Completely regular codes in the n-dimensional rectangular grid

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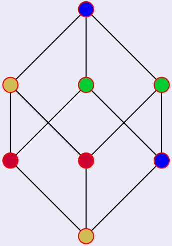
Perfect coloring

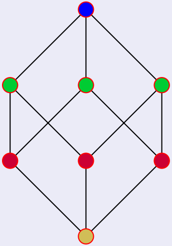
A vertex partition $(V_0, V_1, \ldots, V_{k-1})$ of a r-regular graph is called the perfect k-coloring if for every $i, j \in \{0, 1, \ldots, k-1\}$ there exists the integer α_{ij} such that every vertex of V_i has exactly α_{ij} neighbors in V_j.
Then the matrix $A = (\alpha_{ij})$ is called the parameter matrix.
Obviously, $\alpha_{i,0} + \alpha_{i,1} + \ldots + \alpha_{i,k-1} = r$
$(V_0, V_1, \ldots, V_{k-1}) \longleftrightarrow \varphi : V \to \{0, 1, \ldots, k-1\}$

Examples



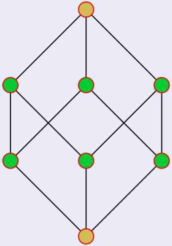
- 0
- 1
- 2
- 3

$$\begin{bmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$



- 0
- 1
- 2
- 3

$$\begin{bmatrix} 0 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$



- 0
- 1

$$\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$$

Distance regular coloring

A perfect coloring $(V_0, V_1, \ldots, V_{k-1})$ is distance regular if its parameter matrix is three-diagonal i.e., it is the coloring by the distance from V_0.

$$A = \begin{bmatrix} a_0 & b_0 & 0 & 0 & \ldots & \ldots & \ldots & 0 \\ c_1 & a_1 & b_1 & 0 & \ldots & \ldots & \ldots & 0 \\ 0 & c_2 & a_2 & b_2 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots & c_{k-2} & a_{k-2} & b_{k-2} \\ 0 & \ldots & \ldots & \ldots & \ldots & 0 & c_{k-1} & a_{k-1} \end{bmatrix}$$

Degree triples

a_i - the inner degree of the i-th color;
b_i - the upper degree of the i-th color;
c_i - the lower degree of the i-th color.
In these terms, any vertex of color i sees c_i vertices of the color $i-1$, a_i vertices of the color i and b_i vertices of the color $i+1$.

Completely regular code

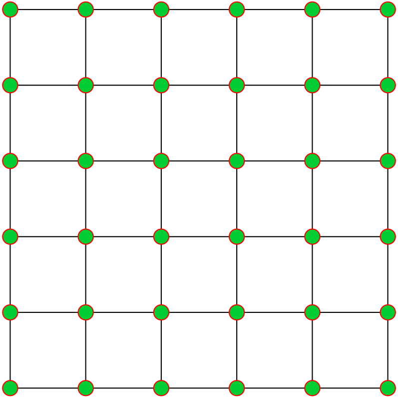
If $(V_0, V_1, \ldots, V_{k-1})$ is a distance regular coloring of a graph then $C = V_0$ is the completely regular code.
Then $k - 1$ is called as the covering radius of C;
the minimum distance between code vertices is called as the code distance of the code C.

Distance regular graph

If for any vertex v of the graph the code $C = \{v\}$ is the completely regular code then the graph is called as distance-regular.
Examples: the n-cube, the Petersen graph.

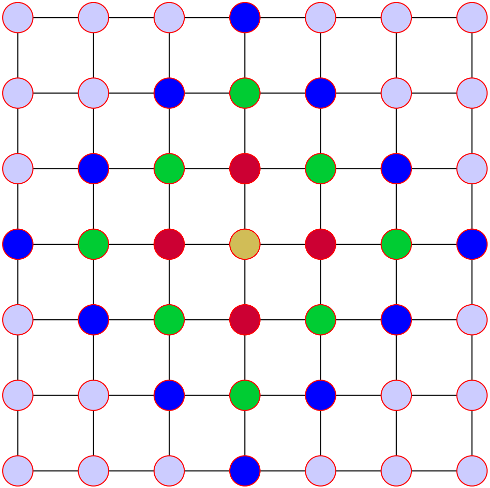
Rectangular grid

The n-dimensional rectangular grid is the graph G_n with:
the vertex set $\mathbb{Z}^n$;
the edge set $\{(x, y) : x, y \in \mathbf{Z}^n, \sum_{i=1}^{n} |x_i - y_i| = 1\}$.



Rectangular grid

If $n \geq 2$ then the n-dimensional rectangular grid is not distance regular.



$n = 1$

If $k \geq 2$ then the distance regular colorings are

$$\ldots, 1, 0, 1, 2, \ldots, k-2, k-1, k-2, \ldots, 2, 1, 0, 1, \ldots$$

$$\ldots, 1, 0, 0, 1, 2, \ldots, k-2, k-1, k-2, \ldots, 2, 1, 0, 0, 1, \ldots$$

$$\ldots, 1, 0, 0, 1, 2, \ldots, k-2, k-1, k-1, k-2, \ldots, 2, 1, 0, 0, 1, \ldots$$

$n = 1$

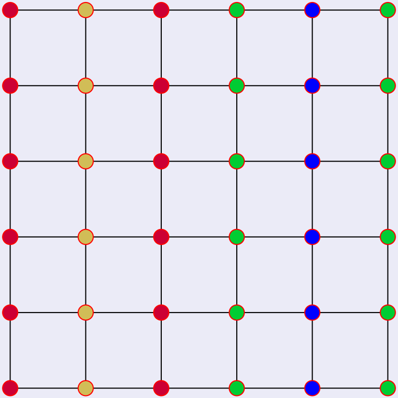
The parmeter matrices are

$$\begin{bmatrix} 02000 & \ldots & \ldots 0 \\ 10100 & \ldots & \ldots 0 \\ 01010 & \ldots & \ldots 0 \\ 00101 & \ldots & \ldots 0 \\ \ldots & \ldots & \ldots \\ 0\ldots & \ldots & 0101 \\ 0\ldots & \ldots & 0020 \end{bmatrix} \quad \begin{bmatrix} 11000 & \ldots & \ldots 0 \\ 10100 & \ldots & \ldots 0 \\ 01010 & \ldots & \ldots 0 \\ 00101 & \ldots & \ldots 0 \\ \ldots & \ldots & \ldots \\ 0\ldots & \ldots & 0101 \\ 0\ldots & \ldots & 0020 \end{bmatrix} \quad \begin{bmatrix} 11000 & \ldots & \ldots 0 \\ 10100 & \ldots & \ldots 0 \\ 01010 & \ldots & \ldots 0 \\ 00101 & \ldots & \ldots 0 \\ \ldots & \ldots & \ldots \\ 0\ldots & \ldots & 0101 \\ 0\ldots & \ldots & 0011 \end{bmatrix}$$

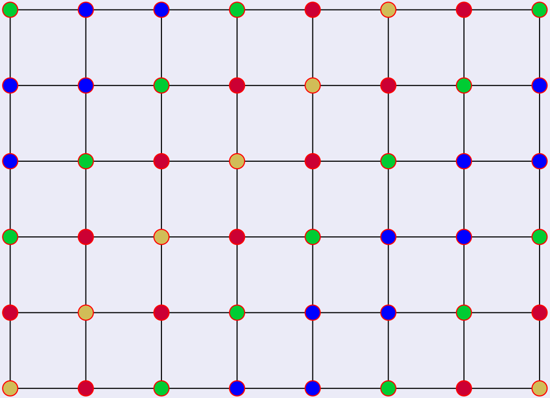
Reducible coloring

A coloring φ of the n-dimensional rectangular grid is called reducible if there exists a coloring ψ of the 1-dimensional rectangular grid such that $\varphi(x_1, x_2, \dots, x_n) = \psi(\delta_1 x_1 + \delta_2 x_2 + \dots + \delta_n x_n),\ \delta_1, \dots, \delta_n \in \{-1, 0, 1\}$

Example



Example



$n = 2$

Theorem The complete list of parameter matrices:

1) six infinite series of reducible matrices;

2) irreducible matrices of order 2:

$$\begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix};$$

3) irreducible matrices of order 3:

$$\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 3 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 \\ 1 & 2 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix};$$

$n = 2$

4) irreducible matrices of order 4:

$$\begin{bmatrix} 0 & 4 & 0 & 0 \\ 1 & 0 & 3 & 0 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 4 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix};$$

5) the irreducible matrix of order 5:

$$\begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}$$

$n = 3$

All feasible parameter matrices are discribed.

Monotonicity

Theorem. If

$$A = \begin{bmatrix} a_0 & b_0 & 0 & 0 & \dots & \dots & \dots & 0 \\ c_1 & a_1 & b_1 & 0 & \dots & \dots & \dots & 0 \\ 0 & c_2 & a_2 & b_2 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \dots & c_{k-2} & a_{k-2} & b_{k-2} \\ 0 & \dots & \dots & \dots & \dots & 0 & c_{k-1} & a_{k-1} \end{bmatrix}$$

is the parameter matrix of an arbitrary disrance regular coloring of n-dimensional rectangular grid then

$$b_0 \geq b_1 \geq \dots \geq b_{k-2},$$

$$c_1 \leq c_2 \leq \dots \leq c_{k-1}.$$

Reducible colorings

Theorem. If $k \geq 4$ and there exists $1 \leq i < j \leq k-2$ such that

$$(c_i, a_i, b_i) = (c_j, a_j, b_j)$$

then

$$(c_i, a_i, b_i) = (c_t, a_t, b_t)$$

for all $t \in \{1, \dots, k-2\}$ and the coloring is reducible.

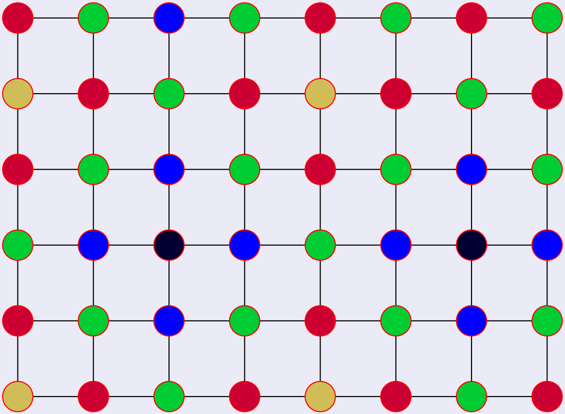
Irreducible colorings

Theorem. Let the distance regular coloring be irreducible and l be the maximal i such that $c_i \leq b_i$. Then
$c_i \neq c_{i+1}$ if $i \leq l$;
$b_i \neq b_{i+1}$ if $i \geq l$.

The number of colors

Theorem. For an arbitrary irreducible distance regular coloring (with k colors) of n-dimensional rectangular grid, it holds $k \leq 2n+1$.
An irreducible distance regular coloring with $2n+1$ colors exists.

$k = 2n+1$



Covering radius

The covering radius of an arbitrary completely regular code in the n-dimensional rectangular grid is at most $2n$.

Code distance

The minimal code distance of an arbitrary completely regular code in the n-dimensional rectangular grid is at most 4.

Density

Let $(V_0, V_1, \ldots, V_{k-1})$ be a coloring. We call as a density of the i-th color, $i = 0, 1, \ldots, k-1$, the following value:

$$p_i = \lim_{r \to \infty} \frac{|V_i \cap B_r|}{|B_r|},$$

where B_r denotes the ball of radius r centered in the all-zero vertex.

Theorem.

Let $(V_0, V_1, \ldots, V_{k-1})$ be an irreducible distance regular coloring of n-dimensional rectangular grid. Then the sequence $p_0, p_1, \ldots, p_{k-1}$ of color densities in unimodal.

$n \geq 4$

Our restrictions:
parameter matrices with the all-zero diagonal: $a_0 = a_1 = \ldots = a_{k-1} = 0$.

1) $k = 2$ $\begin{bmatrix} 0 & 2n \\ 2n & 0 \end{bmatrix}$

2) $k = 3$ $\begin{bmatrix} 0 & 2n & 0 \\ t & 0 & 2n-t \\ 0 & 2n & 0 \end{bmatrix}$, $t = 1, \ldots, n$.

$n = 4, \ k = 4$

We use the linear programming:

Theorem 2.

The matrices

$$\begin{bmatrix} 0 & 8 & 0 & 0 \\ 1 & 0 & 7 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 8 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 8 & 0 & 0 \\ 1 & 0 & 7 & 0 \\ 0 & 3 & 0 & 5 \\ 0 & 0 & 8 & 0 \end{bmatrix}$$

are not feasible for distance regular 4-colorings of 4-dimensional rectangular grid.

Questions

1) Feasible parameter matrices.

2) $a_0, a_1, \ldots, a_{k-1}$.

3) A classification of completely regular codes with a fixed parameter matrix.

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Thank you for your attention!