2.

Fredholm criteria

Theorem

Let assumptions on the coefficients from Theorem 1 hold. Then the following statements are equivalent:

1. Operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ is a Fredholm operator;
2. There exist constants $C > 0$ and $N > 0$ such that:
$$\|u\|_{k+1,\mathcal{R},p,q} \le C\left(\|Pu\|_{k,\mathcal{R},p,q} + \|u\|_{L_p(K_N)}\right), \forall u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n).$$
3. $P(x,\mathbb{D})$ is regular and there exist constants $\delta > 0$ and $M > 0$ such that
$$\left|\sum_{\alpha\in\mathcal{R}} a_\alpha^0(x)\lambda^{1-\max_j(\alpha:\mu^j)}\xi^\alpha\right| \ge \delta(\lambda + |\xi|_{\partial\mathcal{R}}), \forall\xi\in\mathbb{R}^n, \lambda>0, |x|>M.$$

The isotropic case is studied by many authors: Grushin, 1970, Shubin 1970, Bagirov 1975, Rabinovich 1980, Mukhamadiev 1981, Lockhart, McOwen 1983, Schrohe 2000, Volpert 2010, Arutyunov 2013 etc.

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Contents

Properties on the scale of multianisotropic spaces

Corollary 3 (regularity of solutions)

Let $q \in Q^{k,\mathcal{R}}$ and assume the conditions of Theorem 5 hold. If $u \in H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$, $Pu \in H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$, then $u \in H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n)$.

Corollary 4 (index invariance on the scale)

Let $q \in Q^{k,\mathcal{R}}$ and assume the conditions of Theorem 5 hold.
Then, the kernel, cokernel and index of operator $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R},p}(\mathbb{R}^n) \to H_q^{k,\mathcal{R},p}(\mathbb{R}^n)$ are independent of k and p.

Corollary 5 (spectral properties)

Let $q \in Q^{k,\mathcal{R}}$ and $P(x,\mathbb{D}) : H_q^{k+1,\mathcal{R}}(\mathbb{R}^n) \to H_q^{k,\mathcal{R}}(\mathbb{R}^n)$ be an operator from Theorem 5, considered as an unbounded operator in $L_2(\mathbb{R}^n)$: Then one of the following is true:

- $\sigma(P) = \mathbb{C}$;
- $\sigma(P)$ is discrete and $ind(P; H_q^{k,\mathcal{R}}) = 0$.

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Properties on the scale of anisotropic spaces : $q \equiv 1$

Corollary 7 (index invariance on the scale)

Let $q \equiv 1$ and assume the conditions of Theorem 6 hold. Then, the kernel, cokernel and index of operator $P(x, \mathbb{D}) : H^{k+1,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ are independent of k.

Corollary 8 (spectral properties)

Let $q \equiv 1$ and there exist constants $\widetilde{a}_\alpha$ such that $a_\alpha(x) \rightrightarrows \widetilde{a}_\alpha$ when $|x| \to \infty$, $(\alpha : \nu) \leq 1$. Consider $P(x, \mathbb{D}) : H^{k+1,\nu}(\mathbb{R}^n) \to H^{k,\nu}(\mathbb{R}^n)$ as an unbounded operator in $L_2(\mathbb{R}^n)$. Then, $ind(P, H^{k,\nu}) = 0$, and for the essential spectrum of the operator the following holds:

$$\sigma_{ess}(P) = \left\{ \sum_{(\alpha:\nu)\leq 1} \widetilde{a}_\alpha \xi^\alpha, \xi \in \mathbb{R}^n \right\}.$$

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Further Research Questions

1. A priori estimates and Fredholm property results for other weight functions (particularly, $q(x) \to 0$ as $|x| \to \infty$) using integral representation of functions in multianisotropic spaces [1].
2. Solvability conditions and spectral properties for Fredholm and non-Fredholm operators.
3. Extension of the results to more general scales of multianisotropic spaces.
4. Extension to non-linear operators and ΨDOs.
5. Applications of the results for specific boundary value problems.

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Thanks for your attention!