ON THE ACCURACY
OF GAUSSIAN APPROXIMATION
IN HILBERT SPACE

S. V. Nagaev * *and V. I. Chebotarev* **

Abstract

This article is a continuation of the authors' paper [24] with a new approach to studying the accuracy of order $O(1/n)$ of Gaussian approximation in Hilbert space. In contrast to [24], we now study a more general case of the class of sets on which the probability measures are compared, namely, the class of balls with arbitrary centers. The resultant bound depends on the thirteen greatest eigenvalues of the covariance operator T in explicit form; moreover, this dependence is sharper as compared to the bound of [4].

Key words and phrases: Gaussian approximation in Hilbert space, eigenvalues of the covariance operator, discretization of a probability distribution, conditionally independent random variables.

1. Introduction. The main results

Let $\mathbb{H}$ be a separable Hilbert space with an inner product $(\cdot\,,\cdot)$ and the corresponding norm $|\cdot|$ and let $X, X_1, X_2, \ldots$ be independent identically distributed random elements acting in $\mathbb{H}$ with zero mean and covariance operator T. Let $\sigma_1^2 \geq \sigma_2^2 \geq \cdots$ be the eigenvalues of T, and let $e_1, e_2, \ldots$ be the corresponding eigenvectors forming an orthonormal basis. We use the following notations:

$$\Lambda_l = \prod_{j=1}^{l} \sigma_j^2, \quad \sigma^2 = \mathbb{E}\,|X|^2, \quad \beta_\mu = \mathbb{E}\,|X|^\mu,$$

$$S_n = n^{-1/2} \sum_{j=1}^{n} X_j,$$

$$B(a; r) = \big\{ x : x \in \mathbb{H}, \ |x - a| < r \big\},$$

$$\Delta_n(a; r) = \big| \mathbb{P}\big(S_n \in B(a; r)\big) - \Phi\big(B(a; r)\big) \big|,$$

$$\Delta_n(a) = \sup_{r>0} \Delta_n(a; r),$$

where $\Phi(\cdot)$ is the Gaussian distribution in $\mathbb{H}$ with covariance operator T and zero mean. We denote the dimension of $\mathbb{H}$ by d. If $\mathbb{H}$ is an infinite-dimensional space, we write $d = \infty$. The symbols $C(\cdot)$ and $c(\cdot)$ possibly with indices stand for constants depending on the parameters in parentheses, and the symbol c possibly with index denotes an absolute constant. The same notation is permitted for different constants.

We first consider the case of $a = 0$. From the results of [4] it follows that

$$\Delta_n(0) \leq \frac{C(T)}{n} \frac{\beta_4}{\sigma^4}, \tag{1.1}$$

where

$$C(T) = e^{c\sigma^2/\sigma_{13}^2} \qquad \text{if } 13 \leq d \leq \infty \text{ and } \sigma_{13} \neq 0; \tag{1.2}$$

$$C(T) = \frac{\sigma^4}{\sigma_d^4} e^{c\sigma^2/\sigma_9^2} \quad \text{if } 9 \leq d < \infty \text{ and } \sigma_d \neq 0; \tag{1.3}$$

$$C(T) = e^{c\sigma^2/\sigma_9^2} \qquad \begin{array}{l} \text{if the distribution of } X \\ \text{is symmetric and } 9 \leq d \leq \infty. \end{array} \tag{1.4}$$

The formulas (1.2)–(1.4) illustrate the form of the dependence of the error $\Delta_n(0)$ on the covariance operator. In the case of a symmetric distribution of X, (1.4) is the most precise result. In the general case each of the relations (1.2) and (1.3) may turn out to be more precise than other. Thus, if $9 \leq d \leq 13$ and σ_{13}^2 is considerably less than σ_9^2 then (1.3) is more precise than (1.2). But if $d \geq 13$ then it may appear that (1.2) is more precise than (1.3). This is so, for instance, for $d \geq 14$ and sufficiently small σ_d^2.

A result of [11] implies that (1.1) holds for $\sigma^2 = 1$ with

$$C(T) = c\left(\frac{1}{\Lambda_{12}^{1/2}} + \frac{\sigma_1^4}{\sigma_9^8 \Lambda_9^{2/9}} + \frac{1}{\sigma_9^4 \Lambda_9^{1/4}} \right) \quad \text{if } d \geq 12, \ \sigma_{12}^2 \neq 0. \tag{1.5}$$

The dependence on T in (1.5) is more precise than that in (1.2).

We now give the first main result of the present article which somewhat revises the bound of $\Delta_n(0)$ in [24]. This result was obtained for the case $d \geq 13$ and $\sigma_{13} \neq 0$ in terms of the quantities

$$\Gamma_{\mu,l} \equiv \frac{\beta_\mu \sigma^\mu}{\Lambda_l^{\mu/l}}, \quad L_l \equiv \max_{1 \leq j \leq l} \frac{\mathbb{E}\big|(X, e_j)\big|^3}{\sigma_j^3}.$$

Observe that $\Gamma_{\mu,l}/n^{(\mu-2)/2}$ is a generalization of the Lyapunov fraction $\beta_\mu/n^{(\mu-2)/2}\sigma^\mu$.

Theorem 1.1. *There exists an absolute constant c such that*

$$\Delta_n(0) \leq \frac{c}{n}\left(\Gamma_{4,13} + \Gamma_{3,13}^2 + L_9^2\big(\sigma^2/\Lambda_9^{1/9}\big)^2\right). \tag{1.6}$$

R e m a r k 1.1. Neither $\Gamma_{4,13}$ nor $\Gamma_{3,13}^2$ nor $L_9^2\big(\sigma^2/\Lambda_9^{1/9}\big)^2$ is a majorant with respect to the others. For instance, it is incorrect to state that there exists a constant $c > 0$ such that, for every distribution whose the fourth moment of the norm is finite, we have

$$L_9^2\big(\sigma^2/\Lambda_9^{1/9}\big)^2 \leq c\Gamma_{3,13}^2. \tag{1.7}$$

E x a m p l e 1.1. Consider the distribution of X such that

$$\sigma^2 = \sum_{j=1}^k \sigma_j^2 = 1,$$

where

$$\sigma_1^2 = \cdots = \sigma_8^2 = 1/16, \quad \sigma_9^2 = \cdots = \sigma_k^2 = 1/(2(k-8)), \quad k \geq 16,$$

$$\mathbb{E}\big|(X, e_j)\big|^3 \geq c_1\beta_3, \quad 1 \leq j \leq 9, \quad c_1 > 0. \tag{1.8}$$

Note that the condition (1.8) is accessible. This follows, for instance, from a lemma by V. V. Sazonov in [30, p. 85]. Next, the following relations are valid:

$$L_9^2\left(\frac{\sigma^2}{\Lambda_9^{1/9}}\right)^2 \geq \left(\frac{c_1\beta_3}{\sigma_9^3}\right)^2 \frac{c_2}{\sigma_9^{4/9}} = \frac{\beta_3^2 c_3}{\sigma_9^{58/9}} \geq \beta_3^2 k^{29/9}c_4,$$

$$\Gamma_{3,13}^2 = \frac{\beta_3^2}{\Lambda_{13}^{6/13}} = \frac{\beta_3^2 c_5}{\sigma_9^{60/13}} \leq \beta_3^2 k^{30/13}c_6.$$

Hence, for every constant $c > 0$, there exists k such that, in the example under consideration, the inequality opposite to (1.7) holds. It is possible in the case of an infinite-dimensional space $\mathbb{H}$. Note that we may construct an example with a similar property in a finite-dimensional space as well. For instance, if $d = 13$ then it is sufficient to consider the distribution of X such that $\sum_{j=1}^{13} \sigma_j^2 = 1$, $\sigma_1^2, \ldots, \sigma_8^2$ are close to $1/8$, $\sigma_9^2, \ldots, \sigma_{13}^2$ are sufficiently small, and the condition (1.8) is fulfilled.

R e m a r k 1.2. It is clear that, for the case in which the eigenvalue σ_{13}^2 is small as compared with the previous eigenvalues $\sigma_1^2, \ldots, \sigma_{12}^2$, the expression (1.5) reflects the dependence of the error on T in a more exact form than (1.6). On the other hand, there exists a class of covariance operators, for which the bound (1.6) is more precise than the result of [11]. To verify this, it is sufficient to take the distribution in Example 1.1. Then, as immediate calculations show, 1) the quantity $\sigma_1^4/\big(\sigma_9^8 \Lambda_9^{2/9}\big)$ majorizes the quantity $1/\Lambda_{12}^{1/2}$ in the expression (1.5); 2) $L_9^2/\Lambda_9^{2/9}$ majorizes $\Gamma_{3,13}^2$ in the bound (1.6). Moreover, $\beta_4 \sigma_1^4/\big(\sigma_9^8 \Lambda_9^{2/9}\big)$ majorizes $L_9^2/\Lambda_9^{2/9}$.

Let $a \neq 0$. In contrast to $\Delta_n(a)$, we introduce the error of the Gaussian approximation, taking into account the Edgeworth correction of the first order. Put

$$\Delta_{1,n}(a;r) = \mathbb{P}\Big(S_n \in B\big(a;\sqrt{r}\big)\Big) - \Phi\Big(B\big(a;\sqrt{r}\big)\Big) - \frac{1}{\sqrt{n}}Q_1(r;a),$$

$$\Delta_{1,n}(a) = \sup_{r>0}\big|\Delta_{1,n}(a;r)\big|,$$

where $\big(1/\sqrt{n}\big)Q_1(r;a)$ is the Edgeworth correction. In other words, $\Delta_{1,n}(a;r)$ is the remainder in the short Edgeworth expansion (in the Edgeworth expansion of the first order), and $\Delta_{1,n}(a)$ is its norm. Put

$$\widehat{Q}_1(t;a) = \int_0^\infty e^{itr}\,dQ_1(r;a).$$

Note that there are different algorithms for calculating $\widehat{Q}_1(t;a)$, and the proof of the corresponding identities is a particular problem (see [23, Item 1.4], and also Lemma 6.4). We can show that

$$\widehat{Q}_1(t;a) = \frac{s^4}{3!}\mathbb{E}\,\exp\big\{it|Y_0 - a|^2\big\}\big(3|X|^2(X, Y_0 - a) + s^2(X, Y_0 - a)^3\big), \quad (1.9)$$

where Y_0 is a Gaussian random element with the distribution Φ, $s = (2it)^{1/2}$. Define

$$g(t;a) = \mathbb{E}\,\exp\big\{it|Y_0 - a|^2\big\}, \quad g_j(t) = (1 - 2it\sigma_j^2)^{-1/2}.$$

The following representation of $\widehat{Q}_1(t;a)$ in the form of the product of two functions, one of which is the characteristic function $g(t;a)$, is of interest:

$$\widehat{Q}_1(t;a) = g(t;a)P_1(t;a),$$

where

$$P_1(t;a) = -\left\{ 2(it)^2 \mathbb{E}\left[(A_t X, A_t a)(A_t X, A_t X) \right] + \frac{4}{3}(it)^3 \mathbb{E}(A_t X, A_t a)^3 \right\},$$

$$(A_t x, A_t y) = \sum_{j=1}^{\infty} g_j^2(t)(x, e_j)(y, e_j).$$

It is shown in [4, Theorems 1.3 and 1.5] that

$$\Delta_{1,n}(a) \le C(T)\left(1 + \frac{|a|^6}{\sigma^6}\right)\left\{ \frac{\mathbb{E}\left[|X|^4 I\left(|X| \le \sigma\sqrt{n} \right) \right]}{\sigma^4 n} \right. $$
$$\left. + \frac{\mathbb{E}\left[|X|^3 I\left(|X| > \sigma\sqrt{n} \right) \right]}{\sigma^3 \sqrt{n}} \right\}, \qquad (1.10)$$

where $C(T)$ may be taken either of the two expressions: (1.2) or (1.3), in dependence on the conditions on d and σ_l^2. In the case, when $\mathbb{H}$ is a finite-dimensional space, the factor $1 + |a|^6/\sigma^6$ in (1.10) may be replaced with the quantity $1 + |a|^3/\sigma^3$ [4, Theorem 1.5].

Put

$$\beta_\mu(a) = \mathbb{E}\left| (a, X) \right|^\mu, \quad \Gamma_{\mu,l}(a) = \beta_\mu(a)/\Lambda_l^{\mu/l}.$$

The following estimate was proved in [11] under the assumption $\sigma^2 = 1$:

$$\Delta_{1,n}(a) \le \frac{c}{n}\left[\beta_4\left(\frac{1}{\Lambda_{12}^{1/2}} + \frac{\sigma_1^4}{\sigma_9^8 \Lambda_9^{2/9}} + \frac{1}{\sigma_9^4 \Lambda_9^{1/4}} \right) + \frac{\beta_4(a)}{\Lambda_{12}^{1/2}}\left(1 + \beta_2(a) \right) \right]. \quad (1.11)$$

We now formulate the second main result of the present article whose first version was published in [25].

Theorem 1.2. *For every $a \in \mathbb{H}$,*

$$\Delta_{1,n}(a) \le \frac{c}{n}\left[\Gamma_{4,13} + \Gamma_{3,13}^2 + L_9^2\left(\frac{\sigma^2}{\Lambda_9^{1/9}} \right)^2 + \Gamma_{4,9}(a) + \Gamma_{3,13}^2(a) \right]. \qquad (1.12)$$

R e m a r k 1.3. Since $Q_1(r;0) = 0$, Theorem 1.1 is a corollary to Theorem 1.2.

R e m a r k 1.4. Theorem 1.2 gives an estimate of accuracy which is nonuniform with respect to the centers of the balls in the short Edgeworth expansion. The constant is a sum of five summands, each of them is a generalization of the Lyapunov fraction $\beta_\mu/\big(n^{(\mu-2)/2}\sigma^\mu\big)$. Three of them are expressed in terms of a negative power of the product of the first thirteen eigenvalues of the covariance operator. Two other summands (responsible for the behavior of the characteristic function of the squared norm of the sum of the random elements near the point $t = n$) are expressed in terms of a negative power of the product only of the first nine eigenvalues. In addition, it is impossible to remove any of five summands in the resultant bound. This bound depends optimally on the centers of the balls.

Obviously, the bounds of Theorems 1.1 and 1.2 yield a sharper dependence of the error on the covariance operator T as compared with (1.1) and (1.10), where $C(T)$ is taken from (1.2). Moreover, Theorem 1.2 yields the more precise dependence on a of the error with respect to (1.10). The comparison with the inequality (1.11) is carried out in the same way as in Remark 1.2.

In the following remark we discuss the methods that help us to solve the problems of obtaining the estimates (1.6), (1.12), (1.1) and (1.11).

R e m a r k 1.5. To explain some main ideas of the proof, we start with the case $a = 0$. We split our explanation into a few steps.

S t e p 1. Passing to the truncations $\big($see (5.46)$\big)$, we reduce the problem to estimating $\overline{\Delta}_n(0) \equiv \sup_r \left| \mathbb{P}\big(\big|n^{-1/2}\sum_{j=1}^n \overline{X}_j\big|^2 < r\big) - \mathbb{P}\big(|Y_0|^2 < r\big)\right|$.

By the Esseen inequality (see, for instance, [27]) and the bound

$$\sup_r \frac{d}{dr}\mathbb{P}\big(|Y_0|^2 < r\big) \leq \frac{1}{2\sigma_1\sigma_2} \tag{1.13}$$

(see [14] and also Lemma 6.7), for every $\tau_0 > 0$, we have

$$\overline{\Delta}_n(0) \leq c\left(J + \frac{1}{\sigma_1\sigma_2\tau_0}\right), \tag{1.14}$$

where

$$J = \int_{|t|\leq\tau_0} \frac{\big|\bar{g}_n(t) - g(t)\big|}{|t|}dt,$$

$$\bar{g}_n(t) = \mathbb{E}\exp\left\{it\left|n^{-1/2}\sum_{j=1}^n \overline{X}_j\right|^2\right\}, \quad g(t) = g(t;0).$$

For simplicity, we may assume in what follows that

$$J = \int_0^{\tau_0} \frac{\left|\bar{g}_n(t) - g(t)\right|}{t} dt.$$

Take

$$\tau_0 = nt_0, \quad \text{where} \quad t_0 = \frac{c\Lambda_9^{1/9}}{n_0\sigma^4},$$

with n_0 the number in (5.7). The idea of choosing n_0 consists in splitting the sum $\sum_{j=1}^n \overline{X}_j$ into n_0 blocks such that each of them actually possesses the properties of a Gaussian random element. (Without loss of generality we can assume that $n \equiv 0 \pmod{n_0}$.) More exactly, in what follows, we use the representation $\sum_{j=1}^n \overline{X}_j \overset{d}{=} \sqrt{n_0} \sum_{j=1}^{n/n_0} Y_j$, where $Y_1, Y_2, \ldots, Y_{n/n_0}$ are independent copies of the random element

$$Y = n_0^{-1/2} \sum_{j=1}^{n_0} \overline{X}_j.$$

We split the interval $[0, \tau_0]$ into 4 parts by the points τ_1, τ_2, and τ_3 to be defined at the next steps. The choice of the points is determined from the distinction of the methods used for estimating $\bar{g}_n(t) - g(t)$ over the different intervals.

S t e p 2. The bounds of $\bar{g}_n(t) - g(t)$ in [20–22] lead us to the inequality

$$\int_0^{\tau_1} \frac{\left|\bar{g}_n(t) - g(t)\right|}{t} dt \le \frac{c}{n}\left(\frac{\beta_4}{\Lambda_{13}^{2/13}} + \Gamma_{3,13}^2\right) \tag{1.15}$$

if

$$\tau_1 = (n/\Gamma_{4,13})^{1/4}\Lambda_{13}^{-1/13}.$$

For $t \in [\tau_1, \tau_0]$ we use some estimates of the characteristic functions $g(t)$ and $\bar{g}_n(t)$. It is easy to verify that

$$\int_{\tau_1}^{\infty} \frac{\left|g(t)\right|}{t} dt \le \frac{\Gamma_{4,13}}{n}. \tag{1.16}$$

It remains to estimate the integral

$$\int_{\tau_1}^{\tau_0} \frac{\left|\bar{g}_n(t)\right|}{t} dt = \int_{t_1}^{t_0} \frac{\left|\bar{g}_n(nt)\right|}{t} dt$$

with $t_1 = \tau_1/n$.

S t e p 3. Choose t_2 (then $\tau_2 = nt_2$) and evaluate

$$J_2 \equiv \int_{t_1}^{t_2} \frac{|\bar{g}_n(nt)|}{t}\, dt.$$

Taking it into account that

$$\bar{g}_n(t) = \mathbb{E} \exp\left\{ i\frac{n_0 t}{n} \left| \sum_{j=1}^{n/n_0} Y_j \right|^2 \right\} \equiv \widetilde{g}_n(n_0 t/n),$$

we obtain

$$J_2 = \int_{t_1}^{t_2} \frac{\left|\widetilde{g}_n(n_0 t)\right|}{t} dt.$$

Putting

$$t_2 = c_0 \left(2^{l/2} L \sigma(L) \sqrt{2n} \right)^{-1},$$

where $\sigma^2(L) = \mathbb{E}\left\{ \left| Y - \mathbb{E}\{Y/|Y| \le L\} \right|^2 \Big/ |Y| \le L \right\}$, by Lemma 2.8, we have

$$\left| \widetilde{g}_n(n_0 t) \right| \le \frac{c(l)}{\sqrt{1 + \Lambda_l(L)(tn)^l}}$$

for $0 < t < t_2$ and every natural l. Here $\Lambda_l(L) = \prod_{j=1}^{l} \sigma_j^2(L)$, $\sigma_1^2(L) \ge \sigma_2^2(L) \ge \cdots$ are the eigenvalues of the quadratic form

$$B(x; L) = \mathbb{E}\left[(Y - Y', x)^2 ; \ |Y| \vee |Y'| \le L \right],$$

where Y' is an independent copy of Y. It follows that

$$J_2 \le c(l)(\Gamma_{4,13}/n)^{l/8} \Lambda_{13}^{l/26} / \Lambda_l^{1/2}(L).$$

By Lemmas 5.1–5.6, we conclude that

$$\sigma_j^2(L) \ge c(l)\sigma_j^2. \tag{1.17}$$

We obtain this result by the choice of n_0 and the special sets $A_j^{\pm}(L)$ (see Section 4 and Fig. 4.1), $1 \le j \le l$, the probability of hitting in which of the element Y is separated from 0 $\big($see (5.13) and (5.14)$\big)$. Taking an arbitrary integer l from the interval $[8, 13]$ and employing the inequality $\Lambda_{l+1}^{1/(l+1)} \le \Lambda_l^{1/l}$, we infer that

$$J_2 \le c(l)\Gamma_{4,13}/n \tag{1.18}$$

on assuming $\Gamma_{4,13}/n \le 1$ without loss of generality.

S t e p 4. Choose t_3 (in this case $\tau_3 = nt_3$) and estimate $J_4 \equiv \int_{t_3}^{t_0} \frac{|\widetilde{g}_n(n_0t)|}{t}dt$. The definition of the bound t_3 is connected with the behavior of the characteristic function of the squared norm of the sum of independent identically distributed random elements in a neighborhood of the point $t = 1$. The main claim about this behavior is Lemma 4.3 proven in [3].

In order to reduce the problem of choosing t_3 to Lemma 4.3, we consider the random elements taking only finitely many values. Our results lose no generality under this discreteness assumption (see Section 4), since they do not depend on the support of the discrete distribution and depend only on finitely many moments.

Let $\widetilde{F}$ be the distribution of Y. We can write $\widetilde{F} = \sum_{k=0}^{l} F_{k,L}$, where $F_{k,L}$ are the restrictions of $\widetilde{F}$ to $\widetilde{A}_k(L)$ (these sets are defined in Section 4). By Lemma 3.2, the discrete positive measure $F_{k,L}$ is a mixture of some two-point measures with atoms in $A_k^+(L)$ and $A_k^-(L)$ correspondingly. So the n-fold convolution $\widetilde{F}^{n*}$ is the expectation of the convolution of the two-point *random* distributions $G_1 * \cdots * G_n$. We write up this fact in the form of $\widetilde{F}^{n*} = \mathbb{E}_1(G_1 * \cdots * G_n)$. Thus, if $Z_1, \ldots, Z_n$ are random variables with distributions $G_1, \ldots, G_n$ then

$$\widetilde{g}_n(n_0t) = \int \exp\{in_0t|x|^2\}d\widetilde{F}^{n*}(x) = \mathbb{E}_1\mathbb{E}_{G_1,\ldots,G_n}\exp\left\{in_0t\left|\sum_{j=1}^{n}Z_j\right|^2\right\},$$

where $\mathbb{E}_{G_1,\ldots,G_n}$ denotes the expectation with respect to the distributions $G_1, \ldots, G_n$. Consequently,

$$J_4 \leq \int_{\tau}^{\varepsilon} t^{-1}\left|\mathbb{E}_1\mathbb{E}_{G_1,\ldots,G_n}\exp\left\{in_0t\left|\sum_{j=1}^{n}Z_j\right|^2\right\}\right|dt \equiv J_4',$$

where $\tau = n_0t_3$, $\varepsilon = n_0t_0$. If we put

$$\tau/\varepsilon = (B_n\varepsilon)^{-4/l}, \quad \text{where} \quad B_n = L^2\sqrt{n/n_0}, \tag{1.19}$$

then, using Lemma 4.5 for $l = 9$, we obtain

$$J_4' \leq \frac{cn_0}{n}\left(L^2/\Lambda_9^{1/9}\right)^2. \tag{1.20}$$

From (1.19) it follows that

$$t_3 = \left(t_0^5n_0^{-2}L^{-8}\right)^{1/9}n^{-2/9} = c_1\left(\Lambda_9^{5/9}L^{-8}\sigma^{-20}n_0^{-7}\right)^{1/9}n^{-2/9}.$$

It should be noted that the proof of Lemma 4.5 rests on all previous results of Sections 3 and 4, in particular, on Lemmas 4.1, 3.11, and 3.12 and on the construction of the sets $A_j^{\pm}(L)$ as well. Notice also that a bound of the product $f_n(t)f_n(t+\tau)$ is derived in Lemma 4.1, where $f_n(t)$ is the characteristic function of the squared norm of the sum of n independent identically distributed random elements. This claim is in fact the first step in reducing the problem of estimating J_4' to Lemma 4.3. On the other hand, Lemma 4.1 forces us to pass to the bounds (we obtain them in Section 3) of conditional characteristic functions and probabilities, which complicates the proof.

S t e p 5. We estimate the integral

$$J_3 \equiv \int_{t_2}^{t_3} \frac{\left|\widetilde{g}_n(n_0 t)\right|}{t} dt = \int_{n_0 t_2}^{\tau} \frac{\left|\widetilde{g}_n(t)\right|}{t} dt.$$

By Lemma 2.7, the following inequality holds in the interval $[n_0 t_2, \tau]$:

$$\left|\widetilde{g}_n(t)\right| \leq c(l)L^{2l}\Lambda_l^{-1/2}(L)t^{l/2}.$$

From here and (1.17) it follows that

$$J_3 \leq \frac{cn_0}{n}\left(L^2/\Lambda_9^{1/9}\right)^2. \tag{1.21}$$

The estimates (1.14)–(1.16), (1.18), (1.20), and (1.21) imply Theorem 1.1.

The same methods are used in the case of balls with arbitrary centers. To prove Theorem 1.2, it remains to overcome additional difficulties connected with the Edgeworth correction. We will address this in Section 6.

Concluding the remark, we observe that, for proving Theorems 1.1 and 1.2, we use the results and methods of [1–4] and [14–17, 21–23]. On the other hand, we use some new ideas. Lemmas 3.1–3.5, 3.11, and 3.12 are the key assertions. Notice that, in contrast to [4, 11], we avoid using some specific methods of number theory.

The case of $a = 0$ was studied by the authors in [24]. A preliminary version of the present article was published in the preprint [25].

R e m a r k 1.6. We focus attention on some elements of the proof of the bound (1.11) in [11], since its right-hand side depends on twelve eigenvalues of the operator T rather than on the thirteen as in estimate (1.12). It should be noticed that the proof of (1.11) rests on [5, 6] and consists of the two parts: the proofs of two theorems. The general place of these two parts is the method of characteristic functions. The dependence of $\left|\Delta_{1,n}(a)\right|$ on twelve eigenvalues of the covariance operator was obtained in [11, Theorem 1.1],

where, for estimating the integral of the difference of characteristic functions divided by the integration variable, the estimate of the following type is used:

$$\left| \int_{-A}^{A} t^d e^{itb} \left(\prod_{j=1}^{2d+2} g_j(t) \right) dt \right| \leq c \Lambda_{2d+2}^{-1/2}$$

for every $A > 0$, real $b \neq 0$, and integer $d \geq 0$ (see [11, Lemma 2.2]), which differs from the bound (5.48) of the present article. Moreover, alongside the Esseen inequality, the so-called Prawitz inequality [28] is used. The proof of the corresponding bound in [11, Lemma 2.10], revealing the dependence on twelve eigenvalues of T, is based on Lemma 8.4 of [5, 6]. The proof of Theorem 1.2 [11], revealing the dependence on nine eigenvalues of T, is based on Lemmas 6.3–6.7 and 7.1 of [5, 6], where the methods of discreteness and the double large sieve are used.

The results (1.1), (1.2), and (1.12) give the bound $\Delta_{1,n}(a) = O(1/n)$ optimal in n under the condition

$$13 \leq d \leq \infty, \quad \sigma_{13} \neq 0.$$

The bound (1.11) weakens this condition to the following:

$$12 \leq d \leq \infty, \quad \sigma_{12} \neq 0;$$

moreover, as an example in [11, p. 5] shows, if $12 \leq d \leq \infty$ then the bound $\Delta_{1,n}(a) = O(n^{-1})$ cannot depend on less than the first twelve eigenvalues of T.

We mention a result that does not follow from neither (1.1)–(1.4) nor (1.6), (1.11), (1.12).

In what follows, let $I(A)$ be the indicator of a set A.

Theorem 1.3 [21]. *For every $\delta > 0$, $1 < q < 13/12$, and an integer $l \geq 7$, we have*

$$\Delta_n(0) \leq c(l, \delta, q) \left\{ \left(\frac{\Gamma_{4,l}}{n} \right)^{l/(l+4+\delta)} + \left(\frac{\Gamma_{3,l}^2}{n} \right)^{l/(12q)} I(7 \leq l \leq 12) \right.$$
$$\left. + \frac{\Gamma_{3,l}^2}{n} I(l \geq 13) \right\}. \quad (1.22)$$

Note that a bound of $\Delta_n(a; r)$ analogous to (1.22) was obtained in [34], but only for $d = 13$ and with less precise dependence of the error on T. On the other hand, the bound in [34] is nonuniform in $\left| r - |a| \right|$.

R e m a r k 1.7. For the case $\sigma_6 \neq 0$ it is shown in $[15, 17, 31, 36]$ that

$$\Delta_n(a) \leq c\beta_3\left(\sigma^3 + |a|^3\right) \bigg/ \left(\sqrt{n}\,\Lambda_6^{1/2}\right). \tag{1.23}$$

The following estimate is announced by S. V. Nagaev in [17] under the condition $\sigma_4 \neq 0$:

$$\Delta_n(0) \leq c\beta_3\sigma \bigg/ \left(\sqrt{n}\,\Lambda_4^{1/2}\right). \tag{1.24}$$

About the accuracy of (1.23) and (1.24) see, for instance, $[32,\mathrm{p}.25,26]$.

Henceforth, we use the following notations: Z' is an independent copy of Z, $Z^s = Z - Z'$, and $\widetilde{Z} = Z - \mathbb{E}Z$. Notice that the abbreviation Z' will be also used for denoting *a conditionally independent copy* of a random variable Z (see Section 3).

Let

$$B(x; L) = \mathbb{E}\left[(Z^s, x)^2; |Z| \vee \left|Z'\right| \leq L\right], \tag{1.25}$$

where Z is an arbitrary $\mathbb{H}$-valued random variable, and let $\sigma_j^2(L) \geq \sigma_{j+1}^2(L)$, $j = \overline{1, \infty}$, be the eigenvalues of the quadratic form $B(x; L)$.

Also, put

$$\Lambda_l(L) = \prod_{k=1}^{l} \sigma_k^2(L), \quad \delta_l^2(L) = \frac{L^2}{\Lambda_l^{2/l}(L)} \sum_{j=1}^{l} \sigma_j^2(L),$$

$$p_L = \mathbb{P}\left(|Z| \leq L\right), \quad a_L = \mathbb{E}\left(Z/|Z| \leq L\right),$$

$$\sigma^2(L) = \mathbb{E}\left(|Z - a_L|^2/|Z| \leq L\right),$$

$$f_x^Z(t) = \mathbb{E}e^{it(Z,x)}, \quad \int = \int_{\mathbb{H}}.$$

In each particular case we will indicate what we mean by Z.

The article is organized as follows: In Section 2, we obtain some bounds of the characteristic function of the squared norm of the sum of independent identically distributed random elements for the values of the argument t in a neighborhood of zero (Lemmas 2.7 and 2.8). In Section 3, we prove the bounds of conditional characteristic functions. One useful inequality for some quadratic forms on special finite-dimensional sets is proved in Lemma 3.1. In Lemma 3.2, we give a representation of an arbitrary discrete measure in the form of a mixture of two-point probability distributions. Lemmas 3.11 and 3.12 are conditional versions of Lemmas 2.7 and 2.8. In Section 4, grounding on the results of Section 3, we obtain an integral estimate of the characteristic function of the squared norm of the sum of independent two-point random elements.

In Section 5, we compare the eigenvalues $\sigma_j^2(L)$ and σ_j^2 and prove Theorem 1.1 (the case $a = 0$). Sections 6 and 7 are allotted to the proof of Theorem 1.2.

Using statements without proofs, we give precise references to the papers in which these proofs are contained. Sometimes, despite the appropriate references, we repeat the formulations and provide revised proofs (see, for instance, Lemmas 2.6, 3.11, 5.6, and 5.7).

2. Bounds of characteristic functions in the neighborhood of zero

Let $X, X_1, \ldots, X_n, \ldots$ be a sequence of independent identically distributed random elements with values in $\mathbb{H}$. In contrast to the Introduction, we do not assume in this section that $\mathbb{E}X = 0$. We use the notations: $U_{k_1,k_2} = \sum_{k_1}^{k_2} X_j$ and $U_m = U_{1,m}$.

In Section 2, the symbols $\sigma_k^2(L)$, $k = 1, 2, \ldots$, stand for the eigenvalues of the quadratic form (1.25) with $Z \equiv X$.

Lemma 2.1 [21, Lemma 1.3]. *Let ξ be a nonnegative random variable and let $\mathbb{P}(\xi < r) \leq Qr^l$ for $r \geq \varepsilon > 0$. Then*

$$\mathbb{E} \exp\{-\xi^2 t^2\} < \big(c(l)|t|^{-l} + \varepsilon^l\big)Q, \tag{2.1}$$

where $c(l) \leq \Gamma(l/2 + 1)$, $\Gamma(p)$ is the gamma-function. Moreover, for $0 < t < l$,

$$\mathbb{E}\xi^{-t}I(\xi \geq \varepsilon) \leq 2\left(\frac{Qt}{l-t}\right)^{t/l}. \tag{2.2}$$

Lemma 2.2 [16, Lemma 1]. *Let ξ be a real random variable with $\mathbb{E}|\xi|^3 < \infty$ and let η be a bounded positive random variable. Then*

$$\mathbb{E}(1 - \cos \xi)\eta \geq \mathbb{E}\xi^2\eta/2 - \mathbb{E}|\xi|^3\eta/\big(4\sqrt{3}\big).$$

Lemma 2.3. *Let $X, X_1, \ldots, X_m$ be independent identically distributed random vectors with values in $\mathbb{R}^l$, $U = \sum_1^m X_j$, and B_l be a nonnegative symmetric matrix of order $l \times l$ with eigenvalues $b_1^2 \geq \cdots \geq b_l^2 > 0$. Then, for every $L > 0$ and $r > 0$,*

$$\sup_{a \in \mathbb{R}^l} \mathbb{P}\Big(\big(B_l(U^s - a), U^s - a\big)^{1/2} < r \Big) \leq \frac{c(l)(r + \varepsilon_l)^l}{m^{l/2}\Lambda_l^{1/2}(L) \prod_{k=1}^l b_k},$$

where $\varepsilon_l^2 = 32lL^2 \sum_{k=1}^l b_k^2$ and $c(l) = 1/\Gamma(l/2 + 1)$.

Proof. The scheme of the proof is the same as that of Theorem 1 in [16]. (See also the proof of Lemma 2.3 in [25].)

Lemma 2.4. *For every* $1 \le m < n$, $L > 0$, *and* $t \in \mathbb{R}$,

$$\psi(t) \equiv \mathbb{E}\left[\left|f_{U_m^s}^{U_{m+1,n}}(2t)\right|; |U_m| \vee |U_m'| < \sqrt{3}\big/\big(8|t|L\big)\right]$$

$$\le c(l)\left[\left(1 + \Lambda_l(L)\big(|t|\sqrt{m(n-m)}\big)^l\right)^{-1} + \left(\frac{\delta_l(L)}{\sqrt{m}}\right)^l\right]. \qquad (2.3)$$

Proof. The scheme of the proof is the same as that in [16, Lemma 5]. (See also the proof of Lemma 2.4 in [25].)

Lemma 2.5. *Let* Z *and* Y *be independent random elements with values in* $\mathbb{H}$. *Then, for every* $r > 0$, $a \in \mathbb{H}$, *and* $t \in \mathbb{R}$,

$$\left|\mathbb{E}\exp\{it|Z+Y-a|^2\}\right|$$

$$\le \mathbb{P}\big(|Z| \ge r\big)\sup_{b\in\mathbb{H}}\left|\mathbb{E}\exp\{it|Y-b|^2\}\right|$$

$$+ \mathbb{E}^{1/2}\left[\left|f_{Z^s}^Y(2t)\right|; |Z| \vee |Z'| < r\right]. \qquad (2.4)$$

Proof. The assertion of the lemma ensues from the following three relations:

$$E_0 \equiv \mathbb{E}\exp\{it|Z+Y-a|^2\}$$

$$= \mathbb{E}\left[\exp\{it|Z+Y-a|^2\}; |Z| \ge r\right]$$

$$+ \mathbb{E}\left[\exp\{it|Z+Y-a|^2\}; |Z| < r\right]$$

$$\equiv E_1 + E_2,$$

$$|E_1| = \left|\mathbb{E}\left[I\big(|Z| \ge r\big)\mathbb{E}_Y\exp\{it|Z+Y-a|^2\}\right]\right|$$

$$\le \mathbb{P}\big(|Z| \ge r\big)\sup_{b\in\mathbb{H}}\left|\mathbb{E}\exp\{it|Y-b|^2\}\right|,$$

$$|E_2| \le \mathbb{E}^{1/2}\left[\left|f_{Z^s}^Y(2t)\right|; |Z| < r, |Z'| < r\right].$$

(See also [16, Lemmas 8 and 7].)

Put $c_0 = \sqrt{3}/8$.

Lemma 2.6. *Let*

$$Bn^{-1/2} \le |t| \le \frac{c_0}{2\sigma L} \qquad (2.5)$$

and $\{l_i\}_{i=1}^k$ *be a sequence such that*

$$1 \le l_i \le \frac{c_0}{2\sigma L|t|}, \tag{2.6}$$

$$\sum_{i=1}^k l_i^{-2} < 2\left(\frac{t\sigma L}{c_0}\right)^2 n. \tag{2.7}$$

Then

$$\sup_{a \in \mathbb{H}} \left| \mathbb{E} \exp\{it|U_n - a|^2\} \right| < c(l)|t|^{l/2} \widetilde{D}(l) \sum_{j=1}^k l_j^{l/2}/4^j + 4^{-k} \tag{2.8}$$

for every natural l, *where*

$$\widetilde{D}(l) = (\sigma L)^{l/2}\left(B^{-l/2} + (\sigma L)^{l/2}\right)\Big/\Lambda_l^{1/2}(L).$$

Proof (see also [24]). Define the sequence $\{m_i\}_{i=1}^k$ by the formula

$$m_i = \left[\left(c_0/\left(2l_i\sigma Lt\right)^2\right)\right]. \tag{2.9}$$

Note that, by (2.6), the bound $m_i \ge 1$ is valid. Let $\mu_j = \sum_{i=1}^j m_i$. Denote

$$q_j(t) = \sup_{a \in \mathbb{H}} \left| \mathbb{E} \exp\{it|U_{\mu_j+1,n} - a|^2\} \right|.$$

Putting $r = 2\sigma\sqrt{m_j}$, $Z = \widetilde{U}_{\mu_{j-1}+1,\mu_j}$, and $Y = \widetilde{U}_{\mu_j+1,n}$ in Lemma 2.5, we obtain

$$q_{j-1}(t) \le \frac{1}{4}q_j(t) + Q_j, \quad j = 1, \ldots, k, \tag{2.10}$$

where

$$Q_j = \mathbb{E}^{1/2}\left\{\left|f_{U_{m_j}^s}^{\widetilde{U}_{\mu_j+1,n}}(2t)\right|; \ |\widetilde{U}_{m_j}| \vee |\widetilde{U}'_{m_j}| < 2\sigma\sqrt{m_j}\right\}.$$

We have used here the bound $\mathbb{P}(|Z| > r) < \mathbb{E}|Z|^2/r^2 = 1/4$ and the equality $\widetilde{U}_{\mu_{j-1}+1,\mu_j} \overset{d}{=} \widetilde{U}_{m_j}$.

Since, by (2.6) and (2.9), the inequality $r < \sqrt{3}\big/\big(8|t|L\big)$ holds, we may apply Lemma 2.4 for estimating Q_i. As a result, we obtain

$$Q_i < c(l)(a_i + b_i), \tag{2.11}$$

where

$$a_i = \left(1 + \Lambda_l(L)\big(|t|\sqrt{m_i(n-\mu_i)}\big)^l\right)^{-1/2},$$

$$b_i = \left(\frac{\delta_l(L)}{\sqrt{m_i}}\right)^{l/2}.$$

Since $\sum_1^l \sigma_j^2(L) \le 2\sigma^2$, we have

$$\delta_l^2(L) \le \frac{2L^2\sigma^2}{\Lambda_l^{2/l}(L)}.$$

Therefore,

$$b_i < \left(\frac{\sqrt{2}L\sigma}{\Lambda_l^{1/l}(L)\sqrt{m_i}}\right)^{l/2} < c(l)\big(l_i|t|\big)^{l/2}\widetilde{b}_l, \tag{2.12}$$

where

$$\widetilde{b}_l = \left(L\sigma/\Lambda_l^{1/(2l)}\right)^l. \tag{2.13}$$

From (2.7) and (2.9) we conclude that

$$\mu_i < \mu_k \le n/2. \tag{2.14}$$

Hence, taking (2.5) and (2.9) into account, we infer that

$$a_i < \left(\frac{cL\sigma l_i}{\Lambda_l^{1/l}(L)\sqrt{n}}\right)^{l/2} < c(l)\big(l_i|t|\big)^{l/2}\widetilde{a}_l, \tag{2.15}$$

where

$$\widetilde{a}_l = \left(\frac{L\sigma}{B\Lambda_l^{1/l}(L)}\right)^{l/2}. \tag{2.16}$$

From (2.11), (2.12), and (2.15) it follows that

$$q_{i-1}(t) < \frac{1}{4}q_i(t) + c(l)\big(\widetilde{a}_l + \widetilde{b}_l\big)\big(l_i|t|\big)^{l/2}.$$

Successively applying this inequality for $i = 1, \dots, k$, we conclude that

$$q_0(t) \le \frac{1}{4^k}q_k(t) + c(l)|t|^{l/2}\big(\widetilde{a}_l + \widetilde{b}_l\big)\sum_1^k l_j^{l/2}/4^j. \tag{2.17}$$

We are left with using the trivial bound $q_k(t) \le 1$ and replacing $\widetilde{a}_l$ and $\widetilde{b}_l$ by the expressions (2.13) and (2.16). Lemma 2.6 is proven.

Lemma 2.7. *Let*

$$Bn^{-1/2} \le |t| \le \frac{c_0}{4^{1/l}2\sigma L}. \tag{2.18}$$

Then

$$\sup_{a\in\mathbb{H}} \left| \mathbb{E}\exp\big\{it|U_n - a|^2\big\} \right| < c(l)D(l)|t|^{l/2}, \tag{2.19}$$

where

$$D(l) = \left(B^{-l} + (\sigma L)^l\right)\Big/\Lambda_l^{1/2}(L). \tag{2.20}$$

Proof. Put

$$l_i^{l/2} = 2^i \left(\frac{2}{4^{2/l} - 1} \right)^{l/4} \left(\frac{c_0}{2\sigma L|t|\sqrt{n}} \right)^{l/2} + 1. \tag{2.21}$$

Notice that the condition (2.6) is fulfilled if $1 \le i \le k_0$, where

$$k_0 = \min \left\{ i : \frac{1}{4} \le \frac{2^i}{n^{l/4}} \left(\frac{2}{4^{2/l} - 1} \right)^{l/4} \le \frac{1}{2} \right\}, \tag{2.22}$$

and, moreover, if the right inequality in (2.18) holds. We have

$$l_i^2 \ge \frac{2 \cdot 4^{2i/l}}{4^{2/l} - 1} \left(\frac{c_0}{2\sigma L t \sqrt{n}} \right)^2.$$

Hence, the condition (2.7) is valid. Next, by (2.21), we find

$$\sum_1^\infty \frac{l_i^{l/2}}{4^i} \le \left(\frac{2}{4^{2/l} - 1} \right)^{l/4} \left(\frac{c_0}{2\sigma L|t|\sqrt{n}} \right)^{l/2} + 1. \tag{2.23}$$

Notice that, by (2.22),

$$4^{-k_0} \le 16 \left(\frac{2}{4^{2/l} - 1} \right)^{l/2} n^{-l/2}. \tag{2.24}$$

Putting $k = k_0$ in Lemma 2.6 and taking (2.23) and (2.24) into account, we arrive at the bound

$$\sup_{a \in \mathbb{H}} \left| \mathbb{E} \exp\{it|U_n - a|^2\} \right| \le c(l) \left(|t|^{l/2} \widetilde{D}(l) \left[\left(|t|\sigma L\sqrt{n} \right)^{-l/2} + 1 \right] + n^{-l/2} \right).$$

In view of (2.18), we have $\left(|t|\sigma L\sqrt{n} \right)^{-l/2} \le (B\sigma L)^{-l/2}$. Using this bound and considering the following two cases: $B\sigma L \le 1$ and $B\sigma L > 1$, we find that

$$\widetilde{D}(l) \left[\left(|t|\sigma L\sqrt{n} \right)^{-l/2} + 1 \right] \le cD(l)$$

(see (2.20)). According to (2.18), we have $n^{-l/4} \le \left(|t|/B \right)^{l/2}$. Hence,

$$\sup_{a \in \mathbb{H}} \left| \mathbb{E} \exp\{it|U_n - a|^2\} \right| \le c(l)|t|^{l/2} \left(D(l) + B^{-l/2} \right).$$

Using the inequality $\Lambda_l^{1/2}(L) \le c(l)(\sigma L)^{l/2}$, it is easy to see that

$$D(l) + B^{-l/2} \le c(l)D(l).$$

This completes the proof of Lemma 2.7.

Lemma 2.8. *If* $|t| < c_0 \big(2^{l/2} L \sigma(L) \sqrt{2n} \, \big)^{-1}$ *then*

$$\sup_{a \in \mathbb{H}} \Big| \mathbb{E} \exp\big\{ it |U_n - a|^2 \big\} \Big| \leq \frac{c(l)}{\sqrt{1 + \Lambda_l(L) \big(|t|n \big)^l}}. \qquad (2.25)$$

Lemma 2.8 can be proven by analogy with Lemma 10 of [16] (see also [34, Lemma 10]).

3. Preliminaries

Lemma 3.1 [24]. *Let* $\lambda_1, \dots \lambda_N$ *be some positive numbers and let* A_k, $1 \leq k \leq N$, *be a set of vectors* $a = (a_1, a_2, \dots, a_N)$ *such that*

$$|a_j| < \lambda_j/(2N), \quad j \neq k, \quad \lambda_k < |a_k| < (1 + \varepsilon)\lambda_k. \qquad (3.1)$$

Then, for every $x = (x_1, x_2, \dots, x_N)$, *we have*

$$\sum_1^N \inf_{a \in A_k} (x, a)^2 > \frac{1}{4N} \sum_1^N \lambda_j^2 x_j^2, \qquad (3.2)$$

$$\max_k \sup_{a \in A_k} (x, a)^2 < N(1 + \varepsilon)^2 \sum_1^N \lambda_j^2 x_j^2. \qquad (3.3)$$

Lemma 3.2 [24]. *Let* F *be a discrete positive measure concentrated on the union of two finite disjoint sets:* $A = \{x_1, x_2, \dots, x_m\}$ *and* $B = \{y_1, \dots, y_n\}$. *Then there exists a matrix of nonnegative numbers* $\{\varepsilon_{ij}\}$, $i = \overline{1, m}$ *and* $j = \overline{1, n}$, *such that* F *is represented as the mixture*

$$F = \sum_{i=1}^m \sum_{j=1}^n \varepsilon_{ij} F_{ij}, \qquad (3.4)$$

where F_{ij} *are the two-point probability measures defined by the equalities*

$$F_{ij}(x_i) = \frac{a}{a+1}, \qquad a = \frac{F(A)}{F(B)}, \quad i = \overline{1, m}, \quad j = \overline{1, n}, \qquad (3.5)$$
$$F_{ij}(y_j) = \frac{1}{a+1},$$

and, in addition,

$$\sum_{i,j} \varepsilon_{ij} F_{ij}(x_i) = F(A),$$

$$\sum_{i,j} \varepsilon_{ij} F_{ij}(y_j) = F(B).$$

Lemma 3.3 [24]. *Let a random element X take two values: x_1 and x_2, and let $\mathbb{P}(X = x_1) = p$ and $\mathbb{P}(X = x_2) = q$. Then*

$$\mathbb{E}\left\{ \left((X + X')^s, y\right)^2 / X^s = (X^s)' = 0 \right\} = \frac{8p^2 q^2}{(p^2 + q^2)^2}(x_1 - x_2, y)^2, \quad (3.6)$$

$$\mathbb{E}\left\{ \left((X + X')^s, y\right)^2 / X^s = (X^s)' = x_2 - x_1 \right\} = 0. \quad (3.7)$$

Lemma 3.4 [24]. *Let $\left\{ (X_k; Y_k) \right\}$, $k = \overline{1, n}$, be a sequence of independent random variables taking values in $\mathbb{H} \times \mathbb{H}$. Then, for every Borel function $\varphi(\cdot)$ on $\mathbb{H}$, we have*

$$\mathbb{E} \prod_1^n \left| \mathbb{E}\left(\varphi(X_k)/Y_k\right) \right| = \prod_1^n \mathbb{E} \left| \mathbb{E}\left(\varphi(X_k)/Y_k\right) \right|.$$

Definition 3.1 (see [13, 26]). We call a random variable X' the *conditionally independent copy of a random variable X with respect to a σ-algebra $\mathfrak{B}$* if X' coincides with X in distribution, and, for arbitrary Borel functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$, the following equality holds almost surely:

$$\mathbb{E}\left\{ \varphi_1(X)\varphi_2(X')/\mathfrak{B} \right\} = \mathbb{E}\left\{ \varphi_1(X)/\mathfrak{B} \right\} \mathbb{E}\left\{ \varphi_2(X)/\mathfrak{B} \right\}.$$

If $\mathfrak{B}$ is generated by a random variable Y then, according to the conventional notations, we write $\mathbb{E}\{\varphi_1(X)\varphi_2(X')/Y\}$ and say that X' is a *conditionally independent copy of X with respect to Y*.

R e m a r k 3.1. Let X' be a conditionally independent copy of X with respect to Y. Let $(Z; W)$ be an independent copy of $(X; Y)$. If the random elements X and Y are discrete then

$$\mathbb{E}\left\{ \varphi(X; X')/Y \right\} = \mathbb{E}\left\{ \varphi(X; Z)/Y, W \right\} \big|_{Y=W}. \quad (3.8)$$

We use this identity in the proof of Lemma 4.4. Prove (3.8). Let, for instance, X and Y take the values $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ correspondingly. To deduce (3.8), it remains to check the validity of the equality

$$\mathbb{E}\left(\varphi(X; X')/Y = y_k\right) = \mathbb{E}\left(\varphi(X; Z)/Y = W = y_k\right), \quad k = 1, 2, \ldots.$$

But this holds, since, as is easy to see, we have

$$\mathbb{P}\left(X = x_i, Z = x_j / Y = y_k, W = y_k\right)$$
$$= \mathbb{P}\left(X = x_i / Y = y_k\right)\mathbb{P}\left(X = x_j / Y = y_k\right).$$

Let the sequence $\{(X_k; Y_k)\}$ be the same as in Lemma 3.4. Here we use the notations

$$V_{k,m} = \sum_k^m X_j, \quad \widetilde{V}_{k,m} = V_{k,m} - \mathbb{E}V_{k,m}, \quad V_m = V_{1,m}.$$

Let $\mathfrak{B}_{k,m}$ be the σ-algebra generated by the random variables Y_j, $j = \overline{k,m}$, and $\mathfrak{B}_m = \mathfrak{B}_{1,m}$. Put

$$\varphi_{k,m}(t; \mathfrak{B}_{k,m}) = \sup_{b \in \mathbb{H}} \left| \mathbb{E}\left\{ \exp\left(it\big|V_{k,m} - b\big|^2\right)/\mathfrak{B}_{k,m} \right\} \right|,$$

$$\varphi_k(t; \mathfrak{B}_k) = \varphi_{1,k}(t; \mathfrak{B}_{1,k}),$$

$$\varphi_{k,m}(t) = \mathbb{E}\varphi_{k,m}(t; \mathfrak{B}_{k,m}),$$

$$\varphi_k(t) = \varphi_{1,k}(t),$$

$$f_k(t; x, y) = \mathbb{E}\left\{ \exp\left(it(X_k, x)\right)/Y_k = y \right\}.$$

We also use the notation

$$\mathbb{E}_Z f(Z; W) = \mathbb{E}\left\{ f(Z; W)/W \right\}.$$

Lemma 3.5 [24]. *For every $r > 0$, $t \in \mathbb{R}$, and $1 \le k \le n$, the following inequality holds:*

$$\varphi_n(t) < \mathbb{P}\left(|\widetilde{V}_{k+1,n}| > r\right)\varphi_k(t)$$

$$+ \mathbb{E}^{1/2}\mathbb{E}\left\{ \prod_1^k \mathbb{E}_{Y_j}\big|f_j(2t; V_{k+1,n}^s, Y_j)\big| \right.$$

$$\left. \times I\left(|\widetilde{V}_{k+1,n}| \vee |\widetilde{V}'_{k+1,n}| < r\right)/\mathfrak{B}_{k+1,n} \right\}, \qquad (3.9)$$

where $\widetilde{V}'_{k+1,n}$ is the conditionally independent copy of $\widetilde{V}_{k+1,n}$ with respect to $\mathfrak{B}_{k+1,n}$, and $V_{k+1,n}^s = V_{k+1,n} - V'_{k+1,n} \equiv \widetilde{V}_{k+1,n} - \widetilde{V}'_{k+1,n}$.

Proof. Put $I_{k+1,n}(r) = I\left(|\widetilde{V}_{k+1,n}| \vee |\widetilde{V}'_{k+1,n}| < r\right)$. By Lemma 2.5,

$$\varphi_n(t; \mathfrak{B}_n) < \mathbb{P}\left(|\widetilde{V}_{k+1,n}| > r/\mathfrak{B}_{k+1,n}\right)\varphi_k(t; \mathfrak{B}_k)$$

$$+ \mathbb{E}^{1/2}\left\{ \prod_{j=1}^k \big|f_j(2t; V_{k+1,n}^s, Y_j)\big| I_{k+1,n}(r)/\mathfrak{B}_{k+1,n} \right\}$$

$$\equiv D_1 + D_2. \qquad (3.10)$$

By Lemma 3.4,

$$\mathbb{E}^2 D_2 \leq \mathbb{E} D_2^2 = \mathbb{E}\mathbb{E}\left\{\left(\prod_{j=1}^{k} \mathbb{E}_{Y_j} \big| f_j\big(2t; V_{k+1,n}^s, Y_j\big)\big|\right) I_{k+1,n}(r)/\mathfrak{B}_{k+1,n}\right\}. \quad (3.11)$$

We also have

$$\mathbb{E} D_1 = \mathbb{P}\left(\big|\widetilde{V}_{k+1,n}\big| > r\right)\varphi_k(t). \quad (3.12)$$

The inequality (3.9) follows from (3.10)–(3.12).

Lemma 3.6 [24]. *Let* $B(x; L)$ *be the quadratic form defined by the equality* (1.25). *If* $|t| < \sqrt{3}\big/\big(2L|x|\big)$ *then*

$$\big|f_x^Z(t)\big|^2 \equiv \big|\mathbb{E}\exp\{it(Z, x)\}\big|^2 < 1 - t^2 B(x; L)/4 \quad (3.13)$$

for each $L > 0$.

Proof. Observe first that

$$\big|f_x^Z(t)\big|^2 = \mathbb{E}\left\{\exp\big(it(Z^s, x)\big); |Z| \vee |Z'| \leq L\right\}$$
$$+ \mathbb{E}\left\{\exp\big(it(Z^s, x)\big); |Z| \vee |Z'| > L\right\}$$
$$\equiv E_1 + E_2. \quad (3.14)$$

Put $\eta = I\big(|Z| \vee |Z'| \leq L\big)$. Using Lemma 2.2, we obtain

$$E_1 = \mathbb{E}\Big[\big(\cos\big(t(Z^s, x)\big) - 1\big)\eta\Big] + \mathbb{E}\eta$$

$$\leq \mathbb{P}^2\big(|Z| \leq L\big) - \frac{t^2}{2}B(x; L) + \frac{|t|^3}{4\sqrt{3}}\mathbb{E}\Big[\big|(Z^s, x)\big|^3\eta\Big]$$

$$\leq \mathbb{P}^2\big(|Z| \leq L\big) - B(x; L)\left(\frac{t^2}{2} - \frac{L|x||t|^3}{2\sqrt{3}}\right).$$

Hence, for $|t| < \sqrt{3}\big/\big(2L|x|\big)$, the inequality

$$E_1 < \mathbb{P}^2\big(|Z| \leq L\big) - B(x; L)t^2/4 \quad (3.15)$$

holds. On the other hand,

$$E_2 \leq 1 - \mathbb{P}^2\big(|Z| \leq L\big). \quad (3.16)$$

Combining (3.14)–(3.16), we arrive at the estimate (3.13).

Denote

$$f(t; x, y) = \mathbb{E}\left\{\exp\big(it(X, x)\big)/Y = y\right\},$$
$$B(x, y; L) = \mathbb{E}\left\{(X^s, x)^2 I\big(|X| \vee |X'| < L\big)/Y = y\right\},$$

where X' is the conditionally independent copy of X with respect to Y.

Lemma 3.7 [24]. *For every* $|t| \leq \sqrt{3}\big/\big(2L|x|\big)$, *we have*

$$\mathbb{E}\big| f(t; x, Y)\big| \leq \exp\big\{-t^2 \mathbb{E} B(x, Y; L)/8\big\}. \tag{3.17}$$

Denote by $\bar{\sigma}_1^2(k_1, k_2; L) \geq \bar{\sigma}_2^2(k_1, k_2; L) \geq \cdots$ the eigenvalues of the form

$$\overline{B}(k_1, k_2; x; L) = \sum_{j=k_1}^{k_2} \mathbb{E} B_j(x, Y_j; L),$$

where $B_j(x, y; L)$ coincides with the form $B(x, y; L)$ calculated under $X = X_j$. Put

$$\overline{\Lambda}_l(k_1, k_2; L) = \prod_1^l \bar{\sigma}_j^2(k_1, k_2; L), \quad \overline{\Lambda}_l(k; L) = \overline{\Lambda}_l(1, k; L).$$

Lemma 3.8 [24]. *Let* $W_1, W_2, \ldots, W_m$ *be* $\mathbb{R}^l$*-valued random variables conditionally independent with respect to a* σ*-algebra* $\mathfrak{F}$, *let* $U = \sum_1^m W_j$, *and let* B_l *be* $l \times l$ *nonnegative symmetric matrix with eigenvalues* $b_1^2 \geq \cdots \geq b_l^2$. *Then, for every* $L > 0$ *and* $r > 0$,

$$\sup_{a \in \mathbb{R}^l} \mathbb{E}\mathbb{P}\Big(\big(B_l(U^s - a), U^s - a\big)^{1/2} < r/\mathfrak{F}\Big) \leq \frac{c(l)(r + \varepsilon_l)^l}{\overline{\Lambda}_l^{1/2}(m; L) \prod_1^l b_j}, \tag{3.18}$$

where $\varepsilon_l^2 = 32 l L^2 \sum_1^l b_k^2$.

Lemma 3.8 generalizes Lemma 2.3 (see also [16, Theorem 1]).

The following assertion generalizes Lemma 5 of [16]. Recall that $c_0 = \sqrt{3}/8$.

Lemma 3.9 [24]. *For every* $t \neq 0$ *and* $L > 0$, *we have*

$$\mathbb{E}\mathbb{E}\left\{\left(\prod_{j=1}^k \mathbb{E}_{Y_j}\big| f_j(2t; V_{k+1,n}^s, Y_j)\big|\right) I\left(\big|\widetilde{V}_{k+1,n}\big| \vee \big|\widetilde{V}_{k+1,n}'\big| < \frac{c_0}{|t|L}\right)\Big/ \mathfrak{B}_{k+1,n}\right\}$$

$$\leq c(l)\left(|t|^{-l}\big[\overline{\Lambda}_l(k; L)\overline{\Lambda}_l(k+1, n; L)\big]^{-1/2} + \bar{\delta}_l^l(k, n; L)\right), \tag{3.19}$$

where

$$\bar{\delta}_l^2(k, n; L) = \big(\overline{\Lambda}_l(k; L)\overline{\Lambda}_l(k+1, n, L)\big)^{-1/l} L^2 \sum_{j=1}^l \bar{\sigma}_j^2(k; L).$$

Lemma 3.10. *Let X and Y be random elements with values in $\mathbb{H}$ and let X' be a conditionally independent copy of X with respect to Y. Then, for every $L > 0$ and $x \in \mathbb{H}$,*

$$\mathbb{E}\mathbb{E}\left\{(X^s, x)^2 I\big(|X| \vee |X'| \le L\big)/Y\right\}$$
$$= 2\mathbb{E}\Big[\mathbb{E}\big\{(X, x)^2 I\big(|X| \le L\big)/Y\big\}\mathbb{E}\big\{I\big(|X| \le L\big)/Y\big\}$$
$$- \mathbb{E}^2\big\{(X, x)I\big(|X| \le L\big)/Y\big\}\Big], \tag{3.20}$$
$$\mathbb{E}\mathbb{E}\left\{(X^s, x)^2 I\big(|X| \vee |X'| \le L\big)/Y\right\} \le 2\mathbb{E}\big[(X, x)^2 I\big(|X| \le L\big)\big]. \tag{3.21}$$

Proof. The formula (3.20) can be proved straightforward; and the inequality (3.21) follows from (3.10).

Put $\overline{B}_j(x; L) = \mathbb{E}B_j(x, Y_j; L)$. Let $\{\bar{e}_p\}_{p=1}^{\infty}$ be an orthonormal basis in $\mathbb{H}$, let $\bar{\sigma}$, $\gamma_1, \gamma_2, \dots, \gamma_l$ be some positive numbers, and let $\gamma_0 = \min_{1 \le k \le l} \gamma_k$.

Definition 3.2. Let A_k, $1 \le k \le l$, be the sets of Lemma 3.1 with $N = l$. We say that the quadratic form $\overline{B}_j(x; L)$ satisfies the condition $\mathcal{K}(A_k, \gamma_k, l)$ if there exists an element $b_j \in \mathbb{H}$ such that $\left\{(b_j, \bar{e}_p)\right\}_{p=1}^{l} \in A_k$ and the inequality

$$\overline{B}_j(x; L) \ge \gamma_k (b_j, x)^2$$

holds for every $x \in \mathbb{H}$ with $(x, \bar{e}_p) = 0$ for $p > l$.

In the following lemma, it is actually proved that

$$\varphi_n(t) \le c(l)\left(\frac{|t|}{\gamma_0}\right)^{l/2}\frac{\bar{\sigma}^{2l}}{\prod_1^l \lambda_j}$$

for $|t| \ge \frac{1}{\bar{\sigma}^2 n^{1/2}}$, if, for every set A_k, $1 \le k \le l$, there exist sufficiently many quadratic forms $\overline{B}_j(x; L)$, $1 \le j \le n$, satisfying the condition $\mathcal{K}(A_k, \gamma_k, l)$. (Here λ_j are the quantities defining the sets A_k.)

Lemma 3.11 [24]. *Let $X_1, \dots, X_n$ be independent random elements with values in $\mathbb{H}$, let the sets A_k be the same as in Lemma 3.1, and let $N = l \ge 4$. Denote by $\Omega_n(k)$ the set of indices j, $1 \le j \le n$, such that each of the quadratic forms $\overline{B}_j(x; L)$ satisfies the condition $\mathcal{K}(A_k, \gamma_k, l)$. Denote $n_k = \operatorname{card}\Omega_n(k)$. If*

$$\min_{1 \le k \le l} n_k \ge \eta n, \quad 0 < \eta \le 1/l, \tag{3.22}$$

$$\max_{1 \le j \le n} \mathbb{E}|X_j|^2 \le \bar{\sigma}^2, \tag{3.23}$$

then, for

$$Bn^{-1/2} \le |t| \le \frac{c_0 \sqrt{\eta/5}}{4^{1/l}\bar{\sigma}L}, \qquad (3.24)$$

the estimate

$$\varphi_n(t) \le c(l)|t|^{l/2}(\gamma_0\eta^2)^{-l/2}\big(\overline{D}(l) + B^{-l/2}\big) \qquad (3.25)$$

holds, where

$$\overline{D}(l) = \big(B^{-l} + (\bar{\sigma}L)^l\big)\Big/ \prod_{j=1}^{l} \lambda_j.$$

Proof. Let m_i and l_i be defined by the equalities (2.9) and (2.21) respectively (with $\bar{\sigma}$ instead of σ). Put $\mu_j = \sum_{i=1}^{j} m_i$. The number k_0 is defined below. Denote

$$\bar{q}_i(t) = \varphi_{\mu_i+1,n}(t), \quad i = 1, \dots, k.$$

We now use Lemma 3.5, replacing the sets of indices in the intervals $[1, n]$, $[k+1, n]$, and $[1, k]$ by the indices in $[\mu_{i-1}+1, n]$, $[\mu_{i-1}+1, \mu_i]$, and $[\mu_i+1, n]$ respectively. Then

$$\bar{q}_{i-1}(t) < \mathbb{P}\Big(\big|\widetilde{V}_{\mu_{i-1}+1,\mu_i}\big| > r\Big)\bar{q}_i(t) + \overline{Q}_i, \qquad (3.26)$$

where

$$\overline{Q}_i = \mathbb{E}^{1/2}\mathbb{E}\left\{\left(\prod_{j=\mu_i+1}^{n} \mathbb{E}_{Y_j}\big|f_j\big(2t; V^s_{\mu_{i-1}+1,\mu_i}, Y_j\big)\big|\right) \right.$$
$$\left. \times I\Big(\big|\widetilde{V}_{\mu_{i-1}+1,\mu_i}\big| \vee \big|\widetilde{V}'_{\mu_{i-1}+1,\mu_i}\big| \le r\Big)\Big/ \mathfrak{B}_{\mu_{i-1}+1,\mu_i}\right\}.$$

Putting $r = 2\bar{\sigma}\sqrt{m_i}$, we obtain

$$\mathbb{P}\Big(\big|\widetilde{V}_{\mu_{i-1}+1,\mu_i}\big| > r\Big) \le 1/4. \qquad (3.27)$$

On the other hand, since $l_i \ge 1$, from (2.9) we deduce that $r \le c_0/(|t|L)$. Using Lemma 3.9, we obtain

$$\overline{Q}_i \le c(l)\left(\frac{1}{|t|^{l/2}\big[\overline{\Lambda}_l(\mu_i + 1, n; L)\overline{\Lambda}_l(\mu_{i-1} + 1, \mu_i; L)\big]^{1/4}} + \bar{\delta}_{l,i}^{l/2}\right), \qquad (3.28)$$

where

$$\bar{\delta}_{l,i}^2 = \frac{L^2 \sum_{j=1}^{l} \bar{\sigma}_j^2(\mu_i + 1, n; L)}{\big(\overline{\Lambda}_l(\mu_i + 1, n; L)\overline{\Lambda}_l(\mu_{i-1} + 1, \mu_i; L)\big)^{1/l}}.$$

Estimate the right-hand side of (3.28). Denote

$$\Omega_n(k;i) = \Omega_n(k) \cap \big\{ j : \mu_{i-1} + 1 \leq j \leq \mu_i \big\}, \quad m_{ik} = \operatorname{card} \Omega_n(k;i).$$

It is easy that

$$\sum_{j=\mu_{i-1}+1}^{\mu_i} \overline{B}_j(x;L) \geq \sum_{k=1}^{l} \gamma_k \sum_{j \in \Omega_n(k;i)} (b_j, x)^2$$

$$\geq \sum_{k=1}^{l} \gamma_k m_{ik} \inf_{a \in A_k} (a, x)^2$$

$$\geq \gamma_0 M_i \sum_{k=1}^{l} \inf_{a \in A_k} (a, x)^2, \tag{3.29}$$

where $M_i = \min_{1 \leq k \leq l} m_{ik}$. Find a lower bound for M_i. Observe that, by (2.14) and (3.22), the inequalities $m_i < n/2 \leq n_k/(2\eta)$ are fulfilled, i.e., $n_k \geq 2\eta m_i$ for every $1 \leq i \leq k_0$. On the other hand, $2\eta m_i \leq m_i$. Note that $\varphi_n(t)$ does not depend on the order of summands X_i. Hence, we can renumber the summands so that, for every $1 \leq k \leq l$, the set $\Omega_n(k;i)$ includes not less than $[2\eta m_i]$ elements from $\big\{ j : \mu_{i-1} + 1 \leq j \leq \mu_i \big\}$. Thus, $m_{ik} \geq [2\eta m_i]$, $k = \overline{1, l}$. In addition, let l_i satisfy the condition

$$l_i \leq c_0 \sqrt{\eta/5} \, \big(\bar{\sigma} L |t| \big)^{-1}. \tag{3.30}$$

In view of (2.9) and (3.30), we have

$$2\eta m_i \geq 2\eta \big[5/(4\eta) \big] > 2\eta \big(5/(4\eta) - 1 \big) \geq 5/2 - 2/l \geq 2$$

for $l \geq 4$. Consequently,

$$M_i \geq \eta m_i, \quad k = \overline{1, l}. \tag{3.31}$$

Using (3.29), (3.31), and Lemma 3.1, we obtain the inequality

$$\sum_{j=\mu_{i-1}+1}^{\mu_i} \overline{B}_j(x;L) \geq \frac{\gamma_0 \eta^2 m_i}{4} \sum_{j=1}^{l} \lambda_j^2 x_j^2,$$

where $x_j = (x, \bar{e}_j)$, $j \leq l$, $(x, \bar{e}_i) = 0$, and $i > l$. Therefore,

$$\bar{\sigma}_j^2 (\mu_{i-1} + 1, \mu_i; L) \geq \left(\frac{\gamma_0 \eta^2 m_i}{4} \right) \lambda_j^2.$$

Thus,

$$\overline{\Lambda}_l(\mu_{i-1}+1,\mu_i;L) \geq \left(\frac{\gamma_0\eta^2 m_i}{4}\right)^l \prod_1^l \lambda_j^2. \qquad (3.32)$$

Analogously, we have

$$\overline{\Lambda}_l(\mu_i+1,n;L) \geq \left(\frac{\gamma_0\eta^2(n-\mu_i)}{4}\right)^l \prod_1^l \lambda_j^2. \qquad (3.33)$$

We now obtain an upper bound for $\sum_{j=1}^l \bar{\sigma}_j^2(\mu_i+1,n;L)$. By (3.21), we have

$$\sum_{p=\mu_i+1}^n \overline{B}_p(x;L) \leq 2 \sum_{p=\mu_i+1}^n \mathbb{E}\left[(X_p,x)^2 I(|X_p| \leq L)\right].$$

From here it follows that

$$\sum_{j=1}^l \bar{\sigma}_j^2(\mu_i+1,n;L) \leq 2 \sum_{p=\mu_i+1}^n \mathbb{E}\left[|X_p|^2 I(|X_p| \leq L)\right]$$

$$\leq (n-\mu_i)c\min(\bar{\sigma}^2, L^2). \qquad (3.34)$$

Combining (3.28) and (3.32)–(3.34), we obtain

$$\overline{Q}_i < c(l)(a_i+b_i), \qquad (3.35)$$

where

$$a_i = \frac{1}{|t|^{l/2} m_i^{l/4}(n-\mu_i)^{l/4}(\gamma_0\eta^2)^{l/2}\prod_1^l \lambda_j},$$

$$b_i = \frac{(L\bar{\sigma})^{l/2}}{m_i^{l/4}(\gamma_0\eta^2)^{l/2}\prod_1^l \lambda_j}.$$

Since $l_i \leq c_0/(2\sqrt{2}\,\bar{\sigma}L|t|)$ (see (3.30)), by the definition (2.9), we have $m_i \geq \frac{1}{2}\left(c_0/(2l_i\bar{\sigma}Lt)\right)^2$. From here and (2.14) it follows that

$$a_i \leq c(l)\left(\frac{l_i\bar{\sigma}L}{\gamma_0\eta^2\sqrt{n}}\right)^{l/2}\left(\prod_1^l \lambda_j\right)^{-1}.$$

Using the first inequality of (3.24), we find

$$a_i \leq c(l)\left(\frac{l_i\bar{\sigma}L|t|}{B\gamma_0\eta^2}\right)^{l/2}\left(\prod_1^l \lambda_j\right)^{-1} \equiv c(l)\left(\frac{l_i|t|}{\gamma_0\eta^2}\right)^{l/2}\overline{a}_l, \qquad (3.36)$$

where $\overline{a}_l = \left(L\overline{\sigma}/B\right)^{l/2}\left(\prod_1^l \lambda_j\right)^{-1}$. By (2.9), we conclude that

$$b_i \le c(l)\left(\frac{l_i|t|}{\gamma_0\eta^2}\right)^{l/2}\overline{b}_l, \qquad (3.37)$$

where $\overline{b}_l = (L\overline{\sigma})^l\left(\prod_1^l \lambda_j\right)^{-1}$. The relations (3.26), (3.27), and (3.35)–(3.37) imply that

$$\overline{q}_{i-1}(t) \le \frac{1}{4}\overline{q}_i(t) + c(l)(\gamma_0\eta^2)^{-l/2}\left(\overline{a}_l + \overline{b}_l\right)\left(l_i|t|\right)^{l/2}.$$

By analogy with (2.17), we now obtain

$$\overline{q}_0(t) \le 4^{-k_0} + c(l)(\gamma_0\eta^2)^{-l/2}|t|^{l/2}\left(\overline{a}_l + \overline{b}_l\right)\sum_1^k l_j^{l/2}/4^j. \qquad (3.38)$$

Define k_0 by the equality

$$k_0 = \min\left\{i : \frac{1}{4}\left(\frac{\eta}{5}\right)^{l/4} \le \frac{2^i}{n^{l/4}}\left(\frac{2}{4^{2/l}-1}\right)^{l/4}\frac{1}{2^{l/2}} \le \frac{1}{2}\left(\frac{\eta}{5}\right)^{l/4}\right\}. \qquad (3.39)$$

Taking the second inequality in (3.24) into account, we can verify that (3.30) holds. From (3.39) it follows that

$$4^{-k_0} \le n^{-l/2}16(5/\eta)^{l/2}\left(2(4^{2/l}-1)\right)^{-l/2}. \qquad (3.40)$$

By (3.38), (3.40), and (2.23), we have

$$\overline{q}_0(t) \le c(l)\left[\frac{1}{(n\eta)^{l/2}} + \left(\frac{|t|}{\gamma_0\eta^2}\right)^{l/2}\left(\overline{a}_l + \overline{b}_l\right)\left(\left(\frac{1}{|t|\overline{\sigma}L\sqrt{n}}\right)^{l/2} + 1\right)\right]. \qquad (3.41)$$

Using the inequality $\left(|t|\overline{\sigma}L\sqrt{n}\right)^{-l/2} \le \left(B\overline{\sigma}L\right)^{-l/2}$ (see (3.24)) and considering two cases: $B\overline{\sigma}L \le 1$ and $B\overline{\sigma}L > 1$, we find that

$$\left(\overline{a}_l + \overline{b}_l\right)\left[\left(|t|\overline{\sigma}L\sqrt{n}\right)^{-l/2} + 1\right] \le c\overline{D}(l). \qquad (3.42)$$

The bound (3.25) ensues from (3.41), (3.42), and the left inequality in (3.24). Lemma 3.11 is proven.

Lemma 3.12. *Let the conditions (3.22) and (3.23) be fulfilled and let*

$$|t| \le c_0/\left(2^{l/2}\overline{\sigma}L\sqrt{n}\right).$$

Then

$$\varphi_n(t) \le c(l)(\gamma_0\eta^2)^{-l/2}\left(n|t|\right)^{-l/2}\left(\prod_1^l \lambda_j\right)^{-1}.$$

Lemma 3.12 can be proven by the same scheme as Lemma 3.11 with the exception that we use Lemma 2.8 rather than Lemma 2.7.

4. Estimation of a characteristic function in the neighborhood of the unity

Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be arbitrary independent random variables taking values in $\mathbb{H}$ and let $V_n = \sum_1^n Z_j$. Put $Z_j^0 = Z_j + Z_j'$ and $V_n^0 = \sum_1^n Z_j^0$. Let $\mathfrak{A}_n$ be the σ-algebra generated by the random variables Z_j^s, $j = \overline{1,n}$. Define

$$\Psi_n(t; \tau; a) = \left| \mathbb{E} \exp\{it|V_n - a|^2\} \right| \left| \mathbb{E} \exp\{i(t + \tau)|V_n - a|^2\} \right|,$$

$$\Psi(\tau; b; \mathfrak{A}_n) = \left| \mathbb{E}\left\{ \exp\{i\tau|V_n^0 - b|^2\}/\mathfrak{A}_n \right\} \right|.$$

If $G_1, \ldots, G_n$ are the distributions of $Z_1, \ldots, Z_n$ then we say that $\Psi_n(t; \tau; a)$ corresponds to the convolution $\prod_{*1}^n G_j$.

Lemma 4.1 [4]. *For every* $t \in \mathbb{R}$, $a \in \mathbb{H}$, *and* $\tau > 0$, *the following inequality is valid:*

$$\Psi_n(t; \tau; a) \leq \mathbb{E} \sup_{b \in \mathbb{H}} \Psi(\tau/4; b; \mathfrak{A}_n). \tag{4.1}$$

Lemma 4.2. *Let* μ_n *be the number of successes in the Bernoulli trials,* p *be the success probability,* $q = 1 - p$, *and*

$$H(p; \varepsilon) = \varepsilon \log(\varepsilon/p) + (1 - \varepsilon) \log\big((1 - \varepsilon)/(1 - p)\big).$$

Then

$$\mathbb{P}\big(\mu_n \geq n(1 - \varepsilon)\big) \leq \exp\{-nH(q; \varepsilon)\} \quad \text{if } 0 < \varepsilon < q, \tag{4.2}$$

$$\mathbb{P}(\mu_n \leq n\varepsilon) \leq \exp\{-nH(p; \varepsilon)\} \quad \text{if } 0 < \varepsilon < p. \tag{4.3}$$

R e m a r k 4.1. The statement of Lemma 4.2 coincides with Theorem 10 of [7, p. 131] up to notations. The function $H(p; \varepsilon)$ called *information* as well as *entropy* (see, for instance, [12]) was already used in the 1950s in connection with studying the probabilities of large deviations (see, for instance, [9, p. 497; 29, p. 13]).

Define the function

$$M_l(t; A) = \begin{cases} \big(|t|A^2\big)^{-l/2} & \text{if } |t| < A^{-1}, \\ |t|^{l/2} & \text{if } |t| \geq A^{-1}. \end{cases}$$

Lemma 4.3 [3]**.** *Let $\varphi(t)$ be a continuous nonnegative function on $[0, \infty)$, $\varphi(0) = 1$, and let, for every $\tau > 0$, the following inequality hold:*

$$\sup_{t \geq 0}\big(\varphi(t)\varphi(t + \tau)\big) \leq \chi_l M_l(\tau; A),$$

where $\chi_l \geq 1$ does not depend on τ. Then, for $l \geq 9$,

$$\int_{A^{-4/l}}^{1} \frac{\varphi(t)}{t} dt \leq \frac{c\chi_l}{A^2}.$$

Using the scheme of the proof in [3], we can improve Lemma 4.3 as follows:

Lemma 4.3a. *Let the conditions of Lemma 4.3 be fulfilled. Then, for every integer $l \geq 9$ and a real γ such that*

$$1 < \gamma \leq A^{2(l/8-1)}(2\log A)^{-1}, \tag{4.4}$$

the following estimate holds:

$$\int_{A^{-4/l}}^{1} \frac{\varphi(t)}{t} dt \leq \frac{\chi_l}{A^2}\psi(l; \gamma), \tag{4.5}$$

where

$$\psi(l; \gamma) = \frac{2}{l} + \left(1 + \frac{4}{l}\log\gamma\right)\frac{\gamma}{\gamma^{1-8/l} - 1} + \frac{4\gamma\log\gamma}{l(\gamma^{1-8/l} - 1)^2}.$$

R e m a r k 4.2. In contrast to Lemma 4.3, Lemma 4.3a makes it possible to estimate the constant c, coefficient of χ_l/A^2. Indeed, the function $\psi(l; \gamma)$ decreases in l. So $\psi(l; \gamma) \leq \psi(9; \gamma) \equiv \psi(\gamma)$ for $l \geq 9$. We can minimize $\psi(\gamma)$ provided that the inequality (4.4) is fulfilled. The function $\psi(\gamma)$ is positive and continuous for $\gamma > 1$. Moreover, $\lim_{\gamma \to 1+0} \psi(\gamma) = \lim_{\gamma \to +\infty} \psi(\gamma) = +\infty$. Therefore, $\inf_{\gamma > 1} \psi(\gamma) \geq 0$. Computer calculations enable us to claim that

$$121 < \min_{\gamma > 1}\psi(\gamma) = \min_{2.74 \leq \gamma \leq 2.76}\psi(\gamma) < 122.$$

It is easy to see that if $A \geq 109 \cdot 10^6$ then A satisfies the inequality $2.76 \leq A^{1/4}/(2\log A)$, which, in turn, imply the fulfillment of the condition (4.4) for all $1 < \gamma \leq 2.76$ and hence for a minimum point. Thus, in particular, we may assume $c = 122$ in Lemma 4.3 if $A \geq 109 \cdot 10^6$.

Given fixed $l > 9$, we can diminish the constant c in Lemma 4.3, alongside weakening the condition on A.

The analogous arguments lead to the following results in the cases $l = 12$ and $l = 13$:

$$l = 12 \Rightarrow \quad 15 < \min_{\gamma}{}^{*}\psi(12; \gamma) = \min_{3 \leq \gamma \leq 3.01}\psi(12; \gamma) < 15.1 \ \text{ for } \ A \geq 17.1,$$

$$l = 13 \Rightarrow 11.4 < \min_{\gamma}{}^{*}\psi(13; \gamma) = \min_{3.1 \leq \gamma \leq 3.11}\psi(13; \gamma) < 11.5 \ \text{ for } \ A \geq 8,$$

where $\min_{\gamma}{}^{*}$ is the minimum in all γ satisfying (4.4).

Let $X^\circ, X_1^\circ, X_2^\circ, \dots$ be independent identically distributed random vectors taking finitely many values in $\mathbb{H}$. For every $1 \le k \le l$, put

$$A_k^+ = \left\{ x \in \mathbb{H} : \sigma_k \le (x, e_k) \le 2\sigma_k, \ 0 \le (x, e_j) \le \frac{\sigma_j}{2l}, \ j \ne k, \ j \le l \right\},$$

$$A_k^- = \left\{ x \in \mathbb{H} : -2\sigma_k \le (x, e_k) \le -\sigma_k, \ -\frac{\sigma_j}{2l} \le (x, e_j) \le 0, \ j \ne k, \ j \le l \right\},$$

$$\widetilde{A}_k = A_k^+ \cup A_k^-,$$

$$D(L) = \left\{ x \in \mathbb{H} : \sum_{i=l+1}^{\infty} (x, e_i)^2 \le L^2 \right\},$$

$$A_k^+(L) = A_k^+ \cap D(L),$$

$$A_k^-(L) = A_k^- \cap D(L),$$

$$\widetilde{A}_k(L) = A_k^+(L) \cup A_k^-(L)$$

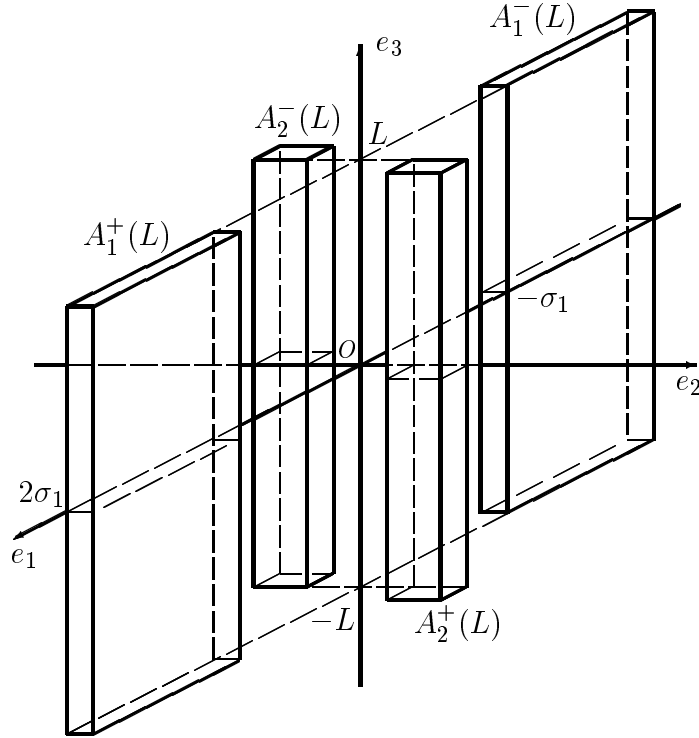(see Fig. 4.1). Notice that $\Phi\big(A_k^\pm(L)\big) > 0$.

Fig. 4.1. The sets $A_k^\pm(L)$ in the case $\mathbb{H} = \mathbb{R}^3$ and $l = 2$

Let F° be a distribution of X°, let $F_{k,L}$ be the restriction of F° to $\widetilde{A}_k(L)$, $k = \overline{1,l}$, and let $F_{0,L}$ be the restriction of F° to $\widetilde{A}_0(L) \equiv \mathbb{H} - \bigcup_{k=1}^{l} \widetilde{A}_k(L)$.

Thus,

$$F^\circ = \sum_{k=0}^{l} F_{k,L} = \sum_{k=0}^{l} \varepsilon_k P_{k,L}, \qquad (4.6)$$

where $\varepsilon_k = F_{k,L}(\mathbb{H}) = F^\circ\big(\widetilde{A}_k(L)\big)$ and $P_{k,L} = F_{k,L}/\varepsilon_k$, $0 \le k \le l$. According to Lemma 3.2, for every k, $1 \le k \le l$, we have

$$F_{k,L} = \sum_{i,j} \varepsilon_{ij}^{(k)} F_{ij,L}^{(k)}, \qquad (4.7)$$

where $\sum_{i,j} = \sum_{i=1}^{m_k^+} \sum_{j=1}^{m_k^-}$, m_k^+ and m_k^- are the numbers of values taken by the random vector X° in the sets $A_k^+(L)$ and $A_k^-(L)$, and $F_{ij,L}^{(k)}$ are the distributions, concentrated at the two points: $x_i^+ \in A_k^+(L)$ and $x_j^- \in A_k^-(L)$,

$$F_{ij,L}^{(k)}(x_i^+) = \frac{F^\circ\big(A_k^+(L)\big)}{F^\circ\big(\widetilde{A}_k(L)\big)} \equiv p_k,$$

$$F_{ij,L}^{(k)}(x_i^-) = \frac{F^\circ\big(A_k^-(L)\big)}{F^\circ\big(\widetilde{A}_k(L)\big)} \equiv q_k.$$

For every fixed k, $1 \le k \le l$, we denote by $\mathcal{F}_{k,L}$ the set of two-point distributions $F_{ij,L}^{(k)}$.

From (4.6) and (4.7) it follows that the convolution $(F^\circ)^{n*}$ is a linear combination of convolutions of the form $G_1 * G_2 * \cdots * G_n$, where each component G_p coincides with one of the two-point distributions $F_{ij}^{(k)}$ or $P_{0,L}$ with some probability, namely,

$$\Pr\left(G_j = F_{pq}^{(k)}\right) = \varepsilon_{pq}^{(k)},$$

$$\Pr\left(G_j \in \mathcal{F}_{k,L}\right) = \varepsilon_k,$$

$$\Pr\left(G_j = P_{0,L}\right) = \varepsilon_0.$$

Here the symbol Pr denotes the probability distribution on the corresponding probability space. Therefore, $(F^\circ)^{n*}$ is an expectation (denote it by $\mathbb{E}_1$) of the convolution of random distributions, i.e.

$$(F^\circ)^{n*} = \mathbb{E}_1\big(G_1 * G_2 * \cdots * G_n\big). \qquad (4.8)$$

Put

$$\overline{p} = \min_{1 \le k \le l} \varepsilon_k,$$

$$N_k = \mathrm{card}\big\{j : 1 \le j \le n, \ G_j \in \mathcal{F}_{k,L}\big\},$$

$$N_0 = \mathrm{card}\big\{j : 1 \le j \le n, \ G_j = P_{0,L}\big\}.$$

Notice that the probabilities ε_k depend on L. Using (4.3) and taking the increase of $H(p; \varepsilon)$ in p into account, we obtain the following lower bound:

$$\Pr\left(\min_{1 \leq k \leq l} N_k > n\varepsilon\right) \geq 1 - \sum_1^l \exp\{-nH(\varepsilon_k; \varepsilon)\}$$

$$\geq 1 - l \exp\{-nH(\overline{p}; \varepsilon)\} \qquad (4.9)$$

for every $\varepsilon \in (0, \overline{p})$. Denote

$$\mathcal{M}_{n,L} = \left\{\prod_{1}^{n} {}_* G_j : \min_{1 \leq k \leq l} N_k > \overline{p}n/2\right\}.$$

According to (4.9), we have

$$\Pr\left(\mathcal{M}_{n,L}\right) \geq 1 - l \exp\{-nH(\overline{p}; \overline{p}/2)\}. \qquad (4.10)$$

Let Z_j be independent random variables with the distributions G_j, where $1 \leq j \leq n$.

Lemma 4.4. *Let* $\Psi_n(t; \tau; a)$ *correspond to the convolution* $\prod_1^n {}_* G_j \in \mathcal{M}_{n,L/4}$ *and let*

$$L \geq \left(8\sigma/\sqrt{3}\right) \vee \sqrt{2\mathbb{E}|X^\circ|^2/\varepsilon_0}, \quad \text{where} \quad \varepsilon_0 = F^\circ\left(\mathbb{H} - \bigcup_{k=1}^{l} \widetilde{A}_k(L/4)\right).$$

Then

$$\left|\Psi_n(t; \tau; a)\right| \leq c(l)w_1(l)w_2(l)M_l(\tau; A), \qquad (4.11)$$

where $A = L^2\sqrt{n}$, $w_1(l) = (\gamma_0 \overline{p}^2)^{-l/2}$, $w_2(l) = L^{2l}/\Lambda_l^{1/2}$, $\overline{p} = \min_{1 \leq k \leq l} \varepsilon_k$, $\varepsilon_k = F^\circ\left(\widetilde{A}_k(L/4)\right)$.

Proof. Bearing Lemma 4.1 in mind, we need to estimate the expectation

$$\mathbb{E} \sup_{b \in \mathbb{H}} \Psi(\tau; b; \mathfrak{U}_n) = \mathbb{E} \sup_{b \in \mathbb{H}} \left|\mathbb{E}\left\{\exp\left\{i\tau\left|\sum_{j=1}^{n} Z_j^0 - b\right|^2\right\}\middle/ \mathfrak{U}_n\right\}\right|,$$

where $\mathfrak{U}_n$ is the σ-algebra generated by Z_j^s, $1 \leq j \leq n$. This expectation coincides with the function $\varphi_n(\tau)$ in Section 3 if we replace the variables X_j and Y_j in the notation of the function by Z_j^0 and Z_j^s respectively. In turn, to estimate $\varphi_n(\tau)$ by means of Lemmas 3.11 and 3.12, we have to check the validity of the conditions (3.22) and (3.23). Observe that, in the case under consideration, $\overline{B}_j(x; L) = \mathbb{E} B_j(x; Z_j^s; L)$, where

$$B_j(x; Z_j^s; L) = \mathbb{E}\left\{(Z_j^0 - Z_j^{0\prime}, x)^2 I(|Z_j^0| \vee |Z_j^{0\prime}| \leq L)/Z_j^s\right\},$$

$Z_j^{0\,\prime}$ is a conditionally independent copy of Z_j^0 with respect to $\mathfrak{U}_n$. From Remark 3.1 it follows that

$$B_j(x; Z_j^s; L) = \mathbb{E}\left\{ (Z_j^0 - W_j^0, x)^2 I\big(|Z_j^0| \vee |W_j^0| \leq L\big) / Z_j^s, W_j^s \right\}\Big|_{Z_j^s = W_j^s},$$

where the random vector $(W_j^0; W_j^s)$ is an independent copy (in the conventional sense) of the random vector $(Z_j^0; Z_j^s)$. For $x^+ \in A_k^+(L/4)$ and $x^- \in A_k^-(L/4)$, the following inequalities hold:

$$2\sigma_k < \big|(x^+ - x^-, e_k)\big| < 4\sigma_k, \quad \big|(x^+ - x^-, e_j)\big| < \sigma_j/l, \ \ j \neq k, \ j \leq l.$$

Therefore, we may assume that

$$\left\{ \{(x^+ - x^-, e_j)\}_1^l : x^+ \in A_k^+(L/4), \ x^- \in A_k^-(L/4) \right\} \subset A_k,$$

with A_k from (3.1) provided that $\lambda_j = 2\sigma_j$, $N = l$, and $\varepsilon = 1$.

Observe also that $\widetilde{A}_k(L/4) \subset B(0; L/2)$ for $L \geq 8\sigma/\sqrt{3}$. Hence, if Z_j takes two values: $x^+ \in A_k^+(L/4)$ and $x^- \in A_k^-(L/4)$, then

$$|Z_j^0|^2 4 \max\big\{|x^+|^2, |x^-|^2\big\} \leq L^2,$$

i.e., $I\big(|Z_j^0| \vee |W_j^0| \leq L\big) = 1$ and $\mathbb{E}|Z_j^0|^2 \leq L^2$. Let $G_j = P_{0,L}$. Since $\mathbb{E}|X^\circ|^2 = \sum_{k=0}^l \varepsilon_k \int |x|^2 P_{k,L}(dx)$, we have $\mathbb{E}|Z_j^0|^2 = 2\mathbb{E}|Z_j|^2 = 2\int |x|^2 P_{0,L}(dx) \leq 2\mathbb{E}|X^\circ|^2/\varepsilon_0$.

Thus, the condition of the lemma implies that we may assume $\bar{\sigma} = L$ in (3.23).

Hence, by Lemma 3.3 for $Z_j \in A_k^{\pm}(L/4)$, we have

$$\overline{B}_j(x; L) = \mathbb{E}\mathbb{E}\left\{ (Z_j^0 - W_j^0, x)^2 / Z_j^s, W_j^s \right\}\Big|_{Z_j^s = W_j^s}$$

$$\geq 8 p_k^2 q_k^2 (x_j^+ - x_j^-, x)^2,$$

where $p_k = \mathbb{P}(Z_j = x_j^+)$, $q_k = \mathbb{P}(Z_j = x_j^-)$, $x_j^+ \in A_k^+(L/4)$, and $x_j^- \in A_k^-(L/4)$. Taking the definition of $\mathcal{M}_{n,L/4}$ into account, we conclude that the condition (3.22) holds with

$$\gamma_k = 8 p_k^2 q_k^2, \quad \eta = \overline{p}/3.$$

Using Lemma 3.11 for $B = c_0/(2^{l/2}L^2)$ and $L = c\sigma$ and Lemma 3.12, we now obtain the estimate

$$\varphi_n(\tau) \leq c(l) w_1(l) w_2(l) M_l(\tau; A),$$

where $A = 2^{l/2}L^2\sqrt{n}/c_0$, which, in view of Lemma 4.1 and the inequality

$$M_l(\tau; A) \leq M_l\left(\tau; L^2\sqrt{n}\right),$$

implies (4.11).

Let Ω_n be the set of random sequences $\omega_n = \{\xi_1, \xi_2, \ldots, \xi_n\}$ such that $\xi_j = i$ if $G_j \in \mathcal{F}_{i,L/4}$ and $\xi_j = 0$ if $G_j = P_{0,L/4}$. We say that $\omega_n \in \mathcal{M}_{n,L/4}$ if $\maltese_1^n G_j \in \mathcal{M}_{n,L/4}$. Denote

$$\widetilde{f}_n(t; a; \omega_n) = \mathbb{E}\left\{\exp\left\{it\left|\sum_1^n Z_j - a\right|^2\right\} \middle/ \omega_n\right\},$$

$$\widetilde{f}_n(t; a) = \mathbb{E}\exp\left\{it\left|\sum_1^n Z_j - a\right|^2\right\}.$$

Lemma 4.5. *For every $a \in \mathbb{H}$, $0 < \tau < \varepsilon$, $\tau/\varepsilon = (A\varepsilon)^{-4/l}$, and $l \geq 9$, we have*

$$I_n \equiv \int_\tau^\varepsilon \frac{\left|\widetilde{f}_n(t; a)\right|}{t} dt \leq c(l)\left(\frac{w_1(l)w_2(l)\varepsilon^{l/2}}{(A\varepsilon)^2} + \exp\{-nH(\overline{p}; \overline{p}/2)\}\log\frac{\varepsilon}{\tau}\right),$$

where A, $w_1(l)$, and $w_2(l)$ are the quantities in Lemma 4.4.

Proof. We first estimate $I_n(\omega_n) = \int_\tau^\varepsilon \left|\widetilde{f}_n(t; a; \omega_n)/t\right| dt$, where $\omega_n \in \mathcal{M}_{n,L/4}$. Obviously,

$$I_n(\omega_n) = \int_{\tau/\varepsilon}^1 \left|\widetilde{f}_n(\varepsilon t; a; \omega_n)/t\right| dt.$$

Using (4.11), we obtain

$$\left|\widetilde{f}_n(\varepsilon t; a; \omega_n)\right| \cdot \left|\widetilde{f}_n\big(\varepsilon(t + \tau); a; \omega_n\big)\right| \leq c(l)w_1(l)w_2(l)M_l(\varepsilon\tau; A) \qquad (4.12)$$

for every t and $\tau > 0$. If $\tau/\varepsilon = (A\varepsilon)^{-4/l}$ then (4.12), the formula

$$M_l(\varepsilon t; A) = \varepsilon^{l/2}M_l(t; \varepsilon A),$$

and Lemma 4.3 imply that

$$I_n(\omega_n) \leq c(l)w_1(l)w_2(l)\varepsilon^{l/2}(A\varepsilon)^{-2}$$

under the condition $\omega_n \in \mathcal{M}_{n,L/4}$. Thus,

$$I_n \leq \mathbb{E}I_n(\omega_n) \leq c(l)w_1(l)w_2(l)\varepsilon^{l/2}(A\varepsilon)^{-2} + \Pr\left(\omega_n \notin \mathcal{M}_{n,L/4}\right)\log\frac{\varepsilon}{\tau},$$

where $\Pr\left(\omega_n \notin \mathcal{M}_{n,L/4}\right) \leq l\exp\{-nH(\overline{p}; \overline{p}/2)\}$ in view of (4.10). Lemma 4.5 is proven.

5. Proof of Theorem 1.1

Denote

$$m_n = [n/4] + 1, \quad Y = \frac{1}{\sqrt{n_0}} \sum_1^{n_0} \overline{X}_j, \quad \overline{X}_j = \begin{cases} X_j, & |X_j| \le \sigma\sqrt{m_n}, \\ 0, & |X_j| > \sigma\sqrt{m_n}, \end{cases}$$

where n_0 is a fixed number to be defined $\big($see (5.6) and (5.7)$\big)$.

Let Y_j be independent identically distributed random vectors, $Y_j \overset{d}{=} Y$. It is clear that

$$\sum_1^n \overline{X}_j = \sqrt{n_0} \sum_1^{n/n_0} Y_j \tag{5.1}$$

if

$$n \equiv 0 \pmod{n_0}.$$

In what follows, we suppose for simplicity that this condition holds.

Our aim is to apply the results of Sections 2 and 3 to the sum $\sum_1^{n/n_0} Y_j$.

The next five lemmas were proved in [24]. However, in those cases, we give a proof whenever the formulation is changed.

Lemma 5.1 [24, Lemma 5.1]. *For every k, the following inequality holds:*

$$\big| \mathbb{E}(Y, e_k) \big| < 2\sigma_k \sqrt{n_0/n}.$$

Lemma 5.2 [24, Lemma 5.2 and 5.3]. *For every k, we have*

$$\mathbb{E}(Y, e_k)^2 < (1 + 4n_0/n)\sigma_k^2. \tag{5.2}$$

Thus,

$$\mathbb{E}|Y|^2 < (1 + 4n_0/n)\sigma^2. \tag{5.3}$$

Proof. Since $\mathbb{E}(Y, e_k)^2 \le \sigma_k^2 + \mathbb{E}^2(Y, e_k)$, it remains to refer to Lemma 5.1 to obtain (5.2). The bound (5.3) is immediate from (5.2).

Lemma 5.3 [24, Lemma 5.4]. *For every k, the following inequality holds:*

$$\Big| \mathbb{E}\big\{ (Y, e_k); |Y| \le \sigma d \big\} \Big| \le \left(2\sqrt{\frac{n_0}{n}} + \frac{1}{d}\sqrt{1 + 4\frac{n_0}{n}} \right)\sigma_k, \quad d > 0. \tag{5.4}$$

Let $\mathbb{R}^l$ be the space of the vectors $(x_1, \dots, x_l)$, where $x_j = (x, e_j)$, $1 \le j \le l$. Denote by Φ_l the standard Gaussian distribution in $\mathbb{R}^l$ and by Φ_0, the distribution function of the standard Gaussian random variable (in $\mathbb{R}^1$).

Lemma 5.4 [24, Lemma 5.5]. *Let l be a natural, $\sigma_l \neq 0$, x be an element of the space $\mathbb{H}$ such that $|(x, e_j)| < \sigma_j/(4l)$, $j = 1, \ldots, l$. Then*

$$\Phi\big(\widetilde{A}_k + x\big) \geq q(l) := 2\left[\Phi_0\left(\frac{1}{4l}\right) - \Phi_0\left(-\frac{1}{4l}\right)\right]^{l-1}\left[\Phi_0\left(2 - \frac{1}{4l}\right) - \Phi_0\left(1 + \frac{1}{4l}\right)\right]$$

for every k, $1 \leq k \leq l$.

Proof. Obviously,

$$\widetilde{A}_k + x \supset \left\{ y \in \mathbb{H} : \left(1 + \frac{1}{4l}\right)\sigma_k \leq |(y, e_k)| \leq \left(2 - \frac{1}{4l}\right)\sigma_k, \right.$$

$$\left. |(y, e_j)| \leq \frac{\sigma_j}{4l}, \ j \neq k, \ j \leq l \right\}. \tag{5.5}$$

Put

$$\pi_k(l) = \left\{ x \in \mathbb{R}^l : 1 + (4l)^{-1} \leq x_k \leq 2 - (4l)^{-1}, \ |x_j| \leq 1/(4l), \ j \neq k \right\}.$$

It remains to notice that (5.5) implies the bound

$$\Phi\big(\widetilde{A}_k + x\big) \geq 2\Phi_l\big(\pi_k(l)\big) = 2\Phi_l\big(\pi_1(l)\big) \equiv q(l).$$

Lemma 5.5 [24, Lemma 5.6]. *Let Y be a random vector in $\mathbb{R}^N$ and let the sets A_k satisfy (3.1). Then*

$$\mathbb{E}(Y, x)^2 \geq \min_k \mathbb{P}(Y \in A_k) \sum_{m=1}^{N} x_m^2 \lambda_m^2/(4N).$$

Define n_0. Let $\mathbb{H} = \mathbb{R}^l$. According to the Berry–Esseen bound in $\mathbb{R}^l$, we have

$$\sup_{B \in \mathfrak{C}_l} \big|\mathbb{P}(S_n \in B) - \Phi(B)\big| \leq \frac{c^0(l)}{\sqrt{n}} \max_{1 \leq j \leq l} \frac{\mathbb{E}\big|(X, e_j)\big|^3}{\sigma_j^3}, \tag{5.6}$$

where $\mathfrak{C}_l$ is the class of all convex Borel sets in $\mathbb{R}^l$. Denote

$$n_0 \equiv n_0(l) = \min\left\{ n \geq 2 : \frac{c^0(l)}{\sqrt{n}} \max_{1 \leq j \leq l} \frac{\mathbb{E}\big|(X, e_j)\big|^3}{\sigma_j^3} \leq \frac{q(l)}{2^l 4} \right\}, \tag{5.7}$$

where $q(l)$ is the constant in Lemma 5.4. Observe that this definition implies the inequality

$$n_0 \leq c(l) \max_{1 \leq j \leq l} \left(\frac{\mathbb{E}\big|(X, e_j)\big|^3}{\sigma_j^3}\right)^2 \tag{5.8}$$

which is used for completing the proof of Theorem 1.1.

Denote $\varepsilon_k = \mathbb{P}\big(Y \in \widetilde{A}_k(L)\big)$, $0 \leq k \leq l$, $p_j = \mathbb{P}\big(Y \in A_j^+(L)\big)/\varepsilon_j$, and $q_j = 1 - p_j$, $1 \leq j \leq l$.

Lemma 5.6. Let $L = \sigma d$, $d > 0$. If

$$d^2 \geq 8 \max\{64l^2, 2^l/q(l)\}, \tag{5.9}$$

$$n_0/n \leq \frac{1}{16} \min\{1/(64l^2), q(l)/2^l\} \tag{5.10}$$

then

$$\mathbb{E}\{(Y^s, x)^2; |Y| \vee |Y'| \leq L\} \geq \frac{q(l)}{16l} \sum_1^l \sigma_j^2(x, e_j)^2 \tag{5.11}$$

and, moreover,

$$\mathbb{E}|Y|^2/\varepsilon_0 \leq c\sigma^2, \tag{5.12}$$

$$\min_{1 \leq k \leq l} \varepsilon_k \geq c(l), \tag{5.13}$$

$$\min_{1 \leq k \leq l} p_k q_k \geq c(l). \tag{5.14}$$

Proof (see also [24, Lemma 5.7]). It is easily seen that

$$\mathbb{E}\{(Y^s, x)^2; |Y| \vee |Y'| \leq L\} = 2p_L \mathbb{E}\{(Y - a_L, x)^2; |Y| \leq L\}, \tag{5.15}$$

where $a_L = \mathbb{E}\{Y/|Y| \leq L\}$ and $p_L = \mathbb{P}(|Y| \leq L)$.

Denote $(x, y)_l = \sum_{j=1}^l (x, e_j)(y, e_j)$. Using the equality $\mathbb{E}\{(Y - a_L, u); |Y| \leq L\} = 0$, which is valid for each $u \in \mathbb{H}$, we obtain

$$\mathbb{E}\{(Y - a_L, x)^2; |Y| \leq L\} \geq \sum_{j=1}^l (x, e_j)^2 \mathbb{E}\{(Y - a_L, e_j)^2; |Y| \leq L\}$$

$$= \mathbb{E}\{(Y - a_L, x)_l^2; |Y| \leq L\}. \tag{5.16}$$

By Lemma 5.5,

$$\mathbb{E}\{(Y - a_L, x)_l^2; |Y| \leq L\}$$

$$\geq \frac{1}{4l} \min_{1 \leq k \leq l} \mathbb{P}\left(\{(Y - a_L, e_j)\}_1^l \in A_k; |Y| \leq L\right) \sum_{j=1}^l \sigma_j^2(x, e_j)^2, \tag{5.17}$$

where A_k are the sets satisfying (3.1) for $\lambda_j = \sigma_j$, $N = l$, and $\varepsilon = 1$. Notice that $\{(u, e_j)\}_1^l \in A_k \Leftrightarrow u \in \widetilde{A}_k$. We have

$$\mathbb{P}(Y - a_L \in \widetilde{A}_k; |Y| \leq L) \geq \mathbb{P}(Y - a_L \in \widetilde{A}_k) - \mathbb{P}(|Y| > L). \tag{5.18}$$

Estimate the quantity $P_1 \equiv \mathbb{P}\big(Y \in \widetilde{A}_k + a_L\big)$. Put $P_0 = \mathbb{P}\big(S_{n_0} \in \widetilde{A}_k + a_L\big)$, $\delta_1 = \big|P_0 - \Phi\big(\widetilde{A}_k + a_L\big)\big|$, $\delta_2 = |P_0 - P_1|$. Obviously,

$$P_1 \geq \Phi\big(\widetilde{A}_k + a_L\big) - \delta_1 - \delta_2. \tag{5.19}$$

We will estimate $\Phi\big(\widetilde{A}_k + a_L\big)$ by means of Lemma 5.4. We have first to verify that

$$\big|(a_L, e_j)\big| \leq \sigma_j/(4l), \quad j = 1, 2, \ldots, l. \tag{5.20}$$

Note that

$$\big|(a_L, e_j)\big| = \big|\mathbb{E}\big\{(Y, e_j); |Y| \leq L\big\}\big|/p_L.$$

We then obtain a lower bound for p_L. From the Chebyshev inequality and the bound (5.3) it follows that

$$\mathbb{P}\big(|Y| > L\big) \leq (1 + 4n_0/n)\sigma^2/L^2 = (1 + 4n_0/n)/d^2. \tag{5.21}$$

Provided that

$$4n_0/n \leq 1, \tag{5.22}$$

$$d^2 \geq 4, \tag{5.23}$$

we obtain

$$p_L \geq 1/2. \tag{5.24}$$

Then the inequality (5.4) yields

$$\big|(a_L, e_j)\big| \leq 2\bigg(2\sqrt{n_0/n} + \frac{1}{d}\sqrt{1 + 4n_0/n}\,\bigg)\sigma_j.$$

Let n and d satisfy the inequality

$$2\sqrt{n_0/n} + \frac{1}{d}\sqrt{1 + 4n_0/n} \leq 1/(8l).$$

To this end, it suffices to suppose that

$$4n_0/n \leq 1/(256l^2), \tag{5.25}$$

$$d \geq 16\sqrt{2}l. \tag{5.26}$$

Since the conditions (5.25) and (5.26) are stronger than (5.22) and (5.23) respectively, the inequality (5.20) holds if (5.25) and (5.26) are fulfilled. Thus, by Lemma 5.4, we obtain

$$\Phi\big(\widetilde{A}_k + a_L\big) \geq q(l) \tag{5.27}$$

if n and d satisfy the conditions (5.25) and (5.26).

Estimate δ_2 and δ_1. It is easily seen that

$$\delta_2 \leq n_0 \mathbb{P}\big(|X| > \sigma\sqrt{m_n}\,\big) \leq 4n_0/n.$$

In addition to (5.25), assume that

$$4n_0/n \leq q(l)/4. \tag{5.28}$$

Then

$$\delta_2 \leq q(l)/4. \tag{5.29}$$

From (5.6) and (5.7) it follows that

$$\delta_1 \leq q(l)/4. \tag{5.30}$$

$\big($While deriving (5.29), we have used that $\widetilde{A}_k$ is the union of two disjoint parallelepipeds.$\big)$

The inequalities (5.19), (5.27), (5.29), and (5.30) imply the bound

$$P_1 \geq q(l)/2 \tag{5.31}$$

if the conditions (5.26) and the inequality

$$4n_0/n \leq \min\big\{1/(256l^2), q(l)/4\big\} \tag{5.32}$$

hold.

By (5.21), we have

$$\mathbb{P}\big(|Y| > L\big) \leq q(l)/4 \tag{5.33}$$

if, in addition to the condition (5.26), we suppose that

$$d^2 \geq 8/q(l). \tag{5.34}$$

Since (5.9) provides the simultaneous validity of (5.26) and (5.34), from (5.18), (5.31), and (5.33), under the conditions of the lemma, it follows that

$$\mathbb{P}\big(Y - a_L \in \widetilde{A}_k; |Y| \leq L\big) \geq q(l)/4. \tag{5.35}$$

Returning to (5.15) and taking (5.16), (5.17), (5.24), and (5.35) into account, we arrive at the bound (5.11).

Now, we prove (5.12)–(5.14). By the definition of $\widetilde{A}_k(L)$ and $A_k^{\pm}(L)$, we have

$$\widetilde{A}_k - \widetilde{A}_k(L) \ \subset \ \Big\{x \in \mathbb{H} : \sum_{j=l+1}^{\infty} (x, e_j)^2 > L^2\Big\} \subset \mathbb{H} - B(0; L),$$
$$A_k^{\pm} - A_k^{\pm}(L) \subset \mathbb{H} - B(0; L).$$

Consequently, for every $1 \leq k \leq l$, the following inequalities hold:

$$\mathbb{P}\big(Y \in \widetilde{A}_k\big) - \mathbb{P}\big(|Y| > L\big) \leq \varepsilon_k \equiv \mathbb{P}\big(Y \in \widetilde{A}_k(L)\big) \leq \mathbb{P}\big(Y \in \widetilde{A}_k\big), \quad (5.36)$$

$$\mathbb{P}\big(Y \in A_k^{\pm}\big) - \mathbb{P}\big(|Y| > L\big) \leq \mathbb{P}\big(Y \in A_k^{\pm}(L)\big) \leq \mathbb{P}\big(Y \in A_k^{\pm}\big). \quad (5.37)$$

By analogy with (5.30), under the condition (5.10), we infer that

$$\left|\mathbb{P}\big(Y \in \widetilde{A}_k\big) - \Phi\big(\widetilde{A}_k\big)\right| \leq 4n_0/n + q(l)/2^{l+1} < q(l)/2^l, \quad (5.38)$$

$$\left|\mathbb{P}\big(Y \in A_k^{\pm}\big) - \Phi\big(A_k^{\pm}\big)\right| \leq 4n_0/n + q(l)/2^{l+2} < q(l)/2^{l+1}. \quad (5.39)$$

We have

$$\Phi\big(\widetilde{A}_k\big) = q_1(l) := 2\big(\Phi_0(2) - \Phi_0(1)\big)\Big(2\Phi_0\big((2l)^{-1}\big) - 1\Big)^{l-1}, \quad (5.40)$$

$$q(l) < 2\big(\Phi_0(2) - \Phi_0(1)\big)\Big(2\Phi_0\big((4l)^{-1}\big) - 1\Big)^{l-1} < q_1(l). \quad (5.41)$$

Since $\Phi_0(2) - \Phi_0(1) < 0.14$ and, for every $x > 0$,

$$2\Phi_0(x) - 1 \leq 2x/\sqrt{2\pi},$$

we have

$$lq_1(l) < 0.3l\left(\frac{1}{l\sqrt{2\pi}}\right)^{l-1} < 0.3. \quad (5.42)$$

From (5.36), (5.38), and (5.40)–(5.42) it follows that

$$\varepsilon_k \leq \Phi\big(\widetilde{A}_k\big) + q(l)/2^l \leq 3q_1(l)/2 < 1/(2l) \quad (5.43)$$

for every $1 \leq k \leq l$. Taking (5.3) and (5.22) into account, from (5.43) we obtain

$$\frac{\mathbb{E}|Y|^2}{\varepsilon_0} \leq \frac{2\sigma^2}{1 - \sum_{k=1}^{l} \varepsilon_k} \leq c\sigma^2.$$

The bound (5.12) is proven.

From (5.33), (5.36), (5.38), and Lemma 5.4 it follows that the following relations are valid for every $1 \leq k \leq l$:

$$\varepsilon_k \geq \Phi\big(\widetilde{A}_k\big) - q(l)/2^l - \mathbb{P}\big(|Y| > L\big) > q(l)/4.$$

This means that the bound (5.13) is valid.

In view of the condition (5.10), the following bound stronger than (5.33) holds:

$$\mathbb{P}\big(|Y| > L\big) \leq 2/d^2 \leq q(l)/(2^l 4) < q_1(l)/(2^l 4).$$

Moreover,

$$\Phi\left(A_k^{\pm}\right) = \left(\Phi_0(2) - \Phi_0(1)\right)\left(\Phi_0\left((2l)^{-1}\right) - 1/2\right)^{l-1} = q_1(l)/2^l.$$

From which, using (5.37) and (5.39), we obtain

$$\mathbb{P}\left(Y \in A_k^{\pm}(L)\right) \geq \Phi\left(A_k^{\pm}\right) - \frac{q(l)}{2^l 2} - \mathbb{P}\left(|Y| > L\right)$$

$$\geq \frac{q_1(l)}{2^l} - \frac{q_1(l)}{2^l 2} - \frac{q_1(l)}{2^l 4} = \frac{q_2(l)}{2^l 4}. \tag{5.44}$$

We arrive at the bound (5.14) by using (5.43), (5.44), and the equality

$$\varepsilon_k - \mathbb{P}\left(Y \in A_k^+(L)\right) = \mathbb{P}\left(Y \in A_k^-(L)\right).$$

Lemma 5.6 is proven.

Let $\sigma_j^2(L)$ be the eigenvalues of the quadratic form (1.25) with $Z = Y$. Without loss of generality we may assume the conditions of Lemma 5.6 to be fulfilled. Hence, (5.11) implies the following:

Corollary 5.1. *The following inequality holds under the assumptions of Lemma 5.6:*

$$\sigma_j^2(L) \geq \frac{q(l)}{16l}\sigma_j^2. \tag{5.45}$$

R e m a r k 5.1. The bound (5.45) holds as before if the truncation of $\overline{X}_j$ is defined as

$$\mathbb{P}\left(\overline{X}_j \in B\right) = \mathbb{P}\left(X_j \in B/|X_j| \leq \sigma\sqrt{m_n}\right). \tag{5.46}$$

Then Lemmas 5.1–5.3 remain valid if n is replaced by nb_n^2 with $b_n = \mathbb{P}\left(|X| \leq \sigma\sqrt{m_n}\right)$, and Lemma 5.6 remains valid if (5.10) is replaced by the condition

$$n_0/(nb_n^2) \leq \frac{1}{64}\min\left\{\frac{1}{16l^2}, \frac{q(l)}{2^l}\right\}.$$

Lemma 5.7. 1. *Let $d \in \mathbb{R}$ and an integer l be such that $l \geq 2d + 3$. Then, for every $\theta > 0$ and $a \in \mathbb{H}$, we have*

$$I_d(\theta) \equiv \int_{|t| \geq \theta} |t|^d |g(t; a)| dt \leq 2^{-l/2+2}\Lambda_l^{-1/2}\theta^{-l/2+d+1}. \tag{5.47}$$

2. *Let $d \geq 0$ and an integer l be such that $l \geq 2d + 3$. Then, for every $a \in \mathbb{H}$,*

$$I_d \equiv \int_{-\infty}^{\infty} |t|^d |g(t; a)| dt < 2^{-d+2}\Lambda_l^{-(d+1)/l}. \tag{5.48}$$

Proof. First of all, notice that $\big|g(t;a)\big| \le \big|g(t)\big|$. The bound (5.47) ensues from the inequality

$$I_d(\theta) \le \frac{1}{2^{l/2}\Lambda_l^{1/2}} \int_{|t| \ge \theta} |t|^{d-l/2} dt.$$

Prove (5.48). By (5.47), for every $\theta > 0$, we have

$$I_d \equiv \int_{|t| \le \theta} + \int_{|t| > \theta} \le \frac{2\theta^{d+1}}{d+1} + 2^{-l/2+2}\Lambda_l^{-1/2}\theta^{-l/2+d+1} \equiv A_1 + A_2.$$

Choose $\theta = 2^{-1+2/l}(d+1)^{2/l}\Lambda_l^{-1/l}$ by putting $A_1 = A_2$. From which we obtain the estimate (5.48). Indeed, in the case $l > 2d+2$, we have

$$I_d \le \frac{4}{d+1}\theta^{d+1} = \frac{4}{d+1}2^{-(d+1)(1+2/l)}(d+1)^{2(d+1)/l}\Lambda_l^{-(d+1)/l} < 4 \cdot 2^{-d}\Lambda_l^{-(d+1)/l}.$$

R e m a r k 5.2. In connection with (5.48) we observe that an estimate is obtained in [11, Lemma 2.2], implying that, for every $A > 0$, a real $b \ne 0$, and an integer $d \ge 0$, we have

$$\left| \int_{-A}^{A} t^d e^{itb} \left(\prod_{j=1}^{2d+2} g_j(t) \right) dt \right| \le c\Lambda_{2d+2}^{-1/2}.$$

Lemma 5.8. *The following bound is valid:*

$$\Delta_n(0) \le \frac{c}{n}\left[\Gamma_{4,13} + \Gamma_{3,13}^2 + \left(\frac{\sigma^2}{\Lambda_9^{1/9}} \right)^2 n_0 \right]. \tag{5.49}$$

Proof $\big($see also [24, the proof of (5.19)]$\big)$. Denote

$$\overline{\Delta}_n(a) = \sup_r \left| \mathbb{P}\left(\frac{1}{\sqrt{n}} \sum_1^n \overline{X}_j \in B(a;r) \right) - \Phi\big(B(a;r)\big) \right|,$$

$$g(t;a) = \int \exp\{it|x-a|^2\}\Phi(dx), \quad g(t) = g(t;0),$$

$$\bar{g}_n(t;a) = \mathbb{E}\exp\left\{ it\left| n^{-1/2}\sum_1^n \overline{X}_j - a \right|^2 \right\}, \quad \bar{g}_n(t) = g_n(t;0),$$

where $\overline{X}_j$ are the truncations defined in (5.46). By the Esseen bound and (1.13), for every $t_0 > 0$, we have

$$\overline{\Delta}_n(0) < J/\pi + 2\pi(nt_0\sigma_1\sigma_2)^{-1}, \tag{5.50}$$

where

$$J = \int_{|t|<nt_0} \left| \frac{\bar{g}_n(t) - g(t)}{t} \right| dt.$$

Let $t_1 = n^{-1} \Lambda_l^{-1/l} (n/\Gamma_{4,l})^{1/4}$. Show that

$$J_1 := \int_{|t|<nt_1} \left| \frac{\bar{g}_n(t) - g(t)}{t} \right| dt \leq \frac{c(l)}{n} \left(\beta_4 / \Lambda_l^{2/l} + \Gamma_{3,l}^2 \right), \quad l \geq 13. \qquad (5.51)$$

Using Lemma 6.1 (see also [21]), we can show that

$$\left| \bar{g}_n(t) - g(t) \right| \leq \frac{1}{n} \left\{ \frac{c\beta_4}{\sigma^4} |g(t/2)| \left((t\sigma^2)^2 + |t|\sigma^2 \right) \right.$$
$$\left. + \widetilde{f}_l(t, n, \gamma, q) \frac{\beta_3^2}{\sigma^6} \left((t\sigma^2)^6 + |t|\sigma^2 \right) \right\} \qquad (5.52)$$

for every $q > 1$, every natural l, and $0 < \gamma < l/2$, where

$$\widetilde{f}_l(t, n, \gamma, q) = c(l, \gamma, q) \left\{ 1 \wedge \left[f_l(t, n, \gamma, q) + 0.9^{n/4} \right] \right\},$$

$$f_l(t, n, \gamma, q) = \left(|t| \Lambda_l^{1/l} \right)^{-l/(2q)} + \left(\frac{\Gamma_{4,l}}{n} \right)^{l/(4q)} + \left(\left(t\Lambda_l^{1/l} \right)^2 \frac{\Gamma_{4,l}}{n} \right)^{\gamma/(2q)}. \qquad (5.53)$$

Assume that

$$\Gamma_{4,l}/n \leq 1. \qquad (5.54)$$

It is easy to verify that

$$0.9^{n/4} \leq c(l; q)(\Gamma_{4,l}/n)^{l/(4q)}$$

and, moreover, in view of (5.54),

$$\left(\left(t\Lambda_l^{1/l} \right)^2 \frac{\Gamma_{4,l}}{n} \right)^{\gamma/2} \leq c(\gamma) \left(1 \wedge \left(|t| \Lambda_l^{1/l} \right)^{-\gamma} \right),$$

$$\left(\frac{\Gamma_{4,l}}{n} \right)^{l/4} \leq \left(|t| \Lambda_l^{1/l} \right)^{-l/2}$$

for $|t| \leq \tau_1 \equiv nt_1$. Hence, if $|t| \leq \tau_1$ then, under the condition (5.54), we obtain

$$\widetilde{f}_l(t, n, \gamma, q) \leq c(l, \gamma, q) \left(1 \wedge \left(|t| \Lambda_l^{1/l} \right)^{-\gamma/q} \right). \qquad (5.55)$$

The bound (5.51) follows from (5.48), (5.52), and (5.55) $\big($see also (6.3)$\big)$.

Estimate the integral

$$J_2 := \int_{nt_1 < |t| < nt_2} \left| \frac{\bar{g}_n(t)}{t} \right| dt,$$

where $t_2 = c_0 / \big(n_0 2^{l'/2} \sigma(L) L \sqrt{2n} \big)$, $n_0 = n_0(l')$ is taken from the definition (5.7) with l' instead of l,

$$\sigma^2(L) = \mathbb{E}\Big\{ \big| Y - \mathbb{E}\{Y/|Y| \le L\} \big|^2 / |Y| \le L \Big\},$$

and $c_0 = \sqrt{3}/8$. Notice that

$$\bar{g}_n(t) = \mathbb{E}\exp\left\{ i\frac{n_0 t}{n} \left| \sum_{j=1}^{n/n_0} Y_j \right|^2 \right\} \equiv \tilde{g}_n(n_0 t/n), \qquad (5.56)$$

where $\tilde{g}_n(t) = \mathbb{E}\exp\big\{ it \big| \sum_1^{n/n_0} Y_j \big|^2 \big\}$. Consequently, employing Lemma 2.8 and taking (5.45) into account, we obtain

$$J_2 \le \frac{c(l')n_0^{l'/2}}{\Lambda_{l'}^{1/2} n^{l'/2}} \int_{n_0 t_1}^{\infty} \frac{dt}{t^{l'/2+1}} \le c_1(l') \left(\frac{\Gamma_{4,l}}{n} \right)^{l'/8}. \qquad (5.57)$$

Put $l' \ge 8$,

$$t_0 = \Lambda_{l'}^{1/l'} / (L^4 n_0), \quad t_3 = t_0 (B_n n_0 t_0)^{-4/l'}, \quad B_n = L^2 \sqrt{n/n_0}. \qquad (5.58)$$

It is easy to see that

$$J_4 := \int_{nt_3 < |t| < nt_0} \left| \frac{\bar{g}_n(t)}{t} \right| dt = \int_{\tau < |t| < \varepsilon} \left| \frac{\tilde{g}_n(t)}{t} \right| dt,$$

where

$$\varepsilon = n_0 t_0 = \Lambda_{l'}^{1/l'} / L^4, \quad \tau = n_0 t_3 = \varepsilon (B_n \varepsilon)^{-4/l'}. \qquad (5.59)$$

Denote $\varepsilon_0 = \mathbb{P}\Big(Y \in \mathbb{H} - \bigcup_1^{l'} \widetilde{A}_k(L/4) \Big)$. By (5.12), there exists a constant $c(l')$ such that $L \ge \big(8\sigma/\sqrt{3} \big) \vee \sqrt{2\mathbb{E}|Y|^2/\varepsilon_0}$ if $L \ge c(l')\sigma$. In view of Lemma 4.5, we find that, for $l' \ge 9$,

$$J_4 \le c(l') \left(\frac{w_1(l')w_2(l')\varepsilon^{l'/2}}{(B_n \varepsilon)^2} + \exp\left\{ -\frac{n}{n_0} H\left(\bar{p}; \frac{\bar{p}}{2} \right) \right\} \log\left(\frac{n}{n_0} \right) \right), \qquad (5.60)$$

where

$$\overline{p} = \min_{1 \le k \le l'} \mathbb{P}\big(Y \in \widetilde{A}_k(L/4)\big), \quad w_1(l') = (\gamma_0\,\overline{p}^2)^{-l'/2}, \quad \gamma_0 = 8 \min_{1 \le k \le l'} p_k^2 q_k^2,$$

$$p_k = \mathbb{P}\big(Y \in A_k^+(L/4)\big)\big/\mathbb{P}\big(Y \in \widetilde{A}_k(L/4)\big), \quad q_k = 1 - p_k.$$

In view of (5.13) and (5.14), we have

$$\overline{p} \ge c_1(l'), \quad \gamma_0 \ge c_2(l'). \tag{5.61}$$

It is easy to check the validity of the equality

$$w_2(l')\varepsilon^{l'/2}(B_n\varepsilon)^{-2} = \frac{n_0}{n}\left(\frac{L^2}{\Lambda_{l'}^{1/l'}}\right)^2. \tag{5.62}$$

Since $H(t; t/2)$ is an increasing function in $[0, 1]$, from (5.60)–(5.62) we infer that

$$J_4 \le c(l')\frac{n_0}{n}\left(\frac{L^2}{\Lambda_{l'}^{1/l'}}\right)^2, \quad l' \ge 9. \tag{5.63}$$

Using Lemma 2.7 with $B = c_0\big/\big(2^{(l'+1)/2}\sigma(L)L\big)$, (5.45), (5.56), and the inequality $\sigma(L) \le L$, we arrive at the bound

$$J_3 := \int_{nt_2 \le |t| < nt_3} \left|\frac{\overline{g}_n(t)}{t}\right| dt = \int_{n_0 t_2 < |t| < \tau} \left|\frac{\widetilde{g}_n(t)}{t}\right| dt \le c(l')\frac{L^{2l'}}{\Lambda_{l'}^{1/2}}\int_0^\tau t^{l'/2-1} dt.$$

By (5.59) and (5.62), the equality $\frac{L^{2l'}\tau^{l'/2}}{\Lambda_{l'}^{1/2}} = \frac{n_0}{n}\left(\frac{L^2}{\Lambda_{l'}^{1/l'}}\right)^2$ is fulfilled. Hence,

$$J_3 \le c(l')\left(\frac{L^2}{\Lambda_{l'}^{1/l'}}\right)^2\frac{n_0}{n}. \tag{5.64}$$

In view of (5.47), for $k \ge 2$,

$$J_5 := \int_{nt_1 \le |t| \le nt_0} \left|\frac{g(t)}{t}\right| dt \le \int_{|t| \ge nt_1} \left|\frac{g(t)}{t}\right| dt \le \left(\frac{\Gamma_{4,l}}{n}\right)^{k/8}. \tag{5.65}$$

Since $J \le \sum_{k=1}^5 J_k$, collecting the bounds (5.51), (5.57), and (5.63)–(5.65), we obtain the inequality

$$J \le \frac{c(l, l')}{n}\left(\Gamma_{3,l}^2 + \Gamma_{4,l} + \left(\frac{L^2}{\Lambda_{l'}^{1/l'}}\right)^2 n_0\right), \quad l \ge 13, \ l' \ge 9. \tag{5.66}$$

Take $l = 13$ and $l' = 9$. The bound (5.49) ensues from (5.50), (5.66), and the inequality

$$\Delta_n(0) \leq \overline{\Delta}_n(0) + \frac{2n\beta_4}{\sigma^4 m_n^2}.\tag{5.67}$$

Notice that without loss of generality we may assume the condition (5.54) with $l = 13$ to be satisfied.

The claim of Theorem 1.1 follows from Lemma 5.8 and the bound (5.8), which completes the proof.

6. Auxiliary statements for the case $a \neq 0$

Denote by $\overline{F}$ the distribution of the random variable $\overline{X}$ in (5.46). Put

$$\overline{P}_{\nu,n}(t;a) = \binom{n}{\nu} \int \exp\left\{it\left|n^{-1/2}x - a\right|^2\right\} \Phi^{*(n-\nu)} * (\overline{F} - \Phi)^{*\nu}(dx),$$

$$R_{k,n}(t;a) = \bar{g}_n(t;a) - g(t;a) - \sum_{\nu=1}^{k-1} \overline{P}_{\nu,n}(t;a),$$

$$\bar{g}_n(t;a) = \mathbb{E}\exp\left\{it\left|n^{-1/2}\sum_{j=1}^{n}\overline{X}_j - a\right|^2\right\},$$

$$\tau_1 = \Lambda_l^{-1/l}(n/\Gamma_{4,l})^{1/4}.$$

We use the notation $\alpha = (\alpha_1, \alpha_2, \dots)$ for the sequence of independent standard normal random variables α_j, $j = 1, 2, \dots$.

First of all, we are interested in an estimate for the integral

$$J_1(a) \equiv \int\limits_{|t|\leq\tau_1} \frac{\left|R_{2,n}(t;a)\right|}{|t|} dt.$$

Lemma 6.1. *Let $|t| \leq 0.4n\sigma_1^{-2}$. Then, for every $q > 1$, every natural l, and γ such that $0 < \gamma < l/2$, the following estimate is valid:*

$$\left|R_{2,n}(t;a)\right| \leq \frac{c(q)\left(\beta_3/\sigma^3 + \left(\beta_3(a)\right)/\sigma^6\right)^2}{n}\widetilde{f}_l(t,n,\gamma,q)\left\{(t\sigma^2)^6 + |t|\sigma^2\right\},\tag{6.1}$$

where

$$\widetilde{f}_l(t,n,\gamma,q) = c(l,\gamma,q)\left\{1 \wedge \left[f_l(t,n,\gamma,q) + 0.9^{n/4}\right]\right\},$$

$$f_l(t,n,\gamma,q) = \left(|t|\Lambda_l^{1/l}\right)^{-l/(2q)} + \left(\frac{\Gamma_{4,l}}{n}\right)^{l/(4q)} + \left(\left(t\Lambda_l^{1/l}\right)^2\frac{\Gamma_{4,l}}{n}\right)^{\gamma/(2q)}$$

(see also (5.53)).

In the case of $a = 0$, the bound (6.1) was found in [22]. For an arbitrary a, the proof is analogous, but more cumbersome (see [8, p. 151]).

It was proved in [22, formula (2.8)] that

$$J_1(0) \leq \frac{c(l)}{n} \Gamma_{3,l}^2$$

for every $l \geq 13$. The following assertion extends this bound.

Lemma 6.2. *Let* (5.54) *be satisfied. If* $l \geq 13$ *then*

$$J_1(a) \leq \frac{c(l)}{n} \left(\Gamma_{3,l}^2 + \Gamma_{3,l}^2(a) \right). \tag{6.2}$$

Proof. It is easy to see that, for $\gamma/q > 6$ and under the condition (5.54),

$$\int_0^{\tau_1} \frac{t^5 \sigma^{12} + \sigma^2}{1 \vee \left(t \Lambda_l^{1/l} \right)^{\gamma/q}} dt \equiv \int_0^{\Lambda_l^{-1/l}} + \int_{\Lambda_l^{-1/l}}^{\tau_1} \leq \frac{c \sigma^{12}}{(\gamma/q - 6) \Lambda_l^{6/l}}. \tag{6.3}$$

Take, for instance, $q = 14/13$ and $\gamma = l/2.01$. Then $\gamma/q > 6$, and (6.2) is immediate from (5.55), (6.1), and (6.3).

Denote

$$I(y; n) = I\left(|y| > \sigma \sqrt{m_n} \right) + (1 - b_n) I\left(|y| \leq \sigma \sqrt{m_n} \right) / b_n,$$

$\beta_{p,\theta}(a) = \beta_p^{(p-\theta)/p} \left(\beta_p(a) \right)^{\theta/p}$, where $0 \leq \theta \leq p$.

Lemma 6.3. *The following bounds are valid:*
1. *If* $0 \leq \lambda \leq p$ *and* $n \geq 5$ *then*

$$\int |x|^{\lambda - \theta} \left| (a, x) \right|^\theta \left| (\overline{F} - F)(dx) \right|$$

$$\leq \mathbb{E} I(X; n) |X|^{\lambda - \theta} \left| (a, X) \right|^\theta$$

$$\leq c(p) \left(\sigma \sqrt{n} \right)^{\lambda - p} \beta_{p,\theta}(a)$$

$$\leq c(p) \left(\sqrt{n} \right)^{\lambda - p} \sigma^{\lambda + \theta} \left(\frac{\beta_p}{\sigma^p} + \frac{\beta_p(a)}{\sigma^{2p}} \right). \tag{6.4}$$

2. *If* $\theta \leq 3 \leq \lambda$ *then*

$$\int |x|^{\lambda - \theta} \left| (a, x) \right|^\theta (\overline{F} + \Phi)(dx)$$

$$\leq c(\lambda) \left(\sigma \sqrt{n} \right)^{\lambda - 3} \beta_{3,\theta}(a)$$

$$\leq c(\lambda) \left(\sqrt{n} \right)^{\lambda - 3} \sigma^{\lambda + \theta} \left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6} \right). \tag{6.5}$$

3. *If $0 \leq \theta \leq \lambda$ and $0 \leq \nu \leq \lambda - \theta$ then*

$$\int |x|^{\lambda - \theta} |(a, x)|^{\theta} (\overline{F} + \Phi)(dx)$$

$$\leq \left(\sigma\sqrt{n}\right)^{\lambda - \theta - \nu} \left\{ \mathbb{E} |\overline{X}|^{\nu} |(a, \overline{X})|^{\theta} + \frac{c(\lambda; \theta)\sigma^{\nu}\beta_2^{\theta/2}(a)}{n^{(\lambda - \theta - \nu)/2}} \right\}. \qquad (6.6)$$

Proof. It is easily seen that

$$\left| (\overline{F} - F)(dy) \right| \leq I(y; n) F(dy), \qquad (6.7)$$

and $b_n^{-1} \leq 5$ for $n \geq 5$. This implies the first inequality in (6.4). Using the Chebyshev and Hölder inequalities, we obtain the following estimates:

$$\mathbb{E} |X|^{\lambda - \theta} |(X, a)|^{\theta} I\left(|X| > \sigma\sqrt{m_n}\right)$$

$$\leq \left(\sigma\sqrt{m_n}\right)^{\lambda - p} \mathbb{E} |X|^{p - \theta} |(X, a)|^{\theta}$$

$$\leq c(p)\left(\sigma\sqrt{n}\right)^{\lambda - p} \beta_{p, \theta}(a), \qquad (6.8)$$

$$\mathbb{E} |X|^{\lambda - \theta} |(X, a)|^{\theta} \leq \left(\mathbb{E} |(X, a)|^{p} \right)^{\theta/p} \left(\mathbb{E} |X|^{((\lambda - \theta)p)/(p - \theta)} \right)^{(p - \theta)/p}$$

$$\leq \left(\mathbb{E} |(X, a)|^{p} \right)^{\theta/p} \left(\mathbb{E} |X|^{p} \right)^{(\lambda - \theta)/p}, \qquad (6.9)$$

$$1 - b_n \leq \left(\sigma\sqrt{m_n}\right)^{\lambda - p} \mathbb{E} |X|^{p - \theta} \leq c(p)\left(\sigma\sqrt{n}\right)^{\lambda - p} \left(\mathbb{E} |X|^{p} \right)^{(p - \lambda)/p}. \qquad (6.10)$$

The estimates (6.7)–(6.10) imply the second inequality in (6.4). The third estimate in (6.4) is easily derived by using the representation

$$\beta_{p, \theta}(a) = \left(\frac{\beta_p}{\sigma^p} \right)^{(p - \theta)/p} \left(\frac{\beta_p(a)}{\sigma^{2p}} \right)^{\theta/p} \sigma^{p + \theta}.$$

We now prove (6.5). Using the equality

$$\overline{F}(dy) = I\left(|y| \leq \sigma\sqrt{m_n}\right) F(dy)/b_n$$

and the Hölder inequality, we find that, for $0 \leq \theta \leq 3 \leq \lambda$,

$$\mathbb{E} |\overline{X}|^{\lambda - \theta} |(\overline{X}, a)|^{\theta} \leq \frac{\left(\sigma\sqrt{m_n}\right)^{\lambda - 3}}{b_n} \mathbb{E} |X|^{3 - \theta} |(X, a)|^{\theta} \leq \frac{\left(\sigma\sqrt{m_n}\right)^{\lambda - 3}}{b_n} \beta_{3, \theta}(a).$$

It is easy to see that

$$\mathbb{E} |Y_0|^{\lambda - \theta} |(Y_0, a)|^{\theta} \leq \left(\mathbb{E} |Y_0|^{\lambda} \right)^{(\lambda - \theta)/\lambda} \left(\mathbb{E} |(Y_0, a)|^{\lambda} \right)^{\theta/\lambda}. \qquad (6.11)$$

Taking it into account that
$$\mathbb{E}|Y_0|^\lambda \leq \sigma^\lambda \mathbb{E}|\alpha_1|^\lambda$$
(see, for instance, [18]) and (Y_0, a) is the real normal random variable with zero mean and the variance (Ta, a), we obtain

$$\left(\mathbb{E}|Y_0|^\lambda\right)^{(\lambda-\theta)/\lambda}\left(\mathbb{E}\big|(Y_0, a)\big|^\lambda\right)^{\theta/\lambda} \leq \left(\sigma^\lambda\mathbb{E}|\alpha_1|^\lambda\right)^{\lambda-\theta}\left((Ta, a)^{\lambda/2}\mathbb{E}|\alpha_1|^\lambda\right)^{\theta/\lambda}$$
$$= c(\lambda)\sigma^{\lambda-\theta}(Ta, a)^{\theta/2}. \tag{6.12}$$

Consequently, if $0 \leq \theta \leq 3 \leq \lambda$ then

$$\int |x|^{\lambda-\theta}\big|(a, x)\big|^\theta\left(\overline{F} + \Phi\right)(dx) \leq \frac{\left(\sigma\sqrt{m_n}\right)^{\lambda-3}}{b_n}\beta_{3,\theta}(a) + c(\lambda)\sigma^{\lambda-\theta}(Ta, a)^{\theta/2}$$
$$\leq b_n^{-1}\left(\sigma\sqrt{n}\right)^{\lambda-3}\beta_{3,\theta}(a)c_1(\lambda). \tag{6.13}$$

The bound (6.5) is proven. The inequality (6.6) ensues from the bounds (6.11), (6.12), and the inequality

$$\mathbb{E}|\overline{X}|^{\lambda-\theta}\big|(\overline{X}, a)\big|^\theta \leq \left(\sigma\sqrt{n}\right)^{\lambda-\theta-\nu}\mathbb{E}|\overline{X}|^\nu\big|(\overline{X}, a)\big|^\theta, \quad \nu \leq \lambda - \theta.$$

For every $x, y \in \mathbb{H}$, define

$$(y, \alpha + sx) = \sum_{j=1}^\infty (y, e_j)\alpha_j + s(x, y),$$

where $s = (2it)^{1/2}$.

Lemma 6.4 [25, Lemma 6.4]. *For every natural M,*

$$\mathbb{E}_\alpha\left[(y, \alpha + x)^M \exp\big\{\lambda(y, \alpha + x)\big\}\right]$$
$$= \exp\big\{(\lambda y, \lambda y + 2x)/2\big\}\sum_{m=0}^{[M/2]}\binom{M}{2m}\mathbb{E}\alpha_1^{2m}|y|^{2m}(y, \lambda y + x)^{M-2m}.$$

Lemma 6.5. *Let $a, b \in \mathbb{H}$, k, and N be nonnegative integers.*
1. The following bounds hold:

$$\left|\mathbb{E}\exp\big\{it|Y_0 - b|^2\big\}(y, Y_0 - a)^k|Y_0|^{2N}\right|$$
$$\leq c_1(k; N)\big|g(c_2(k; N)t)\big|\sum_{j=0}^k\big|(a, y)\big|^{k-j}\sigma^{j+2N}|y|^j m_j(t; b); \tag{6.14}$$

$$\left|\mathbb{E}\exp\big\{it|Y_0 - b|^2\big\}(y, Y_0 - a)^k(Y_0, a)^N\right|$$
$$\leq c_1(k; N)\big|g\big(c_2(k; N)t\big)\big|(Ta, a)^{N/2}\sum_{j=0}^k\big|(a, y)\big|^{k-j}\sigma^j|y|^j m_j(t; b), \tag{6.15}$$

where $m_j(t; b) = 1$ if j is even and $m_j(t; b) = \min\big\{1, |t|(Tb, b)^{1/2}\big\}$ if j is odd.

2. *The following equalities are valid:*

$$\mathbb{E}\exp\{it|Y_0 - a|^2\}(y, Y_0 - a)^k = g(t; a)\mathbb{E}(A_t y, Y_0 - A_t a)^k; \quad (6.16)$$

$$\mathbb{E}\left(y, \alpha + s\left(A_t\sqrt{T}\,\alpha' - A_t^2 a\right)\right)^k = \mathbb{E}(A_t y, \alpha - sA_t a)^k; \quad (6.17)$$

$$\mathbb{E}\left[\exp\{it|Y_0 - a|^2\}\mathbb{E}_\alpha\left(y, \alpha + s(Y_0 - a)\right)^k\right] = g(t; a)\mathbb{E}(A_t y, \alpha - sA_t a)^k. \quad (6.18)$$

Proof. We only sketch the proof. The bounds (6.14) and (6.15) are proved by means the representation

$$Y_0 \overset{d}{=} \frac{1}{\sqrt{M}}\sum_{j=1}^{M} Y_{0j},$$

where M is a sufficiently large number depending on k and N and Y_{0j} are independent copies of Y_0 (about this method see also [33, 34]). We can prove the equality (6.16) by using the representation $(Y_0, x) = \sum_1^\infty \sigma_j\alpha_j(x, e_j)$, changing variables in the multivariate integral and applying the Cauchy theorem (see [23]). The formula (6.17) is proved by induction on the dimension of $\mathbb{H}$, beginning from the equality $\mathbb{E}\left(\alpha_1 + sg_1(t)\sigma_1\alpha_1'\right)^k = g_1^k(t)\mathbb{E}\alpha_1^k$ (we may first consider the case $a = 0$). The formula (6.18) is a consequence of (6.16) and (6.17).

In what follows, we use the notations

$$\widetilde{Y}_0 = \sqrt{1 - 1/n}\,Y_0, \quad \mathbb{E}_{\overline{W}}f(x) = \int f(x)d\big(\overline{F} - \Phi\big)(x)$$

($f : \mathbb{H} \to \mathbb{C}$ is an arbitrary Borel function). We call $\overline{W}$ the generalized random variable with the distribution $\overline{F} - \Phi$. We assume that $\overline{W}$ is independent of all random variables introduced earlier. In particular,

$$\mathbb{E}\left[f_1\big(\overline{W}\big)f_2(\alpha)f_3\big(\widetilde{Y}_0\big)\right] = \mathbb{E}f_1\big(\overline{W}\big) \cdot \mathbb{E}f_2(\alpha) \cdot \mathbb{E}f_3\big(\widetilde{Y}_0\big).$$

Lemma 6.6. *The following estimate holds:*

$$\left|\overline{P}_{1,n}(t; a)\right| \le 2n\left|\mathbb{E}\exp\left\{it\big|\widetilde{Y}_0\big|^2\right\}\right|. \quad (6.19)$$

Moreover, if $n \ge 5$ and the condition

$$|t| \le 0.9n/\sigma_1^2 \quad (6.20)$$

is fulfilled then

$$\left|\overline{P}_{1,n}(t; a)\right| \le \frac{c_1}{\sqrt{n}}\left|g(c_2 t)\right|\left(\left(|t|\sigma^2\right)^3 + |t|\sigma^2\right)\left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6}\right). \quad (6.21)$$

Proof. First of all, notice that (6.19) is immediate from the representation

$$\overline{P}_{1,n}(t;a) = n \int \mathbb{E} \exp\left\{it\left|\widetilde{Y}_0 + n^{-1/2}y - a\right|^2\right\}(\overline{F} - \Phi)(dy) \qquad (6.22)$$

and the bound $\sup_{x\in\mathbb{H}}\left|\mathbb{E}e^{it|\widetilde{Y}_0+x|^2}\right| \le \left|\mathbb{E}e^{it|\widetilde{Y}_0|^2}\right|$.

We give the proof of (6.21) in abbreviated form. Applying the formula

$$\exp\left\{it\left|n^{-1/2}y\right|^2\right\} = \mathbb{E}_\alpha \exp\left\{sn^{-1/2}(y,\alpha)\right\}$$

(see, for instance, [22]), from (6.22) we obtain

$$\overline{P}_{1,n}(t;a) = n\mathbb{E} \exp\left\{it\left|\widetilde{Y}_0 - a\right|^2\right\}\mathbb{E}_{\overline{W}}\mathbb{E}_\alpha \exp\left\{sn^{-1/2}\xi\right\}, \qquad (6.23)$$

where $\xi = \left(\overline{W}, \alpha + s(\widetilde{Y}_0 - a)\right)$. Change the order of integration in (6.23) (we bear $\mathbb{E}_{\overline{W}}$ and $\mathbb{E}_\alpha$ in mind). Note that we are able to do this if t satisfies the condition $|t| < n/\sigma_1^2$. By the Taylor formula, we have

$$\mathbb{E}_{\overline{W}}\exp\left\{\frac{s\xi}{\sqrt{n}}\right\} = \sum_{j=1}^{2}\left(\frac{s}{\sqrt{n}}\right)^j \frac{\mathbb{E}_{\overline{W}}\xi^j}{j}$$

$$+ \frac{1}{2}\left(\frac{s}{\sqrt{n}}\right)^3 \int_0^1 (1-\lambda)^2 \mathbb{E}_{\overline{W}}\left[\xi^3 \exp\left\{\frac{\lambda s\xi}{\sqrt{n}}\right\}\right]d\lambda. \qquad (6.24)$$

Changing the order of integration again and using the equalities

$$\mathbb{E}_\alpha(y,\alpha+sx) = s(y,x), \quad \mathbb{E}_\alpha(y,\alpha+sx)^2 = |y|^2 + s^2(y,x)^2$$

and Lemma 6.4, from (6.23) and (6.24) we infer that

$$\overline{P}_{1,n}(t;a) = A + R, \qquad (6.25)$$

where

$$A = n\int \mathbb{E}\left\{\exp\left\{it\left|\widetilde{Y}_0 - a\right|^2\right\}\left[\frac{s^2(y,\widetilde{Y}_0 - a)}{\sqrt{n}}\right.\right.$$

$$\left.\left.+ \frac{s^2}{2n}\left(|y|^2 + s^2(y,\widetilde{Y}_0 - a)^2\right)\right]\right\}(\overline{F} - F)(dy),$$

$$R = \frac{s^3}{2\sqrt{n}}\int_0^1 (1-\lambda)^2 \int \mathbb{E}\left\{\exp\left\{it\left|\widetilde{Y}_0 - a + \lambda_n y\right|^2\right\}\left[s^3(y,\lambda_n y + \widetilde{Y}_0 - a)^3\right.\right.$$

$$\left.\left.+ 3s|y|^2\left(y,\lambda_n y + \widetilde{Y}_0 - a\right)\right]\right\}(\overline{F} - \Phi)(dy)d\lambda,$$

with $\lambda_n = \lambda\sqrt{n}$. By (6.14), for $n \geq 2$, we have

$$\left| \mathbb{E} \exp\left\{ it\big|\widetilde{Y}_0 - a\big|^2 \right\} (y, \widetilde{Y}_0 - a) \right|$$

$$\leq c_1 \big|g(c_2 t)\big| \Big(\big|(a, y)\big| + \sigma|y|\min\{1, |t|\beta_2^{1/2}(a)\} \Big), \tag{6.26}$$

$$\left| \mathbb{E} \exp\left\{ it\big|\widetilde{Y}_0 - a\big|^2 \right\} (y, \widetilde{Y}_0 - a)^2 \right| \leq c_1 \big|g(c_2 t)\big| \big((a, y)^2 + \sigma^2|y|^2\big). \tag{6.27}$$

From (6.4) with $p = 3$, (6.26), and (6.27) it follows that

$$|A| \leq \frac{c_1}{\sqrt{n}} \big|g(c_2 t)\big| \big((t\sigma^2)^2 + |t|\sigma^2\big) \left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6}\right), \quad n \geq 5. \tag{6.28}$$

Moreover, using (6.14) and (6.5), we obtain

$$|R| \leq \frac{c_1}{\sqrt{n}} \big|g(c_2 t)\big| \left(\big(|t|\sigma^2\big)^3 + (t\sigma^2)^2 \right) \left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6}\right), \quad n \geq 5. \tag{6.29}$$

The bound (6.21) follows from (6.25), (6.28), and (6.29).

Lemma 6.7 [14]. *Let Y_0 be a Gaussian random element in $\mathbb{H}$ with $\mathbb{E}Y_0 = 0$ and a covariance operator T. Let $\sigma_1 \geq \sigma_2 \geq \cdots$ be the eigenvalues of T. Then*

$$\sup_{r > 0, a \in \mathbb{H}} \frac{d}{dr}\left[\mathbb{P}\big(|Y_0 - a|^2 < r\big) \right] \leq \frac{1}{2\sigma_1\sigma_2}.$$

Lemma 6.8. *Let*

$$n^{-1/2}\big(\Gamma_{3,l} + \Gamma_{3,l}(a)\big) \leq 1, \tag{6.30}$$

where $l \geq 9$, $n \geq 5$, and $a \in \mathbb{H}$. Then

$$\sup_{r > 0} \left| \frac{d}{dr}\left[\Phi\Big(B\big(a, \sqrt{r}\big)\Big) + \overline{Q}_{1,n}(r; a) \right] \right| \leq c\Lambda_l^{-1/l}.$$

Proof. Integrate $\big|\overline{P}_{1,n}(t; a)\big|$. Using (6.21) for $|t| \leq cn/\sigma_1^2$, (6.19) for $|t| > cn/\sigma_1^2$ $\big($on account of the condition (6.30)$\big)$, and also (5.48), we arrive at the bound

$$\int_{-\infty}^{\infty} \big|\overline{P}_{1,n}(t; a)\big| dt \leq c\Lambda_l^{-1/l}. \tag{6.31}$$

Since

$$\sup_{r > 0} \left| \frac{d}{dr}\overline{Q}_{1,n}(r; a) \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \big|\overline{P}_{1,n}(t; a)\big| dt,$$

the claim of Lemma 6.8 follows from Lemma 6.7 and the inequality (6.31).

Lemma 6.9. *For every* $a \in \mathbb{H}$ *and* $k \geq 7$,

$$I(\tau_1) := \int_{\tau_1 \leq |t| \leq 0.9n/\sigma_1^2} \left| \frac{\overline{P}_{1,n}(t;a)}{t} \right| dt$$

$$\leq c \max \left\{ \left(\frac{\Gamma_{4,l}}{n} \right)^{k/8}, \left(\frac{\Gamma_{3,k}^2 + \Gamma_{3,k}^2(a)}{n} \right)^{k/12} \right\}.$$

Proof. Using (6.21) and (5.47), we find that, for $k \geq 7$,

$$I(\tau_1) \leq \frac{c}{\Lambda_k^{1/2} \tau_1^{k/2-3} \sqrt{n}} \left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6} \right) \leq \frac{c_1}{\sqrt{n}} \left(\Gamma_{3,k} + \Gamma_{3,k}(a) \right) \left(\frac{\Gamma_{4,l}}{n} \right)^{k/8-3/4}.$$

First considering the case of $\frac{\Gamma_{3,k}+\Gamma_{3,k}(a)}{\sqrt{n}} \leq \left(\frac{\Gamma_{4,l}}{n} \right)^{3/4}$ and then the reverse inequality, we arrive at the assertion of Lemma 6.9.

Denote

$$\widetilde{g}(t;a) = \mathbb{E} \exp \left\{ it \big| \widetilde{Y}_0 - a \big|^2 \right\}, \quad \tau = (1 - 1/n)t, \quad \widetilde{A}_t = A_\tau,$$

$$(A_t x, \alpha - s A_t y) = \sum_{j=1}^{\infty} g_j(t)(x, e_j)\alpha_j - s(A_t x, A_t y), \quad x, y \in \mathbb{H}.$$

Lemma 6.10. *Let the condition (6.20) be fulfilled and let* $n \geq 5$. *Then*

$$\overline{P}_{1,n}(t;a) = n^{-1/2} \widehat{\widetilde{Q}}_1(t;a) + R, \tag{6.32}$$

where

$$\widehat{\widetilde{Q}}_1(t;a) = \widetilde{g}(t;a) \frac{s^3}{3!} \mathbb{E}_\alpha \mathbb{E}_{\overline{W}} \left(\widetilde{A}_t \overline{W}, \alpha - s\widetilde{A}_t a \right)^3$$

$$= \frac{s^4}{3!} \mathbb{E} \left[\exp \left\{ it \big| \widetilde{Y}_0 - a \big|^2 \right\} \mathbb{E}_{\overline{W}} \left(3 \big| \overline{W} \big|^2 \left(\overline{W}, Y_0 - a \right) + s^2 \left(\overline{W}, Y_0 - a \right)^3 \right) \right],$$

$$|R| \leq \frac{c_1}{n} \big| g(c_2 t) \big| \left((t\sigma^2)^4 + |t|\sigma^2 \right) \left[\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8} \right].$$

Proof. We use the representation (6.23). By the Taylor formula, we have

$$\mathbb{E}_{\overline{W}} \exp \left\{ s\xi/\sqrt{n} \right\} = A + r_1 + r_2, \tag{6.33}$$

where

$$A = \frac{s^3 \mathbb{E}\, \overline{W} \xi^3}{3! n^{3/2}},$$

$$r_1 = \frac{s^4}{3! n^2} \int_0^1 (1-\lambda)^3 \mathbb{E}\, \overline{W} \left[\xi^4 \exp\{\lambda_n s \xi\} \right] d\lambda, \quad \lambda_n = \lambda/\sqrt{n},$$

$$r_2 = \sum_{j=1}^{2} \frac{s^j}{j n^{j/2}} \mathbb{E}\, \overline{W} \xi^j.$$

Using (6.23) and (6.33) and taking the condition $|t| < n/\sigma_1^2$ into account, we obtain

$$\overline{P}_{1,n}(t; a) = B + R_1 + R_2, \tag{6.34}$$

where

$$B = \frac{s^3}{3! n^{1/2}} \mathbb{E} \left[\exp\left\{ it |\widetilde{Y}_0 - a|^2 \right\} \mathbb{E}_\alpha \mathbb{E}\, \overline{W} \xi^3 \right],$$

$$R_1 = \frac{s^4}{3! n} \mathbb{E} \left[\exp\left\{ it |\widetilde{Y}_0 - a|^2 \right\} \int_0^1 (1-\lambda)^3 \mathbb{E}_\alpha \mathbb{E}\, \overline{W} \left[\xi^4 \exp\{\lambda_n s \xi\} \right] d\lambda \right],$$

$$R_2 = \sum_{j=1}^{2} \frac{s^j}{j n^{j/2-1}} \mathbb{E} \left[\exp\left\{ it |\widetilde{Y}_0 - a|^2 \right\} \mathbb{E}_\alpha \mathbb{E}\, \overline{W} \xi^j \right].$$

Estimate $|R_1|$. Change the order of integration: $\mathbb{E}_\alpha \mathbb{E}\, \overline{W} = \mathbb{E}\, \overline{W} \mathbb{E}_\alpha$. By Lemma 6.4,

$$\mathbb{E}_\alpha \left[\xi^4 \exp\{\lambda_n s \xi\} \right]$$

$$= \exp\left\{ it \left(\lambda_n \overline{W}, \lambda_n \overline{W} + 2 (\widetilde{Y}_0 - a) \right) \right\} \sum_{k=0}^{2} c_k |\overline{W}|^{2k} \left[s \left(\overline{W}, \lambda_n \overline{W} + \widetilde{Y}_0 - a \right) \right]^{4-2k}.$$

Using this equality, it is easy to show that

$$|R_1| \le \frac{c_1}{n} \int_0^1 \left\{ \sum_{m=0}^{2} |t|^{4-m} \int |y|^{2m} \left| \mathbb{E} e^{it |\widetilde{Y}_0 - a + \lambda_n y|^2} \left(y, \lambda_n y + \widetilde{Y}_0 - a \right)^{4-2m} \right| \right.$$

$$\left. \times \left(\overline{F} + \Phi \right)(dy) \right\} (1-\lambda)^3 d\lambda$$

$$\le \frac{c_2}{n} \int_0^1 \left\{ \sum_{m=0}^{2} |t|^{4-m} \sum_{l=0}^{4-2m} n^{-(2-m-l/2)} \int |y|^{2(4-m-l)} \right.$$

$$\left. \times \left| \mathbb{E} e^{it |\widetilde{Y}_0 - a + \lambda_n y|^2} \left(y, \widetilde{Y}_0 - a \right)^l \right| \left(\overline{F} + \Phi \right)(dy) \right\} (1-\lambda)^3 d\lambda.$$

Therefore, by (6.14),

$$|R_1| \le \frac{c_3}{n} |g(c_4 t)| \sum_{m=0}^{2} |t|^{4-m} \sum_{l=0}^{4-2m} n^{-(2-m-l/2)}$$

$$\times \sum_{j=0}^{l} \sigma^j \int |y|^{2(4-m-l)+j} |(a, y)|^{l-j} (\overline{F} + \Phi)(dy).$$

Using (6.6) with $\theta = l - j$, $\lambda = 8 - 2m - l$, and $\nu = 4 - l + j$, we obtain

$$|R_1| \le \frac{c_5}{n} |g(c_4 t)| \sum_{m=0}^{2} \left(|t| \sigma^2 \right)^{4-m}$$

$$\times \sum_{l=0}^{4-2m} \sum_{j=0}^{l} \frac{\mathbb{E} |\overline{X}|^{4-l+j} |(a, \overline{X})|^{l-j} + \sigma^{4-l+j} (Ta, a)^{(l-j)/2}}{\sigma^{4+l-j}}.$$

It is easy to check that, for any four positive numbers a, b, c, and d, the following inequality holds:

$$\sum_{m=0}^{2} a^{4-m} \sum_{l=0}^{4-2m} \sum_{j=0}^{l} (b^{4-l+j} c^{l-j} + d^{l-j}) \le 44 (a^2 \vee a^4)(c^4 \vee b^4 \vee d^4 \vee 1).$$

Consequently,

$$|R_1| \le \frac{c_6}{n} |g(c_4 t)| \left((t\sigma^2)^4 + (t\sigma^2)^2 \right) \left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8} \right). \tag{6.35}$$

Estimate $|R_2|$. By (6.7), (6.4), and (6.5), we have

$$|R_2| \le |s| \sqrt{n} \left| \int \mathbb{E} \left[\exp\left\{ it |\widetilde{Y}_0 - a|^2 \right\} \mathbb{E}_\alpha \left(y, \alpha + s(\widetilde{Y}_0 - a) \right) \right] (\overline{F} - F)(dy) \right|$$

$$+ \frac{|s|^2}{2} \left| \int \mathbb{E} \left[\exp\left\{ it |\widetilde{Y}_0 - a|^2 \right\} \mathbb{E}_\alpha \left(y, \alpha + s(\widetilde{Y}_0 - a) \right)^2 \right] (\overline{F} - F)(dy) \right|$$

$$\le c_1 |g(c_2 t)| \mathbb{E} I(X; n) \left\{ |t| \sqrt{n} \left[|(a, X)| + \sigma |X| \right] \right.$$

$$\left. + t^2 \left[(a, X)^2 + \sigma^2 |X|^2 \right] + |t| |X|^2 \right\}$$

$$\le c_3 |g(c_2 t)| \left((t\sigma^2)^2 + |t| \sigma^2 \right) \left\{ \frac{\beta_4}{n\sigma^4} + \mathbb{E} I(X; n) \left[\sqrt{n} \frac{|(a, X)|}{\sigma^2} + \frac{(a, X)^2}{\sigma^4} \right] \right\}$$

$$\le \frac{c_3}{n} |g(c_2 t)| \left((t\sigma^2)^2 + |t| \sigma^2 \right) \left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8} \right). \tag{6.36}$$

Since, in view of (6.18),

$$\mathbb{E}\left[\exp\left\{it\big|\widetilde{Y}_0 - a\big|^2\right\}\mathbb{E}_\alpha \mathbb{E}_{\overline{W}}\,\xi^3\right] = \widetilde{g}(t;a)\mathbb{E}_\alpha \mathbb{E}_{\overline{W}}\big(\widetilde{A}_t\,\overline{W}, \alpha - s\widetilde{A}_t a\big)^3,$$

the claim of Lemma 6.10 follows from (6.34)–(6.36).

Lemma 6.11. *For every $n \geq 2$ and t satisfying the condition* (6.20), *the following equality is valid:*

$$n^{-1/2}\,\widehat{\widehat{Q}}_1(t;a) = n^{-1/2}\widehat{Q}_1(t;a) + R,$$

$$|R| \leq \frac{c_1}{n}\big|g(c_2 t)\big|\left(\big(t\sigma^2\big)^4 + \big(|t|\sigma^2\big)^3\right)\left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^4}\right).$$

Proof. According to the Taylor formula, we have

$$\widehat{\widehat{Q}}_1(t;a) = \frac{s^3}{3!}\left[\mathbb{E}\exp\big\{it|Y_0 - a|^2\big\}\right.$$

$$\left.\times\,\mathbb{E}_\alpha \mathbb{E}_{\overline{W}}\big(\overline{W}, \alpha + s(Y_0 - a)\big)^3 - \frac{1}{n}\int_0^1 f'(\theta)\Big|_{\theta = \frac{\lambda}{n}}\,d\lambda\right], \quad (6.37)$$

where

$$f(\theta) = \mathbb{E}\exp\left\{it\big|\sqrt{1-\theta}\,Y_0 - a\big|^2\right\}\mathbb{E}_\alpha \mathbb{E}_{\overline{W}}\left(\overline{W}, \alpha + s\big(\sqrt{1-\theta}\,Y_0 - a\big)\right)^3$$

$$= \mathbb{E}\exp\left\{it\big|\sqrt{1-\theta}\,Y_0 - a\big|^2\right\}$$

$$\times\,\mathbb{E}_{\overline{W}}\left[3s\big|\overline{W}\big|^2\big(\overline{W}, \sqrt{1-\theta}\,Y_0 - a\big) + s^3\big(\overline{W}, \sqrt{1-\theta}\,Y_0 - a\big)^3\right].$$

Estimate

$$f'(\theta) \equiv \mathbb{E}\left\{\exp\left\{it\big|\sqrt{1-\theta}\,Y_0 - a\big|^2\right\}\right.$$

$$\times\left[it\left(-|Y_0|^2 + \frac{(Y_0, a)}{\sqrt{1-\theta}}\right)\left[3s\big|\overline{W}\big|^2\big(\overline{W}, \sqrt{1-\theta}\,Y_0 - a\big)\right.\right.$$

$$\left. + s^3\big(\overline{W}, \sqrt{1-\theta}\,Y_0 - a\big)^3\right]$$

$$\left.\left. - \frac{3s\big|\overline{W}\big|^2\big(\overline{W}, Y_0\big)}{2\sqrt{1-\theta}} - \frac{3s^3\big(\overline{W}, \sqrt{1-\theta}\,Y_0 - a\big)^2\big(\overline{W}, Y_0\big)}{2\sqrt{1-\theta}}\right]\right\}$$

by means of (6.14) and (6.15). As a result of rather bulky computations, we find

$$\frac{|s|^3}{n}\big|f'(\theta)\big| \le \frac{c_1|g(c_2t)|}{n}\Big((t\sigma^2)^4 + (|t|\sigma^2)^3\Big)$$

$$\times \int\left[\frac{|y|^3}{\sigma^3} + \frac{|y|^2|(a,y)|}{\sigma^4}\left(1 + \frac{(Ta,a)^{1/2}}{\sigma^2}\right)\right.$$

$$\left. + \frac{|y|(a,y)^2}{\sigma^5} + \frac{|(a,y)|^3}{\sigma^6}\left(1 + \frac{(Ta,a)^{1/2}}{\sigma^2}\right)\right](\overline{F}+\Phi)(dy)$$

$$\underset{\text{(by (6.5))}}{<} \frac{c_3|g(c_2t)|}{n}\Big((t\sigma^2)^4 + (|t|\sigma^2)^3\Big)\left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6}\right)\left(1 + \frac{(Ta,a)^{1/2}}{\sigma^2}\right)$$

$$\le \frac{c_4|g(c_2t)|}{n}\Big((t\sigma^2)^4 + (|t|\sigma^2)^3\Big)\left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8}\right). \tag{6.38}$$

Taking (1.9), the properties of the random sequence α, and the distribution Φ into account, we derive on assuming (6.20) that

$$A \equiv \left|\frac{s^3}{3!}\mathbb{E}e^{it|Y_0-a|^2}\mathbb{E}_\alpha\mathbb{E}\,\underset{W}{}\big(\overline{W},\alpha+s(Y_0-a)\big)^3 - \widehat{Q}_1(t;a)\right|$$

$$= \frac{|s|^4}{3!}\left|\mathbb{E}e^{it|Y_0-a|^2}\int\Big[3|x|^2(x,Y_0-a)+s^2(x,Y_0-a)^3\Big](\overline{F}-F)(dx)\right|. \tag{6.39}$$

From (6.14) and (6.7) we conclude that

$$A \le c_1t^2\big|(c_2t)\big|\mathbb{E}I(X;n)\left[|X|^2|(X,a)| + |X|^3\sigma m(t;a)\right.$$

$$\left. + |t|\Big(|X|^3\sigma^3 m(t;a) + |X|^2\sigma^2|(X,a)| + |(X,a)|^3\Big)\right],$$

where

$$m(t;a) = \min\big\{1, |t|(Ta,a)^{1/2}\big\}.$$

Applying the trivial bound $m(t;a) \le 1$, we come to the inequality

$$A \le c_1t^2\big|g(c_2t)\big|\mathbb{E}I(X;n)\left[\sum_{\theta=0}^{1}|X|^{3-\theta}\big|(X,a)\big|^\theta\sigma^{1-\theta}\right.$$

$$\left. + |t|\sum_{\theta=0}^{3}|X|^{3-\theta}\big|(X,a)\big|^\theta\sigma^{3-\theta}\right].$$

From the inequality (6.4) with $p = 4$ and $\lambda = 3$ it follows that

$$A \leq \frac{ct^2\sigma^4}{\sqrt{n}}|g(c_2 t)|\left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8}\right)\left(1 + |t|\sigma^2\right). \qquad (6.40)$$

The claim of Lemma 6.11 ensues from (6.37)–(6.40).

Lemma 6.12. *The following inequality is valid for $l \geq 9$ and $n \geq 5$:*

$$\sup_{r>0}\left|\overline{Q}_{1,n}(r;a) - n^{-1/2}Q_1(r;a)\right| \leq \frac{c(l)}{n}\left[\Gamma_{4,l} + \Gamma_{4,l}(a) + \Gamma_{3,l}^2 + \Gamma_{3,l}^2(a)\right].$$

Proof. We have

$$\sup_{r>0}\left|\overline{Q}_{1,n}(r;a) - \frac{Q_1(r;a)}{\sqrt{n}}\right| \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\left|\overline{P}_{1,n}(t;a) - \widehat{Q}_1(t;a)/\sqrt{n}\right|}{|t|}dt. \qquad (6.41)$$

By (1.9) and (6.14), we conclude that

$$\left|\widehat{Q}_1(t;a)\right| \leq c_1|g(c_2 t)|\left(\left(|t|\sigma^2\right)^3 + (t\sigma^2)^2\right)\left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6}\right). \qquad (6.42)$$

Consequently, in view of (5.47),

$$\frac{1}{\sqrt{n}}\int_{|t|>cn/\sigma_1^2}\frac{\left|\widehat{Q}_1(t;a)\right|}{|t|}dt \leq c(l)\left(\frac{\beta_3}{\sigma^3} + \frac{\beta_3(a)}{\sigma^6}\right)\left(\frac{\sigma^2}{n\Lambda_l^{1/l}}\right)^{l/2}n^{5/2}$$

$$\leq c(l)\left(\frac{\Gamma_{3,l}^2 + \Gamma_{3,l}^2(a)}{n}\right)^{l/4-1} \qquad (6.43)$$

for every $l \geq 7$. Here we have used the evident bound $\sigma^2/\Lambda_l^{1/l} \leq \Gamma_{3,l}$. By analogy with (6.43), we can derive from (6.19) that, for $l \geq 3$,

$$\int_{|t|>cn/\sigma_1^2}\frac{\left|\overline{P}_{1,n}(t;a)\right|}{|t|}dt \leq c(l)\left(\frac{\sigma^2}{n\Lambda_l^{1/l}}\right)^{l/2}n \leq c(l)\left(\frac{\Gamma_{3,l}^2}{n}\right)^{l/4}. \qquad (6.44)$$

Next, applying Lemmas 6.10 and 6.11 and taking (5.48) into account, we obtain

$$\int_{|t|\leq cn/\sigma_1^2}\frac{\left|\overline{P}_{1,n}(t;a) - n^{-1/2}\widehat{Q}_1(t;a)\right|}{|t|}dt$$

$$\leq \frac{c(l)}{n}\left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8}\right)\int_{|t|\leq cn/\sigma_1^2}|g(c_2 t)|\left((t\sigma^2)^4 + |t|\sigma^2\right)\frac{dt}{|t|}$$

$$\leq \frac{c_1(l)}{n}\left(\frac{\beta_4}{\sigma^4} + \frac{\beta_4(a)}{\sigma^8}\right)\frac{\sigma^8}{\Lambda_l^{4/l}}$$

$$= \frac{c_1(l)}{n}\left[\Gamma_{4,l} + \Gamma_{4,l}(a)\right] \qquad (6.45)$$

for $l \geq 9$. The assertion of Lemma 6.12 follows from (6.41)–(6.45).

7. Proof of Theorem 1.2

Taking the proof of Theorem 1.1 (see Section 5) into account, we conclude that it suffices to obtain the estimate

$$\Delta_{1,n}(a) \le \frac{c}{n}\left[\Gamma_{4,13} + \Gamma_{3,13}^2 + \left(\frac{\sigma^2}{\Lambda_9^{1/9}}\right)^2 n_0 + \Gamma_{4,9}(a) + \Gamma_{3,13}^2(a)\right]. \qquad (7.1)$$

It is obvious that

$$\Delta_{1,n}(a) \le \widetilde{\Delta}_{1,n}(a) + \sup_{r>0}\left|\overline{Q}_{1,n}(r;a) - n^{-1/2}Q_1(r;a)\right|, \qquad (7.2)$$

where

$$\widetilde{\Delta}_{1,n}(a) = \sup_{r>0}\left|\mathbb{P}\Big(S_n \in B\big(a,\sqrt{r}\,\big)\Big) - \Phi\Big(B\big(a,\sqrt{r}\,\big)\Big) - \overline{Q}_{1,n}(r;a)\right|.$$

The quantity $\sup_{r>0}\left|\overline{Q}_{1,n}(r;a) - n^{-1/2}Q_1(r;a)\right|$ is estimated in Lemma 6.12.
   By analogy with (5.67), we have

$$\widetilde{\Delta}_{1,n}(a) \le \overline{\widetilde{\Delta}}_{1,n}(a) + c\beta_4/(n\sigma^4), \qquad (7.3)$$

where

$$\overline{\widetilde{\Delta}}_{1,n}(a) = \sup_{r>0}\left|\mathbb{P}\left(\left|n^{-1/2}\sum_{j=1}^{n}\overline{X}_j - a\right|^2 < r\right) - \Phi\Big(B\big(a;\sqrt{r}\,\big)\Big) - \overline{Q}_{1,n}(r;a)\right|.$$

By the Esseen inequality, the following relation is valid for every $\tau_0 > 0$:

$$\overline{\widetilde{\Delta}}_{1,n}(a) \le \frac{1}{\pi}J + \frac{4\pi}{\tau_0}\sup_{r>0}\left|\frac{d}{dr}\left[\Phi\Big(B\big(a;\sqrt{r}\,\big)\Big) + \overline{Q}_{1,n}(r;a)\right]\right|, \qquad (7.4)$$

where

$$J = \int_{|t|\le\tau_0}\left|\frac{R_{2,n}(t;a)}{t}\right|dt.$$

The quantity $\sup_{r>0}\left|\frac{d}{dr}\left[\Phi\big(B\big(a;\sqrt{r}\,\big)\big)+\overline{Q}_{1,n}(r;a)\right]\right|$ is estimated in Lemma 6.8. Collecting the bounds (7.2)–(7.4) and using Lemmas 6.8 and 6.12, under the condition (6.30), we obtain

$$\Delta_{1,n}(a) \le J + \frac{c}{\tau_0\Lambda_{\widetilde{l}}^{1/\widetilde{l}}} + \frac{c(\widetilde{l})}{n}\left[\Gamma_{4,\widetilde{l}} + \Gamma_{4,\widetilde{l}}(a) + \Gamma_{3,\widetilde{l}}^2 + \Gamma_{3,\widetilde{l}}^2(a)\right], \qquad (7.5)$$

where $n \ge 5$ and $\widetilde{l} \ge 9$.

Put $\tau_j = nt_j$, $j = 0, 1, 2, 3$, where t_j are the quantities defined in the proof of Lemma 5.8. Denote

$$J_1 = \int_{|t| \leq \tau_1} \left| \frac{R_{2,n}(t; a)}{t} \right| dt,$$

$$J_j = \int_{\tau_{j-1} < |t| < \tau_j} \left| \frac{\bar{g}_n(t; a)}{t} \right| dt, \quad j = 2, 3, 4,$$

$$J_5 = \int_{\tau_1 \leq |t| \leq \tau_0} \frac{\left| g(t; a) \right| + \left| \overline{P}_{1,n}(t; a) \right|}{|t|} dt.$$

By Lemmas 2.7, 2.8, and 4.5, the bounds for J_j ($j = 2, 3, 4$), i.e. (5.57), (5.63), and (5.64), hold as before. The integral J_1 is estimated in Lemma 6.2 (the inequality (6.2)). The integral J_5 is estimated by means of (5.65) and Lemma 6.9. As a result, since $J \leq \sum_{k=1}^{5} J_k$, we obtain the inequality

$$J \leq c(l; l') \left[\left(\frac{\Gamma_{4,l}}{n} \right)^{l'/8} + \left(\frac{\sigma^2}{\Lambda_{l'}^{1/l'}} \right)^2 \frac{n_0}{n} + \frac{\Gamma_{3,l}^2 + \Gamma_{3,l}^2(a)}{n} \right]$$
$$+ c \max \left\{ \left(\frac{\Gamma_{4,l}}{n} \right)^{k/8}, \left(\frac{\Gamma_{3,k}^2 + \Gamma_{3,k}^2(a)}{n} \right)^{k/12} \right\}, \tag{7.6}$$

where $l \geq 13$, $l' \geq 9$, and $k \geq 7$. Notice that $\Lambda_l^{1/l}$ decreases in l. Consequently, $\Gamma_{\mu,l}$ increases in this argument. Put $l = k = 13$ and $l' = \tilde{l} = 9$. Returning to (7.5), we come to (7.1) under the conditions (6.30) and (5.54) (with $l = 13$). It is easy to see that the bound (7.1) is valid as well in the case, when even one of these conditions is violated. This conclusion ensues from the inequality

$$\left| \Delta_{1,n}(a) \right| \leq 1 + \frac{1}{\sqrt{n}} \int_{-\infty}^{\infty} \frac{\left| \widehat{Q}_1(t; a) \right|}{|t|} dt \leq 1 + \frac{c}{\sqrt{n}} \left(\Gamma_{3,13} + \Gamma_{3,13}(a) \right)$$

which, in turn, follows from the definition of $\Delta_{1,n}(a)$, the inversion formula, (5.47), and (6.42). Theorem 1.2 is proven.

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