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Sokolova Galina
student, g.sokolova@g.nsu.ru
Sobolev Institute of Mathematics SB RAS
Novosibirsk State University

SMITH FORM FOR COMPANION MATRIX OF SUPERPOSITION OF POLINOMIALS AND ITS APPLICATION TO KNOT THEORY

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Assume that $\mathscr{R}$ is a non-trivial commutative ring with unity.

Definition 1

Companion matrix C_g of monic polynomial

$$g(t) = t^n + g_{n-1}t^{n-1} + \ldots + g_1 t + g_0 \in \mathscr{R}[t],$$

is defined as a matrix of order n of the form

$$C_g = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -g_0 & -g_1 & -g_2 & \ldots & -g_{n-1} \end{pmatrix}.$$

Polynomial $g(t) \in \mathscr{R}[t]$ is both *the characteristic polynomial* and *the minimal polynomial* for the companion matrix C_g.

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Remark 1.

1. Not every square matrix is similar to a companion matrix.

2. Every square matrix is similar to a block-diagonal matrix, on the diagonal of which the blocks are a companion matrix and are called *Frobenius cell.*

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Remark 1.

3. L.N. Vaserstein and E. Wheland [1] showed that an arbitrary matrix is similar to the product of two matrices each similar to a companion matrix.

[1] Vaserstein L.N., Wheland E. Commutators and Companion Matrices over Rings of Stable Rank 1 // Linear Algebra and its Applications. 1990. V. 142. P. 263–277.

Suppose

- $\mathscr{R}$ is a non-trivial commutative ring with unity;
- $f(t) \in \mathscr{R}[t]$ is an arbitrary polynomial;
- $g(t) \in \mathscr{R}[t]$ is a monic polynomial of degree $n \geq 2$, $n \in \mathbb{N}$;
- $C_g \in \mathscr{R}^{n\times n}$ is the companion matrix for $g(t)$.

Remark 2.

Matrix polynomials $f(C_g)$ over the ring $\mathscr{R}$ forms a commutative ring, which is called *the companion ring of the polynomial* $g(t)$ and denote by $\mathscr{R}_g$.

1. [a] If $g(t) = t^n$, then $\mathscr{R}_g$ is the commutative ring of lower triangular Toeplitz matrices of order n with elements in $\mathscr{R}$.
2. [b] If $g(t) = t^n - 1$, then $\mathscr{R}_g$ is the commutative ring of order n circulant matrices with elements in $\mathscr{R}$.
3. [b] If $g(t) = t^n + 1$, then $\mathscr{R}_g$ is the commutative ring of order n skew-circulant matrices with elements in $\mathscr{R}$.

[a] Bini D.A., Pan V.Y., Polynomial and Matrix Computations. Fund. Algorithms. V. 1, Birkhauser. Boston. MA. 1994.

[b] Davis P.J. Circulant Matrices. New York: AMS Chelsea Publishing. 1994.

Ring of polynomials in a companion matrix

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When $\mathscr{R}$ is an integral domain, polynomial $g(t)$ has n roots (counted with multiplicities) in some appropriate extension of $\mathscr{R}$ and, for $f(t) \in \mathscr{R}[t]$, the determinant of $f(C_g)$ may be expressed as the resultant

$$\det f(C_g) = \prod_{\lambda:\, g(\lambda)=0} f(\lambda) = \operatorname{Res}(f(t), g(t)).$$

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When $\mathscr{R}$ is an integral domain, polynomial $g(t)$ has n roots (counted with multiplicities) in some appropriate extension of $\mathscr{R}$ and, for $f(t) \in \mathscr{R}[t]$, the determinant of $f(C_g)$ may be expressed as the resultant

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Definition 2

Matrices A and B are *elementary equivalent* over $\mathscr{R}$, if

$$\exists N,\, M \;:\; A = NBM, \quad \text{moreover} \quad \det N = 1, \det M = 1.$$

We will write $A \sim B$.

Theorem

[a] Let $g(t) \in \mathscr{R}[t]$ be a monic polynomial, and let $f(t) \in \mathscr{R}[t]$. Suppose that $f(t) = F(t)h(t)$ and $g(t) = G(t)h(t)$, where $h(t)$ is a monic polynomial of degree m. Then

$$f(C_g) \sim F(C_G) \oplus \mathbb{O}_{m\times m},$$

where $\mathbb{O}_{m\times m}$ is zero matrix of degree m.

[a] Noferini V., Williams G. Matrices in companion rings, Smith forms, and the homology of 3-dimensional Brieskorn manifolds // Journal of Algebra. 2021. V. 587, P. 1–19.

Matrix $F(C_G)$ has invariant factors $s_1, s_2, \ldots, s_r$ if and only if $f(C_g)$ has invariant factors $s_1, s_2, \ldots, s_r$ and 0 (repeated m times).

Corollary

Let $g(t) \in \mathscr{R}[t]$ be a monic polynomial, and let $f(t) \in \mathscr{R}[t]$. Suppose that $f(t) = F(t)h(t)$ and $g(t) = G(t)h(t)$, where $h(t)$ is a monic polynomial of degree m. Then

$$\mathrm{Det}\, f(C_g) = \mathrm{Res}(F(t), G(t)),$$

where Det is *essential determinant divisor*.

Theorem A

Let $g(t) \in \mathscr{R}[t]$ be a monic polynomial and let $f(t) \in \mathscr{R}[t]$. Suppose that there exists polynomials $F(t) \in \mathscr{R}[t]$ and $G(t) \in \mathscr{R}[t]$ such that $f(t) = F \circ h(t)$ and $g(t) = G \circ h(t)$, where $h(t)$ is monic polynomial of degree m. Then the following elementary equivalence relation holds

$$f(C_g) \sim \operatorname{diag}(\underbrace{F(C_G), F(C_G), \ldots, F(C_G)}_{m}).$$

Lemma

Let $h(t) \in \mathscr{R}[t]$ and $G(t) \in \mathscr{R}[t]$ be a monic polynomials of degree n and m respectively. Let $\Lambda \in \mathscr{R}^{mn \times mn}$ be a matrix of order mn of the form

$$\Lambda = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \\ \vdots \\ \Lambda_m \end{pmatrix}, \quad \Lambda_k = [\mathbb{O}_{n\times(k-1)} \,|\, \widetilde{\Lambda} \,|\, \mathbb{O}_{n\times(m-k)}], \quad k = 1, 2, \ldots, m,$$

where

$$\widetilde{\Lambda} = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ c_h(0) & c_h(1) & \ldots & c_h(m-1) & 1 & \ldots & 0 \\ c_{h^2}(0) & c_{h^2}(1) & \ldots & c_{h^2}(m-1) & c_{h^2}(m) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{h^{n-1}}(0) & c_{h^{n-1}}(1) & \ldots & c_{h^{n-1}}(m-1) & c_{h^{n-1}}(m) & \ldots & 1 \end{pmatrix},$$

and the symbol $c_z(k)$ denotes the k-th coefficient of the polynomial $z(t)$. Then Λ is a unimodular matrix, and $h(C_{G\circ h}) \sim \operatorname{diag}(\underbrace{C_G, C_G, \ldots, C_G}_{m})$.

Note that the statement of Lemma 2.1 is true if and only if

$$\Lambda h(C_{G\circ h}) = \operatorname{diag}(\underbrace{C_G, C_G, \dots, C_G}_{m})\Lambda.$$

1. Since the determinant of the Vandermonde matrix does not vanish, the following relation holds

$$\Lambda h(C_{G\circ h}) \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = \operatorname{diag}(\underbrace{C_G, C_G, \dots, C_G}_{m})\Lambda \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix}. \qquad (1)$$

2. Consider the right part of the equality (1).
For each k-th block of the matrix Λ we have the following relation

$$C_G\Lambda_k \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = C_G \begin{pmatrix} z^{k-1} \\ z^{k-1}h \\ z^{k-1}h^2 \\ \vdots \\ z^{k-1}h^n \end{pmatrix} = \begin{pmatrix} z^{k-1}h \\ z^{k-1}h^2 \\ \vdots \\ z^{k-1}h^{n-1} \\ z^{k-1}(h^n - G\circ h) \end{pmatrix} = \ell_k.$$

2. Thus, the right part of the equality being proved has the form

$$\operatorname{diag}(\underbrace{C_G, C_G, \ldots, C_G}_{m})\Lambda \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = [\ell_1, \ell_2, \ldots, \ell_m]^T,$$

where

$$\ell_k = \begin{pmatrix} z^{k-1}h \\ z^{k-1}h^2 \\ \vdots \\ z^{k-1}h^{n-1} \\ z^{k-1}(h^n - G \circ h) \end{pmatrix}.$$

Let us remind

$$\Lambda h(C_{G\circ h})\begin{pmatrix}1\\z\\z^2\\\vdots\\z^{mn}\end{pmatrix}=\mathrm{diag}(\underbrace{C_G,C_G,\ldots,C_G}_{m})\Lambda\begin{pmatrix}1\\z\\z^2\\\vdots\\z^{mn}\end{pmatrix}. \tag{1}$$

3. Consider the left part of the equality (1). Note that

$$h(C_{G\circ h})=M+\sum_{k=0}^{m}c_h^k\widetilde{C}_{G\circ h}^k, \tag{2}$$

where the symbol $\widetilde{C}_{G\circ h}^k$ denotes the k-th power of the companion matrix $C_{G\circ h}^k$ such that

$$\widetilde{C}_{G\circ h}^k=C_{G\circ h}^k-\begin{pmatrix}\mathbb{O}_{mn-k\times k} & \mathbb{E}_{mn-k\times mn-k}\\ \mathbb{O}_{k\times k} & \mathbb{O}_{k\times mn-k}\end{pmatrix},$$

Let us remind

$$\Lambda h(C_{G\circ h})\begin{pmatrix}1\\ z\\ z^2\\ \vdots\\ z^{mn}\end{pmatrix}=\mathrm{diag}(\underbrace{C_G, C_G, \ldots, C_G}_{m})\Lambda\begin{pmatrix}1\\ z\\ z^2\\ \vdots\\ z^{mn}\end{pmatrix}. \qquad (1)$$

3. Consider the left part of the equality (1). Note that

$$h(C_{G\circ h}) = M + \sum_{k=0}^{m} c_h^k \widetilde{C}_{G\circ h}^k, \qquad (3)$$

where M is a Toeplitz matrix

$$M=\begin{pmatrix}
c_h(0) & c_h(1) & c_h(2) & \ldots & c_h(m-1) & 1 & 0 & \ldots & 0\\
0 & c_h(0) & c_h(1) & \ldots & \ldots & c_h(m-1) & 1 & \ldots & 0\\
0 & 0 & c_h(0) & \ldots & \ldots & \ldots & c_h(m-1) & \ldots & 0\\
 & & & & & & \vdots & \ddots & \vdots\\
\vdots & \vdots & \vdots & & \ddots & & & & c_h(m-1)\\
 & & & & & & \vdots & \ddots & \vdots\\
0 & 0 & 0 & \ldots & \ldots & \ldots & \ldots & \ldots & c_h(0)
\end{pmatrix}.$$

$$h(C_{G\circ h}) = M + \sum_{k=0}^{m} c_h^k \widetilde{C}_{G\circ h}^k, \tag{2}$$

The following vector is the product of the k-th block of the matrix Λ and the matrix M

$$\Lambda_k M \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = \begin{pmatrix} z^{k-1}h \\ z^{k-1}h^2 \\ \vdots \\ z^{k-1}h^{n-1} \\ z^{k-1}(h^n - z^{mn} - \sum_{j=1}^{k} c_{h^n}^{mn-j} z^{mn-j}) \end{pmatrix}.$$

The relation

$$\Lambda_k \sum_{k=0}^{m} c_f^k \widetilde{C}_{G\circ h}^k \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = \begin{pmatrix} \mathbb{O}_{(mn-1)\times 1} \\ z^{k-1}(z^{mn} + \sum_{j=1}^{k} c_{h^n}^{mn-j} z^{mn-j} - G\circ g) \end{pmatrix}.$$

is satisfied for the second matrix of the equality (2)

Thus, for the k-th block of the transformation matrix Λ, we have

$$\Lambda_k(M + \sum_{k=0}^{m} c_h^k \widetilde{C}_{G\circ h}^k) \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = \begin{pmatrix} z^{k-1}h \\ z^{k-1}h^2 \\ \vdots \\ z^{k-1}h^{n-1} \\ z^{k-1}(h^n - G\circ h) \end{pmatrix} = \ell_k,$$

This means that the vector on the left side

$$\Lambda(M + \sum_{k=0}^{m} c_h^k \widetilde{C}_{G\circ h}^k) \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{mn} \end{pmatrix} = [\ell_1, \ell_2, \ldots, \ell_m]^T.$$

is equal to the vector obtained in step 2.

Lemma

Let $h(t) \in \mathscr{R}[t]$ and $G(t) \in \mathscr{R}[t]$ be a monic polynomials of degree n and m respectively. Let $\Lambda \in \mathscr{R}^{mn\times mn}$ be a matrix of order mn of the form

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and the symbol $c_z(k)$ denotes the k-th coefficient of the polynomial $z(t)$. Then Λ is a unimodular matrix, and $h(C_{G\circ h}) \sim \operatorname{diag}(\underbrace{C_G, C_G, \ldots, C_G}_{m})$.

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Let $g(t) \in \mathscr{R}[t]$ be a monic polynomial and let $f(t) \in \mathscr{R}[t]$. Assume that $f(t) = F \circ h(t)$ and $g(t) = G \circ h(t)$, where $h(t)$ is monic polynomial of degree m. Then

$$f(C_g) \sim \operatorname{diag}(\underbrace{F(C_G), F(C_G), \ldots, F(C_G)}_{m}).$$

Matrix $F(C_G)$ has invariant factors $(s_1, s_2, \ldots, s_r)$ if and only if $f(C_g)$ has invariant factors $(s_1, s_2, \ldots, s_r)$ repeated m times.

Corollary

Let $g(t) \in \mathscr{R}[t]$ be a monic polynomial and let $f(t) \in \mathscr{R}[t]$. Assume that $f(t) = F \circ h(t)$ and $g(t) = G \circ h(t)$, where $h(t)$ is monic polynomial of degree m. Then

$$\operatorname{Res}(f, g) = (\operatorname{Res}(F, G))^m,$$

and

$$\operatorname{Det}(f(C_g)) = \operatorname{Det}(\operatorname{diag}(\underbrace{F(C_G), F(C_G), \ldots, F(C_G)}_{m})),$$

where Det is *essential determinant divisor*.

Definition 2

The Chebyshev polynomials $\mathcal{T}_n(t)$ of the first and $\mathcal{U}_n(t)$ *of the second kind* are polynomials of degree n in t, given by the formulas

$$\mathcal{T}_n(t) = \cos n\theta, \quad \mathcal{U}_n(t) = \frac{\sin(n+1)\theta}{\sin\theta},$$

where $\theta = \arccos t$ and $n \in \mathbb{N}$.

[2] From the formula for the difference of sines

$$\sin(n+1)\theta - \sin(n-1)\theta = 2\sin\theta\cos n\theta,$$

it follows that the polynomials of the first and second kinds are related by the relationship

$$\mathcal{U}_{n+1}(t) - \mathcal{U}_{n-1}(t) = 2\mathcal{T}_n(t).$$

[2] Mason J.C., Handscomb D.C. Chebyshev Polynomials. CRC Press. Boca Raton. 2003.

Definition 2

The Chebyshev polynomials $\mathcal{W}_n(t)$ of the fourth kind is polynomials of degree n in t, given by the formula

$$\mathcal{W}_n(t) = \frac{\sin(n + \frac{1}{2})\theta}{\sin \theta/2},$$

where $\theta = \arccos t$ and $n \in \mathbb{N}$.

[3] From the formula for the sum of sines

$$\sin(n+1)\theta - \sin n\theta = 2\cos(\theta/2)\sin(n\theta + \theta/2),$$

it follows that the polynomials of the fourth and second kinds are related in the following identity

$$\mathcal{U}_n(t) + \mathcal{U}_{n-1}(t) = \mathcal{W}_n(t).$$

[3] Mason J.C., Handscomb D.C. Chebyshev Polynomials. CRC Press. Boca Raton. 2003.

Theorem B

[a] Suppose M_n is an n-fold cyclic covering of the sphere $\mathbb{S}^3$, branches over a two-bridge knot K, and let $\mathscr{L}$ and $\mathscr{L}'$ be matrices

$$\mathscr{L} = (F(C_G))^m \cdot \mathcal{W}_m \left(1 + \frac{1}{2F(C_G)}\right), \quad \text{and} \quad \mathscr{L}' = (F(C_G))^{m-1} \mathcal{U}_{m-1} \left(1 + \frac{1}{2F(C_G)}\right).$$

Then

1. if $n = 2m+1$, $n \in \mathbb{Z}$, then first homology group $H_1(M_{2m+1}, \mathbb{Z})$ splits into direct sum of two copies of the Abelian group $V = \text{coker}\mathscr{L}$, i.e.
$$H_1(M_{2m+1}, \mathbb{Z}) = V \oplus V;$$
2. if $n = 2m$, $n \in \mathbb{Z}$, then the covering map $\varphi : M_{2m} \to M_2$ induces a surjective homomorphism $\varphi_* : H_1(M_{2m}, \mathbb{Z}) \to H_1(M_2, \mathbb{Z})$, whose kernel splits into direct sum of two copies of the Abelian group $V' = \text{coker}\mathscr{L}'$,
$$\text{Ker}\, \varphi_* = V' \oplus V'.$$

[a] Plans A. Aportacion al estudio de los grupos de homologia de los recubrimientos ciclicos ramificados correspondiente a un nudo, Rev. Real. Acad. Cienc. Exact., Fisica y Nat. Madrid, 47 (1953), 161–193.

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