Stone dualities

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An algebraic structure $\langle L; \vee, \wedge \rangle$ is a **lattice** if

- $\vee^2$ and $\wedge^2$ are idempotent, commutative, associative operations;
- for all $a, b \in L$, $a \wedge (a \vee b) = a = a \vee (a \wedge b)$.

A lattice L is **distributive** if, for all $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

Not each lattice is distributive! N_5 and M_3 are the “minimal” non-distributive lattices.

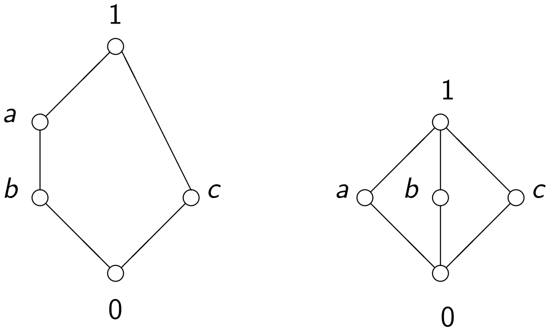

Figure: Lattices N_5 and M_3

Theorem (M. H. Stone)

The category of distributive $(0,1)$*-lattices with* $(0,1)$*-homomorphisms is dually equivalent to the category of spectral spaces with spectral maps.*

Theorem (M. H. Stone)

The category of Boolean algebras with homomorphisms is dually equivalent to the category of Boolean spaces with continuous maps.

One can generalize these results of M. H. Stone in two directions:

- dualities for distributive posets;
- dualities for $(0, 1)$-lattices which are close to distributive.

Dualities for distributive posets

For a poset $\langle P; \leq \rangle$ and $X \subseteq P$:

$L(X)$ is the set of all lower bounds of X;

$U(X)$ is the set of all upper bounds of X.

Distributive posets

A *c*-**poset** is a structure $\mathcal{P} = \langle P; \leq, \varphi \rangle$ such that:

- $\langle P; \leq \rangle$ is a poset;
- φ is an algebraic closure operator on P which defines a **completion** of $\langle P; \leq \rangle$; that is,

$$\varphi\colon P \to \operatorname{Id}\mathcal{P}, \quad \varphi\colon x \mapsto \varphi(x)$$

is an order embedding of $\langle P; \leq \rangle$ into the complete lattice $\operatorname{Id}\mathcal{P}$ of φ-closed subsets of P.

Corollary

If $\mathcal{P} = \langle P; \leq, \varphi \rangle$ is a c-poset then $\varphi(x) = L(x)$ for all $x \in P$.

A c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$ is **distributive** or just a **distributive poset**, if the following condition is satisfied:

- the lattice $\operatorname{Id} \mathcal{P}$ is distributive.

Any φ-closed subset of P is a φ-**ideal** of $\langle P; \leq \rangle$ or just an **ideal** of $\mathcal{P}$.

A set $F \subseteq P$ is a **filter** of $\langle P; \leq \rangle$ if it is an upper cone which is down-directed with respect to $\leq$.

Lemma (Ts. Batueva, MS)

For a c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$ and a proper ideal I of $\mathcal{P}$, TFAE.

1. $P \backslash I$ *is a filter of* $\langle P; \leq \rangle$.
2. I *is a* $\cap$*-prime element in* $\operatorname{Id} \mathcal{P}$.
3. $L(a_0, a_1) \subseteq I$ *implies that* $a_i \in I$ *for some* $i < 2$.

An ideal I of $\mathcal{P}$ is **prime** if it satisfies one of the equivalent statements of Lemma above.

Theorem (Ts. Batueva, MS)

Let $\mathcal{P} = \langle P; \leq, \varphi \rangle$ be a distributive c-poset, let $I \subseteq P$ be a nonempty ideal of $\mathcal{P}$, and let $F \subseteq P$ be a nonempty filter such that $I \cap F = \varnothing$. Then there is a prime ideal $Q \subseteq P$ such that $I \subseteq Q$ and $Q \cap F = \varnothing$.

The category DP

Let **DP** denote the category whose objects are distributive c-posets and whose morphisms are mappings $f\colon P_0 \to P_1$, where $\mathcal{P}_0 = \langle P_0; \leq, \varphi_0 \rangle$ and $\mathcal{P}_1 = \langle P_1; \leq, \varphi_1 \rangle$ are distributive c-posets, which satisfy the following condition:

- f is **proper**; that is, $f^{-1}(I)$ is a prime ideal of $\mathcal{P}_0$ for each prime ideal I of $\mathcal{P}_1$;

DP-morphisms preserve meets, joins, 0, 1. In particular, they are monotone.

Lemma

Let $\mathcal{S}_0$ *and* $\mathcal{S}_1$ *be distributive* $(0,1)$*-lattices. Then* $f\colon \mathcal{S}_0 \longrightarrow \mathcal{S}_1$ *is a* **DP***-morphism if and only if* f *is a* $(0,1)$*-lattice homomorphism.*

Consider the following full subcategories of **DP**:

- the category $\mathbf{DP}_1$ whose objects are distributive posets with 1;
- the category $\mathbf{DP}_0$ whose objects are distributive posets with 0;
- the category $\mathbf{DP}_{0,1}$ whose objects are distributive posets with $0, 1$;
- the category $\mathbf{DSL}^{\wedge}$ whose objects are distributive $\wedge$-semilattices;
- the category $\mathbf{DSL}^{\wedge}_1$ whose objects are distributive $\wedge$-semilattices with 1;
- the category $\mathbf{DSL}^{\wedge}_0$ whose objects are distributive $\wedge$-semilattices with 0;
- the category $\mathbf{DSL}^{\wedge}_{0,1}$ whose objects are distributive $\wedge$-semilattices with $0, 1$;

- the category $\mathbf{DSL}^{\vee}$ whose objects are distributive $\vee$-semilattices;
- the category $\mathbf{DSL}^{\vee}_{1}$ whose objects are distributive $\vee$-semilattices with 1;
- the category $\mathbf{DSL}^{\vee}_{0}$ whose objects are distributive $\vee$-semilattices with 0;
- the category $\mathbf{DSL}^{\vee}_{0,1}$ whose objects are distributive $\vee$-semilattices with $0, 1$;
- the category $\mathbf{DL}$ whose objects are distributive lattices;
- the category $\mathbf{DL}_1$ whose objects are distributive 1-lattices;
- the category $\mathbf{DL}_0$ whose objects are distributive 0-lattices;
- the category $\mathbf{DL}_{0,1}$ whose objects are distributive $(0, 1)$-lattices.

Spectra of posets [Yu. L. Ershov, MS]

Let $\mathcal{P} = \langle P; \leq, \varphi \rangle$ be a c-poset.
$\operatorname{Spec}\mathcal{P}$ is the set of all prime ideals of $\mathcal{P}$. For each $a \in P$, we put

$$V_a = \{I \in \operatorname{Spec}\mathcal{P} \mid a \notin I\}.$$

The space $\mathbb{S}\mathrm{pec}\,\mathcal{P} = \langle \operatorname{Spec}\mathcal{P}, \mathcal{T}, \mathcal{B} \rangle$, where $\mathcal{T}$ denotes the topology with the basis $\mathcal{B} = \{V_a \mid a \in P\}$, is the **spectrum of poset** $\mathcal{P}$.

The space $\mathbb{S}\mathrm{pec}\,\mathcal{S}$ is called the **spectrum of a join-semilattice** $\langle S; \vee \rangle$, where $\mathcal{S} = \langle S; \vee, \psi \rangle$ and

$$\psi(X) = \{s \in S \mid s \leq a \vee \ldots \vee a_n \text{ for some } n < \omega,\ a_0, \ldots, a_n \in X\}.$$

The **spectrum of a lattice** $\langle L; \vee, \wedge \rangle$ is the spectrum of its join-semilattice reduct $\langle L; \vee \rangle$.

Lemma

For a c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$, the following statements hold.

1. *If $a \wedge b$ exists in $\mathcal{P}$ for some $a, b \in P$ then $V_{a \wedge b} = V_a \cap V_b$. If $\mathcal{P}$ is distributive then $V_a \cap V_b = V_c$ for some $c \in P$ implies that $c = a \wedge b$ in $\mathcal{P}$.*
2. *If $\langle P; \leq \rangle$ is a join-semilattice and $\varphi = \psi$ then $V_{a \vee b} = V_a \cup V_b$ for all $a, b \in P$. If $\mathcal{P}$ is distributive then $V_a \cup V_b = V_c$ for some $c \in P$ implies that $c = a \vee b$ in $\mathcal{P}$.*

Sober spaces

Let $\mathbb{X}$ be a T_0-space.

A subset $Y \subseteq X$ is **irreducible** if $Y \subseteq F_0 \cup F_1$ for some closed sets F_0, F_1 implies that $Y \subseteq F_i$ for some $i < 2$.

$\mathbb{X}$ is **sober** if for each nonempty closed irreducible set $F \subseteq X$, there is $x \in X$ such that $F = \downarrow x$.

$\mathbb{X}$ is **almost sober** if for each proper closed irreducible set $F \subseteq X$, there is $x \in X$ such that $F = \downarrow x$.

Proposition (Yu. L. Ershov, MS)

For a c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$, the following statements hold.

1. $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is almost sober.
2. $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is sober whenever P has 0.
3. If $\mathcal{P}$ is down-directed distributive and $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is sober then $\langle P; \leq \rangle$ has 0.

Proposition (Yu. L. Ershov, MS)

For a distributive c-poset $\mathcal{P} = \langle P; \leq, \varphi \rangle$, the following statements hold.

1. The set V_a is compact in $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ for every $a \in S$. In particular, $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is compact whenever $\langle P; \leq \rangle$ has 1.
2. If $\mathcal{P}$ is up-directed and $\mathbb{S}\mathrm{pec}\,\mathcal{P}$ is compact then $\langle P; \leq \rangle$ has 1.

For a T_0-space $\mathbb{X}$ and $\mathcal{F} \subseteq \mathcal{T}(\mathbb{X})$, define a closure operator $\varphi_{\mathcal{F}}$ on $\mathcal{F}$ as follows. If $\mathcal{X} \subseteq \mathcal{F}$ then

$$\varphi_{\mathcal{F}}(\mathcal{X}) = \{U \in \mathcal{F} \mid U \subseteq \bigcup \mathcal{X}\}.$$

If $\mathcal{F}$ consists of compact sets, then $\varphi_{\mathcal{F}}$ is algebraic.

Lemma

If a family $\mathcal{B}$ *of compact open sets in* $\mathbb{X}$ *forms a base of* $\mathcal{T}(\mathbb{X})$ *then the c-poset* $\mathfrak{B} = \langle \mathcal{B}; \subseteq, \varphi_{\mathcal{B}} \rangle$ *is distributive.*

Spaces with base

Definition

A triple $\mathbb{X} = \langle X, \mathcal{T}, \mathcal{B} \rangle$ is a **topological space with base** or just a **space with base**, if

1. $\langle X, \mathcal{T} \rangle$ is a T_0-space and $\mathcal{B}$ forms a base of $\mathcal{T}$;
2. $\langle X, \mathcal{T} \rangle$ is sober and $\mathcal{B}$ is down-directed if and only if $\varnothing \in \mathcal{B}$;
3. X is compact in $\langle X, \mathcal{T} \rangle$ and $\mathcal{B}$ is up-directed if and only if $X \in \mathcal{B}$.

$\mathcal{B}$ is a **multiplicative base** if $\mathcal{B}$ is closed under finite nonempty intersections. $\mathcal{B}$ is an **additive base** if $\mathcal{B}$ is closed under finite nonempty unions.

$\mathcal{K}(\mathbb{X})$ is the set of all compact sets in $\mathbb{X}$.

Lemma

Let $\mathbb{X}$ be a space with additive base and let $\mathcal{B}(\mathbb{X}) \subseteq \mathcal{K}(\mathbb{X})$. Then

$$\mathcal{B}(\mathbb{X}) = \begin{cases} \mathcal{K}(\mathbb{X}), & \textit{if } \varnothing \in \mathcal{B}(\mathbb{X}); \\ \mathcal{K}(\mathbb{X}) \backslash \{\varnothing\}, & \textit{if } \varnothing \notin \mathcal{B}(\mathbb{X}). \end{cases}$$

Almost [semi]spectral spaces with base

Definition

A space with base $\mathbb{X}$ is an **almost semispectral space with base**, if $\langle \mathbb{X}, \mathcal{T}(\mathbb{X}) \rangle$ is an almost sober space, and $\mathcal{B}(\mathbb{X})$ consists of open compact sets.

$\mathbb{X}$ is an **almost spectral space with base**, if $\langle \mathbb{X}, \mathcal{T}(\mathbb{X}) \rangle$ is an almost sober space, and $\mathcal{B}(\mathbb{X})$ is a multiplicative base of $\mathcal{T}(\mathbb{X})$ consisting of open compact sets.

$\mathbb{X}$ is a **[semi]spectral space with base**, if $\mathbb{X}$ is almost [semi]spectral space with base, $\varnothing, X \in \mathcal{B}(\mathbb{X})$, and $\langle \mathbb{X}, \mathcal{T}(\mathbb{X}) \rangle$ is a compact sober space.

The category AS

Let **AS** be the category whose objects are almost semispectral spaces with base and whose morphisms are **spectral** mappings:

> if $f : \mathbb{X} \to \mathbb{Y}$, where $\mathbb{X}, \mathbb{Y} \in$ **AS** then $f^{-1}(U) \in \mathcal{B}(\mathbb{X})$ for all $U \in \mathcal{B}(\mathbb{Y})$.

Lemma

If $\mathbb{X}, \mathbb{Y}$ *are almost semispectral spaces with base and* $f : \mathbb{X} \longrightarrow \mathbb{Y}$ *is spectral then* f *is continuous.*

We consider the following full subcategories of the category **AS**:

- the category $\mathbf{AS}_c$ whose objects are compact almost semispectral spaces with base;
- the category $\mathbf{AS}_s$ whose objects are sober almost semispectral spaces with base;
- the category **S** whose objects are semispectral spaces with base;
- the category **ASp** whose objects are almost spectral spaces with base;
- the category $\mathbf{ASp}_c$ whose objects are compact almost spectral spaces with base;
- the category $\mathbf{ASp}_s$ whose objects are sober almost spectral spaces with base;
- the category **Sp** whose objects are spectral spaces with base;

- the category **AsSpec** whose objects are almost semispectral spaces;
- the category $\mathbf{AsSpec}_c$ whose objects are compact almost semispectral spaces;
- the category $\mathbf{AsSpec}_s$ whose objects are sober almost semispectral spaces;
- the category **sSpec** whose objects are semispectral spaces;
- the category **ASpec** whose objects are almost spectral spaces;
- the category $\mathbf{ASpec}_c$ whose objects are compact almost spectral spaces;
- the category $\mathbf{ASpec}_s$ whose objects are sober almost spectral spaces;
- the category **Spec** whose objects are spectral spaces.

Theorem (Yu. L. Ershov, MS)

For a T_0-space $\mathbb{X}$, the following holds.

1. $\mathbb{X}$ *is a semispectral space if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(0,1,\vee)$*-semilattice.*
2. $\mathbb{X}$ *is a compact almost semispectral space if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(1,\vee)$*-semilattice.*
3. $\mathbb{X}$ *is sober almost semispectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(0,\vee)$*-semilattice.*
4. $\mathbb{X}$ *is almost semispectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $\vee$*-semilattice.*

Theorem (Yu. L. Ershov, MS)

For a T_0-space $\mathbb{X}$, the following holds.

1. $\mathbb{X}$ *is spectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* $(0, 1)$*-lattice.*
2. $\mathbb{X}$ *is compact almost spectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* 1*-lattice.*
3. $\mathbb{X}$ *is sober almost spectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive* 0*-lattice.*
4. $\mathbb{X}$ *is almost spectral if and only if* $\mathbb{X}$ *is homeomorphic to the spectrum of a distributive lattice.*

These two theorems extend to a full duality.

The functor T

$$\mathrm{T}\colon \mathbf{DP} \to \mathbf{AS};$$
$$\mathrm{T}\colon \mathcal{P} \mapsto \mathbb{S}\mathrm{pec}\,\mathcal{P};$$
$$\text{if } f\colon \mathcal{P}_0 \to \mathcal{P}_1 \text{ then } \mathrm{T}(f)\colon I \mapsto f^{-1}(I).$$

The functor P

$$\mathrm{P}\colon \mathbf{AS} \to \mathbf{DP};$$
$$\mathrm{P}\colon \mathbb{X} \mapsto \langle \mathcal{B}(\mathbb{X}); \subseteq, \varphi_{\mathcal{B}} \rangle;$$
$$\text{if } f\colon \mathbb{X}_0 \to \mathbb{X}_1 \text{ then } \mathrm{T}(f)\colon U \mapsto f^{-1}(U).$$

Theorem

The categories **DP** *and* **AS** *are dually equivalent via* P *and* T. *The categories* $\mathbf{DP}_{fin}$ *and* $\mathbf{AS}_{fin}$ *are therefore also dually equivalent.*

An instance:

Corollary

P *and* T *establish the dual equivalence of categories* $\mathbf{DSL}_0^{\wedge}$ *and* $\mathbf{ASp}_s$*.*

Dualities for $(0, 1)$-lattices: quasivarieties generated by finite $(0, 1)$-lattices

The quasivariety $\mathbf{SP}(N_5)$

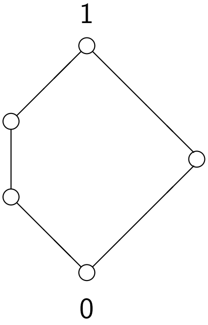

Figure: Lattice N_5

Definition (W. A. Dziobiak, MS)

A structure $\mathbb{S} = \langle X, Y, \leq, f \rangle$ is an N_5-**space**, if

(s1) $X \cap Y = \varnothing$ and $X \cup Y \neq \varnothing$;

(s2) $\leq$ is a partial order on $X \cup Y$;

(s3) $f \colon Y \to X^2$ is a function and for all $y \in Y$ with $f(y) = (a, b)$, the following conditions hold:

- $a \leq y$ and $\{a, b\}$, $\{y, b\}$ are antichains;
- if $a, b \leq z$ for some $z \in X \cup Y$ then $y \leq z$;
- if $z \leq y$ for some $z \in X \cup Y$ then either $z \leq a$ or $z \leq b$, or $z \in Y$ and $\{u, v\} \ll \{a, b\}$ where $f(z) = (u, v)$.

Definition (W. A. Dziobiak, MS)

Let $\mathbb{S} = \langle X, Y, \leq, f \rangle$ and $\mathbb{S}' = \langle X', Y', \leq, f \rangle$ be N_5-spaces. Then $\varphi \colon \mathbb{S} \to \mathbb{S}'$ is a N_5-**morphism**, if the following conditions hold:

(m1) φ maps $X \cup Y$ into $X' \cup Y' \cup \{\{a, b\} \mid a, b \in X'\}$;

(m2) if $u, v \in X \cup Y$ are such that $\varphi(u), \varphi(v) \in X' \cup Y'$ and $u \leq v$ then $\varphi(u) \leq \varphi(v)$;

(m3) for all $x \in X$, $\varphi(x) \in X'$;

(m4) for all $y \in Y$ with $f(y) = (a, b)$, the following holds:

- if $\varphi(y) \in X'$ then either $\varphi(y) = \varphi(a)$ or $\varphi(y) \leq \varphi(b)$;
- if $\varphi(y) \in Y'$ then $f\big(\varphi(y)\big) = \big(\varphi(a), \varphi(b)\big)$;
- if $\varphi(y) \notin X' \cup Y'$ then $\varphi(y) = \big\{\varphi(a), \varphi(b)\big\}$ is an antichain and $\{\varphi(a), \varphi(b)\} \ll \varphi(z)$ for all $z \in X \cup Y$ with $y \leq z$.

Objects in N_5 are bi-algebraic $(0,1)$-lattices belonging to $\mathbf{SP}(N_5)$. Morphisms in N_5 are complete $(0,1)$-lattice homomorphisms.

Objects in B_5 are N_5-spaces. Morphisms in B_5 are N_5-morphisms.

Theorem (W. A. Dziobiak, MS)

The categories N_5 and B_5 are dually equivalent.

Introducing a topology compatible with the N_5-structure, one extends this result to a duality between $\mathbf{SP}(N_5)$ and the category of topological N_5-spaces as follows.

Objects in L_5 are $(0,1)$-lattices belonging to $\mathbf{SP}(N_5)$. Morphisms in L_5 are $(0,1)$-lattice homomorphisms.

Objects in T_5 are spectral N_5-spaces. Morphisms in T_5 are spectral N_5-morphisms.

Theorem (W. A. Dziobiak, MS)

The categories L_5 *and* T_5 *are dually equivalent.*

Similar duality results can also be established for:

- the quasivariety $\mathbf{SP}(L_4)$ (A. O. Basheyeva, MS);
- the quasivariety $\mathbf{SP}(L_6)$ (A. O. Basheyeva, MS);
- the quasivariety $\mathbf{SP}(M_3)$ (A. E. Izyurova, MS).

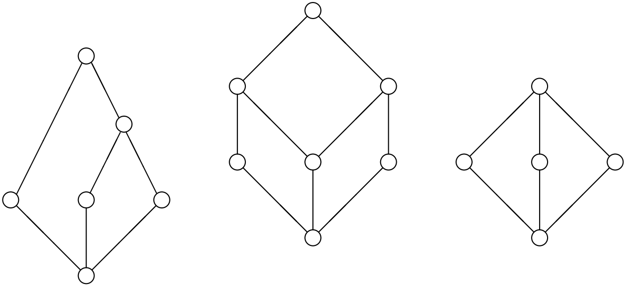

Figure: Lattices L_4, L_6, and M_3

Restrictions of all the dualities discussed to distributive lattices yield the Stone duality.