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A REFINEMENT OF THE ERROR ESTIMATE OF THE NORMAL APPROXIMATION IN A HILBERT SPACE

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Introduction. The Formulation of the Results. Fundamental Notations

Let H be a separable real Hilbert space, let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables with values in H, $\mathbf{E}X_1 = 0$, $\mathbf{E}|X_1|^4 < \infty$ ($|\cdot|$ is the norm in H), let T be the covariance operator of X_1, let Y_1 be a Gaussian random variable with values in H with the same covariance operator and $\mathbf{E}Y_1 = 0$, and let

$$\Delta_n = \sup_r \left| \mathbf{P}\left(\left| n^{-1/2} \sum_1^n X_j \right| < r \right) - \mathbf{P}(|Y_1| < r) \right|.$$

We introduce the notations: σ_j^2 are the eigenvalues of the operator T, $\sigma_j^2 \geq \sigma_{j+1}^2$, $j = 1, 2, \ldots$, $\{e_i\}_1^\infty$ is an orthonormal basis of the eigenvectors of the operator T, $\Lambda_l = \prod_1^l \sigma_i^2$, $\sigma^2 = \mathbf{E}|X_1|^2$, $\beta_k = \mathbf{E}|X_1|^k$, $k \geq 3$.

The symbols $c(\cdot)$, c, with or without indices, denote positive constants, depending only on the arguments indicated in the parentheses, and absolute constants, respectively. We allow the same notation for different constants.

In 1982, B. A. Zalesskii [1] has proved that for any $\delta > 0$ one has $\Delta_n = O(n^{-1+\varepsilon})$. Taking into account the remark made at the end of [1], this result can be formulated in the following manner: for any $\varepsilon > 0$ we have

$$\Delta_n \leqslant A_\varepsilon n^{-1+\varepsilon}, \tag{1}$$

if $\sigma^2_N \neq 0$ for sufficiently large $N = N(\varepsilon)$. In 1984, V. Yu. Bentkus [2] has obtained a more general result, from which there follows the estimate (1) for

$$\Delta_n(a) = \sup_r \left| \mathbf{P}\left(\left| n^{-1/2} \sum_1^n X_j - a \right| < r \right) - \mathbf{P}(|Y_1 - a| < r) - Q_{1,n}(a, r) \right|,$$

where $Q_{1,n}(a, r)$ is the first term of the asymptotic expansion and $Q_{1,n}(0, r) \equiv 0$.

It is natural to consider the problem of the dependence of A_ε on ε and on the distribution of X_1. In the special case when $H = R_l$ and T is the identity operator, the answer to this question can be found in Esseen's paper [3]: $\Delta_n \leq c(l)\beta_4^{3/2} n^{-l/(l+1)}$. Our purpose is to solve this problem in the case of an arbitrary Hilbert space.

We set

$$\Gamma_{1,l} = \beta_4 \sigma^4 \Lambda_l^{-4/l}, \quad \Gamma_{2,l} = \beta_3 \sigma^3 \Lambda_l^{-3/l}.$$

Without loss of generality, we can assume that

$$\Gamma_{1,l}/n \leqslant 1, \quad \Gamma_{2,l}/\sqrt{n} < 1. \tag{2}$$

THEOREM 1. For any $\delta > 0$ and integer $l \geq 7$ we have the estimate

$$\Delta_n \leqslant c(l, \delta) \left[(\Gamma_{1,l}/n)^{l/(l+4+\delta)} + \begin{cases} (\Gamma_{2,l}/\sqrt{n})^{2l/13}, & \text{if } 7 \leqslant l \leqslant 12, \\ \Gamma_{2,l}^2/n, & \text{if } l \geqslant 13 \end{cases} \right].$$

We denote $v(x) = \mathbf{E}\exp\{i(X_1, x)\}$, $x(l) = \sum_1^l (x, e_j) e_j$, where $(\cdot, \cdot)$ is the inner product.

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THEOREM 2. For any $\varepsilon_0 > 0$, $\delta > 0$ and integer $\ell \geq 13$ we have the estimate

$$\Delta_n \leqslant c_1(l)\,\Gamma_{1,l}/n + c(l,\delta)\left[\max\left(1, \left(\varepsilon_0\sigma_1\Gamma_{1,l}^{1/2}\right)^{l/2}\right)\left(\Gamma_{1,l}/n\right)^{l/(16+\delta)} + \Gamma_{2,l}^{2}/n\right] + c(l)\Big(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\Big)^{n/4}\ln(n/\Gamma_{1,l}).$$

We set

$$\varepsilon_0 = \left(\sigma_1\Gamma_{1,l}^{1/2}\right)^{-1}, \quad a = \sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|.$$

From Theorem 2 there follows

Corollary. For any $\delta > 0$ and integer $\ell \geq 13$ we have

$$\Delta_n \leqslant c_1(l)\,\Gamma_{1,l}/n + c(l,\delta)\left[\left(\Gamma_{1,l}/n\right)^{l/(16+\delta)} + \Gamma_{2,l}^{2}/n\right] + c(l)\,a^{n/4}\ln(n/\Gamma_{1,l}).$$

We do not propose to obtain an explicit form for $c(\ell)$ and $c(\ell, \delta)$ in the theorems and the corollaries, although this is feasible within the framework of our paper. If δ is fixed (for example, $\delta = 1/2$), then preliminary computations, with the use of I. F. Pinelis' result [4], show that these constants are bounded from above by the quantity $(c\ell)^{2\ell}$.

In the sequel we shall use the following notations. Let m be a natural number, $m \leq m_n \equiv [n/4] + 1$, where $[\cdot]$ is the integral part of a number. We define the random variables $X_k^{(m)}$ by the equality

$$\mathbf{P}\left(X_k^{(m)} \in A\right) = \mathbf{P}\left(X_k \in A / |X_k| < \sigma\sqrt{m}\right).$$

We set

$$\overline{X}_k = X_k^{(m_n)}, \quad \bar{g}_n(t) = \mathbf{E}\exp\left\{it\left|n^{-1/2}\sum_{1}^{n}\overline{X}_k\right|^2\right\},$$

$$g(t) = \mathbf{E}\exp\{it|Y_1|^2\}, \quad b^2(x) = \mathbf{E}(X_1, x)^2, \quad b_l^2(x) = \mathbf{E}(X_1, x(l))^2.$$

We shall denote one-dimensional random variables by ξ and η (possibly with indices), other random variables by X, Y, Z, V, W, an independent copy by X', and the symmetrization of X by $X^s = X - X'$.

By η_0 we denote the random variable with density

$$p(r) = (3/8\pi)[\sin(r/4)/(r/4)]^4, \quad -\infty < r < \infty.$$

If A is a set, then by I(A) we shall mean the indicator of this set.

If x and y are some complex functions, then the equality $x = O(|y|)$ means that $|x| \leq c|y|$.

In Sec. 1 we obtain estimates for $|\bar{g}_n(t)|$ (Lemmas 1.9, 1.10), and in Sec. 2 estimate for $|\bar{g}_n(t) - g(t)|$ (Lemma 2.5). For this an important role is played by the modifications of Gotze's Lemma (3.37) of [5], realized in Lemmas 1.4 and 2.1. In Sec. 3 we prove Theorems 1 and 2.

1. Auxiliary Statements. An Estimate of $|\bar{g}_n(t)|$

LEMMA 1.1. If Z_1, Z_2 are independent random variables with values in a locally convex space $\mathfrak{X}$ and if $\|\cdot\|$ is a seminorm in $\mathfrak{X}$, then for any $r_1, r_2 > 0$ we have

$$\mathbf{P}(\|Z_1\| < r_1)\,\mathbf{P}(\|Z_2\| < r_2) \leqslant \mathbf{P}(\|Z_1 + Z_2\| < r_1 + r_2).$$

Proof. The assertion of the lemma follows from the obvious inequality

$$\mathbf{P}(\|Z_1\| < r_1, \|Z_2\| < r_2) \leqslant \mathbf{P}(\|Z_1\| + \|Z_2\| < r_1 + r_2).$$

LEMMA 1.2. Let ξ be a random variable such that $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^4 < \infty$. Then

$$|\mathbf{E}\exp\{it\xi\}| \leqslant \exp\{-t^2\mathbf{E}\xi^2/4\}$$

for $|t| \leq (3\mathbf{E}\xi^2/2\mathbf{E}\xi^4)^{1/2}$.

Proof. We consider $\xi^s = \xi - \xi'$. Expanding $\mathbf{E}\exp\{it\xi^s\}$ by using the Taylor formula, we obtain

$$|\mathbf{E}\exp\{it\xi\}|^2 = \mathbf{E}\exp\{it\xi^s\} \leqslant 1 - t^2\mathbf{E}\xi^2 + t^4\mathbf{E}(\xi^s)^4/4! \leqslant 1 - t^2\mathbf{E}\xi^2/2 \tag{1.1}$$

for $|t| \le 2(3E\xi^2/E(\xi^s)^4)^{1/2}$. It is easy to see that

$$E(\xi^s)^4 = 2(E\xi^4 + 3(E\xi^2)^2) \leqslant 8E\xi^4. \tag{1.2}$$

The assertion of the lemma follows from (1.1), (1.2) and the inequality $1 - x \le \exp\{-x\}$.

Remark. There exist both one-dimensional and multidimensional variants of Lemma 1.2. We mention, for example, [6] (Lemma 12, n = 1) and [7] (Theorems 8.5-8.8, n = 1).

LEMMA 1.3. Let $\xi \ge 0$ and let F_ξ be the distribution function of ξ. If for some $\varepsilon > 0$ we have $F_\xi(r) \le Qr^\ell$ for $r \ge \varepsilon$, then

$$E\exp\{-\xi^2 t^2\} \leqslant (|t|^{-l}c(l) + \varepsilon^l)Q, \tag{1.3}$$

where $c(\ell) = \ell^{(\ell+1)/2}(2e)^{-1/2}$. In addition,

$$E\xi^{-t}I(\xi \geqslant \varepsilon) \leqslant 2\left(\frac{tQ}{l-t}\right)^{t/l} \tag{1.4}$$

for $0 < t < \ell$.

Proof. We have

$$E\exp\{-\xi^2 t^2\} = \int_0^\infty \exp\{-r^2 t^2\}\, dF_\xi(r) \equiv \int_0^\varepsilon + \int_\varepsilon^\infty. \tag{1.5}$$

Integrating by parts, we obtain

$$\int_\varepsilon^\infty \leqslant 2t^2 \int_\varepsilon^\infty F_\xi(r)\exp\{-r^2t^2\}\, r\,dr \leqslant 2t^2 Q \int_\varepsilon^\infty \exp\{-r^2t^2\}\, r^{l+1}dr.$$

Extending the domain of integration and performing the change of variable $r^2t^2 = x^2/2$, we find that

$$\int_\varepsilon^\infty \leqslant |t|^{-l} Q 2^{-l/2} J_l, \tag{1.6}$$

where $J_\ell = \int_0^\infty \exp\{-x^2/2\}\, x^{l+1}dx.$

Making use of the expression of J_ℓ in terms of the Gamma function and of known estimates, it is easy to show that $J_\ell \le \ell^{(\ell+1)/2}e^{-\ell}\cdot c$.

Now, it is easy to see that

$$\int_0^\varepsilon \leqslant F_\xi(\varepsilon) \leqslant Q\varepsilon^l. \tag{1.7}$$

The estimate (1.3) follows from (1.5)-(1.7).

We prove (1.4). We have

$$E\xi^{-t}I(\xi \geqslant \varepsilon) = \int_\varepsilon^\infty r^{-t}dF_\xi(r) \leqslant \int_\varepsilon^\infty F_\xi(r)\, t r^{-t-1}dr. \tag{1.8}$$

Let $\varepsilon_1 \equiv ((\ell - t)/t)^{1/\ell}Q^{-1/\ell} \ge \varepsilon$. Then

$$\int_\varepsilon^\infty r^{-t-1}F_\xi(r)\, dr = \int_\varepsilon^{\varepsilon_1} + \int_{\varepsilon_1}^\infty. \tag{1.9}$$

Obviously,

$$\int_{\varepsilon_1}^\infty \leqslant \varepsilon_1^{-t}t^{-1} \text{ and} \int_\varepsilon^{\varepsilon_1} \leqslant Q\int_\varepsilon^{\varepsilon_1} r^{l-t-1}dr \leqslant \frac{Q\varepsilon_1^{l-t}}{l-t}$$

for $\ell > t$. From the last inequalities, (1.8), (1.9), and the definition of ε_1 there follows that

$$\mathrm{E}\xi^{-l}I(\xi\geqslant\varepsilon)\leqslant 2\varepsilon_1^{-t}\leqslant 2\left(\frac{tQ}{l-t}\right)^{t/l}, \tag{1.10}$$

if $\varepsilon_1 \geq \varepsilon$. Inequality (1.10) holds also for $\varepsilon_1 < \varepsilon$. The lemma is proved.

In the sequel we shall use the notation

$$f_x(t)=\mathrm{E}\exp\{it(Y,x)\}.$$

LEMMA 1.4. Let X, Y, V be independent random variables, X + Y = Z, and let $g_1(\cdot)$, $g_2(\cdot)$ be measurable functions in H. Then

$$|\mathrm{E}\exp\{it|Z+V|^2\}g_1(X)g_2(V)|\leqslant c\mathrm{E}^{1/2}|f_{X^s}(2t)g_1(X)g_1(X')|\,\mathrm{E}|g_2(V)|.$$

Proof. Obviously,

$$\mathrm{E}\exp\{it|Z+V|^2\}g_1(X)g_2(V)=\mathrm{E}g_2(V)\mathrm{E}\{\exp\{it|Z+V|^2\}g_1(X)/V\}.$$

Making use of the properties of the conditional mathematical expectations and of Cauchy's inequality, we obtain for any $a \in$ H that

$$|\mathrm{E}\exp\{it|Z+a|^2\}g_1(X)|\leqslant \mathrm{E}^{1/2}|\mathrm{E}\{g_1(X)\exp\{it(|X|^2+$$
$$+2(X,a)+2(X,Y))\}/Y\}|^2=\mathrm{E}^{1/2}g_1(X)\overline{g_1(X')}\exp\{it(|X|^2-|X'|^2+$$
$$+2(X^s,a))\}\,\mathrm{E}\{\exp\{2it(X^s,Y)\}/X,X'\}\leqslant \mathrm{E}^{1/2}|f_{X^s}(2t)g_1(X)g_1(X')|,$$

where $\overline{g}_1$ is the complex conjugate function of g_1. From here there follows the assertion of the lemma.

Let m and μ be natural numbers such that

$$m\leqslant\mu\leqslant 2m\leqslant 2m_n=2([n/4]+1). \tag{1.11}$$

We ask additionally that

$$m\geqslant 21. \tag{1.12}$$

We denote $\alpha_\mu = \sqrt{\mu/n}$, $\alpha = \sqrt{m/n}$. From (1.11) there follows

$$\alpha^2\leqslant\alpha_\mu^2\leqslant 2\alpha^2. \tag{1.13}$$

For the Lemmas 1.5-1.8 we set

$$X=n^{-1/2}\left(\sum_{j=n-\mu+1}^{n-\mu+\nu}Y_j+\sum_{j=n-\mu+\nu+1}^{n}X_j^{(m)}\right),$$

where $0 \leq \nu \leq \mu$, Y_j are independent and distributed in the same way as Y_1.

We define the quantities M_ℓ and $\varepsilon(\ell)$ by the equalities

$$\mathrm{P}(|\eta_0|<M_l)=(1/2)^l,\quad \varepsilon(l)=M_l\left(\sum_{j=1}^{l}\sigma_j^2/a_j^2\right)^{1/2}. \tag{1.14}$$

Here $a_j = \sigma_j(3n/2\beta_{4j}\ell^3)^{1/2}$, $\beta_{4j} = \mathrm{E}(X_1, e_j)^4$.

LEMMA 1.5. For all $0 \leq \nu \leq \mu$ we have the following estimates: if $r \geq \varepsilon(\ell)$, then

$$\mathrm{P}(b_l(X^s)<r)\leqslant c(l)\Lambda_l^{-1}\alpha_\mu^{-l}r^l; \tag{1.15}$$

if $0 < r < \varepsilon(\ell)$, then

$$\mathrm{P}(b_l(X^s)<r)\leqslant c(l)\Lambda_l^{-1}\alpha_\mu^{-l}\varepsilon^l(l). \tag{1.16}$$

Proof. First we prove the lemma for $\nu = 0$. Let $W=\sum_1^l\eta_je_j$, where η_j, $j = \overline{1, \ell}$ are independent random variables, distributed in the same way as η_0/a_j.

For seminorm in H we consider $b_\ell(x)$, $x \in$ H. From Lemma 1.1 there follows

$$\mathrm{P}(b_l(X^s)<r-\varepsilon(l))\leqslant \mathrm{P}(b_l(X^s+W)<r)/\mathrm{P}(b_l(W)<\varepsilon(l)) \tag{1.17}$$

for $r \geq \varepsilon(\ell)$.

Now, by virtue of (1.14) and the definition of W, we have

$$\mathrm{P}(b_l(W)<\varepsilon(l))=\mathrm{P}\left(\sum_1^l\eta_j^2\sigma_j^2<\varepsilon^2(l)\right)\geqslant P^l(|\eta_0|<M_l)=\frac{1}{2}.$$

Then from (1.17) we obtain

$$\mathbf{P}(b_l(X^s) < r - \varepsilon(l)) \leqslant 2\mathbf{P}(b_l(X^s + W) < r) = 2\int_{R_l} p(x) I\left(x \in R_l: \sum_1^l x_j^2\sigma_j^2 < r^2\right) dx \tag{1.18}$$

for $r \geq \varepsilon(\ell)$, where $p(\cdot)$ is the distribution density of the random variable $X^s(\ell) + W$, $x = \sum_1^l x_j e_j$, $dx = dx_1 dx_2 \ldots dx_l$.

We find a uniform estimate of $p(x)$. We denote $A = \{\theta \in R_\ell: |\theta_j| \leq a_j, j = \overline{1, \ell}\}$, where $\theta_j = (\theta, e_j)$. Since $\mathbf{E}\exp\{it\eta_0\} = 0$ for $|t| > 1$, it follows that for $\theta \in \bar{A} \equiv R_\ell \setminus A$ we have

$$\mathbf{E}\exp\{i(\theta, W)\} = \prod_{j=1}^l \mathbf{E}\exp\{i\eta_j\theta_j\} = 0.$$

By virtue of the inversion formula and the independence of X^s and W, we have

$$p(x) = (2\pi)^{-l}\int_{R_l} \exp\{-i(\theta, x)\}\mathbf{E}\exp\{i(\theta, X^s)\}\mathbf{E}\exp\{i(\theta, W)\} d\theta \leqslant (2\pi)^{-l}\int_{R_l} |\mathbf{E}\exp\{i(\theta, X)\}|^2 I(A) d\theta. \tag{1.19}$$

We have

$$|\mathbf{E}\exp\{i(\theta, X)\}|^2 = |\mathbf{E}\exp\{i(\theta, X_1^{(m)} n^{-1/2})\}|^{2\mu}.$$

By Lemma 4 of [8] we have

$$|\mathbf{E}\exp\{i(\theta, X_1^{(m)})\}|^{2\mu} \leqslant |\mathbf{E}\exp\{i(\theta, X_1)\}|^{2\mu} \exp\{4\rho_\theta\mu\} + (1/2 + \rho_\theta)^{2\mu}, \tag{1.20}$$

where $\rho_\theta = \mathrm{var}[\mathbf{P}((\theta, X_1^{(m)}) < r) - \mathbf{P}((\theta, X_1) < r)]$.

From (1.19), (1.20) there follows

$$p(x) \leqslant (2\pi)^{-l}\int_{R_l} [|\mathbf{E}\exp\{in^{-1/2}(\theta, X_1)\}|^{2\mu} \exp\{4\rho_\theta\mu\} + (1/2 + \rho_\theta)^{2\mu}] I(A) d\theta. \tag{1.21}$$

Since for $\theta \in A$ we have

$$\mathbf{E}(X_1, \theta)^4 \leqslant l^3 \sum_1^l \theta_j^4 \beta_{4j} \leqslant 3n\mathbf{E}(X_1, \theta)^2/2;$$

Lemma 1.2 implies

$$|\mathbf{E}\exp\{in^{-1/2}(X_1, \theta)\} I(A)| \leqslant \exp\{-\mathbf{E}(X_1, \theta)^2/4n\} = \exp\left\{-\sum_1^l \theta_j^2\sigma_j^2/4n\right\}. \tag{1.22}$$

It is easy to see that

$$\rho_\theta \leqslant 2\mathbf{P}(|X_1| \geqslant \sigma\sqrt{m})/\mathbf{P}(|X_1| < \sigma\sqrt{m}) \leqslant 2/(m-1).$$

Then from (1.21), (1.22) we obtain the estimate

$$p(x) \leqslant (2\pi)^{-l}\int_{R_l}\left[\exp\left\{-\mu\sum_1^l \theta_j^2\sigma_j^2/2n\right\}\exp\{4\rho_\theta\mu\} + (1/2 + \rho_\theta)^{2\mu} I(A)\right] d\theta \leqslant$$

$$\leqslant (2\pi)^{-l/2}\alpha_\mu^{-l}\Lambda_l^{-1/2}\exp\left\{\frac{8\mu}{m-1}\right\} + \left(1/2 + \frac{2}{m-1}\right)^{2\mu}\cdot \mu^{l/2}(2^{1/2}/\pi l^2)^l \alpha_\mu^{-l}\Lambda_l^{-1/2}.$$

From here, by virtue of (1.11) and (1.12) there follows

$$p(x) \leqslant c(l)\alpha_\mu^{-l}\Lambda_l^{-1/2}. \tag{1.23}$$

The inequalities (1.18) and (1.23) lead to the assertion of Lemma 1.5 for $\nu = 0$.

Let $0 < \nu \leq \mu$. Then either X contains at least half of the terms of the form $X_j^{(m)}$ or in X one has more than half of the terms of the form Y_j. It is well known that the convolution of the densities does not exceed the maximum of any of the components of this convolution of densities. Therefore, in the first case, for the estimation of $p(x)$ it is sufficient that in the previous computations we replace μ by $[\mu/2]$, while in the second case we use the fact $[\mu/2]$ terms are Gaussian. In both cases the estimate (1.23) holds; consequently, also Lemma 1.5 holds.

For Lemma 1.6 we set $Y = n^{-1/2} \sum_{1}^{n-2m_n} \overline{X}_j$.

LEMMA 1.6. For any t > 0 nd $0 \le \nu \le \mu$ we have the estimate

$$\mathrm{E}|X|^t \leqslant c(t)(\sigma\alpha_\mu)^t. \tag{1.24}$$

In addition, for any t > 0 we have

$$\mathrm{E}|Y|^t \leqslant c(t)\sigma^t. \tag{1.25}$$

Proof. Let $\nu = 0$. From [9] there follows that for $t \ge 2$ we have

$$\mathrm{E}|X|^t \leqslant c(t) n^{-t/2} [\mu^{t/2}\sigma^t + (\mu \mathrm{E}|X_1^{(m)}|^2)^{t/2} + (\mu|\mathrm{E}X_1^{(m)}|)^t + \mu\mathrm{E}|X_1^{(m)}|^t]. \tag{1.26}$$

It is easy to see that

$$\mathrm{E}|X_1^{(m)}|^t = \mathrm{E}|X_1|^t I(|X_1| < \sigma\sqrt{m})/\mathrm{P}(|X_1| < \sigma\sqrt{m}) \leqslant 2\sigma^t m^{-1+t/2}$$

for $t \ge 2$ and

$$|\mathrm{E}X_1^{(m)}|^t = (|\mathrm{E}X_1 I(|X_1| \geqslant \sigma\sqrt{m})|/\mathrm{P}(|X_1| < \sigma\sqrt{m}))^t \leqslant (2\sigma m^{-1/2})^t.$$

From the last inequalities and also from (1.26) and (1.13) there follows (1.24) for $t \ge 2$, $\nu = 0$. Applying Holder's inequality, we obtain that the estimate (1.24) is valid also for $0 < t < 2$, $\nu = 0$. Inequality (1.25) is proved in a similar manner.

If $0 < \nu \le \mu$, then applying a result from [9] to $n^{-1/2} \sum_{n-\mu+1}^{n-\mu+\nu} Y_j$ and $n^{-1/2} \sum_{n-\mu+\nu+1}^{n} X_j^{(m)}$

separately, we obtain again (1.26). The lemma is proved.

LEMMA 1.7. For each $0 < \gamma < \ell/2$ we have the estimate

$$\mathrm{P}(b_l(X^s) \geqslant \varepsilon(l), |X^s|^4/b^2(X^s) \geqslant |t|) \leqslant c(l,\gamma)|t|^{-\gamma}(\Lambda_l^{-1/l}\alpha_\mu\sigma^2)^{2\gamma}.$$

Proof. We denote $A = \{b_\ell(X^s) \ge \varepsilon(\ell)\}$. By Chebyshev's inequality, for any $\gamma > 0$ we have

$$\mathrm{P}(|X^s|^4/b^2(X^s) > |t|, A) \leqslant \mathrm{E}(I(A)|X^s|^{4\gamma}/b^{2\gamma}(X^s))|t|^{-\gamma} \leqslant \mathrm{E}(I(A)|X|^{4\gamma}b^{-2\gamma}(X^s))(16|t|^{-1})^\gamma. \tag{1.27}$$

By virtue of Hölder's inequality, we have

$$\mathrm{E}[I(A)|X|^{4\gamma}b^{-2\gamma}(X^s)] \leqslant \mathrm{E}^{1/q}|X|^{4\gamma q}\mathrm{E}^{1/p}[I(A)b^{-2\gamma p}(X^s)], \tag{1.28}$$

where $p, q > 0$, $p^{-1} + q^{-1} = 1$.

Making use of Lemmas 1.3 (inequality (1.4)) and 1.5, we obtain that for any $0 < \gamma < \ell/2p$ we have

$$\mathrm{E}^{1/p}b^{-2\gamma p}(X^s)I(A) \leqslant c(l,\gamma,p)(\Lambda_l^{-1/l}\alpha_\mu^{-1})^{2\gamma}. \tag{1.29}$$

We set $0 < \gamma < \ell/2$, $p = \ell/2\gamma$. The assertion of Lemma 1.7 follows from (1.27)-(1.29) and Lemma 1.6.

For Lemma 1.8 we introduce the notations: $Y = n^{-1/2} \sum_{1}^{n-2m} \overline{X}_j$ or $Y = n^{-1/2} \sum_{1}^{n-2m} Y_j$.

LEMMA 1.8. Let ℓ be a natural number and let $0 < \gamma < \ell/2$. Then we have the estimate

$$\mathrm{E}^{1/2}|f_{X^s}(t)| \leqslant c(l)[(\Lambda_l^{1/2}\alpha_\mu^{l/2}|t|^{l/2}+1)^{-1} + (\Gamma_{1,l}/\alpha_\mu^2 n)^{l/4}] + c(l,\gamma)(\alpha_\mu\Lambda_l^{1/2}|t|\Gamma_{1,l}^{1/2}/n^{1/2})^\gamma + \left(\frac{3}{5}\right)^{\frac{n}{4}-1}. \tag{1.30}$$

In addition, if $m = m_n$, then for any p > 1 we have

$$\mathrm{E}^{1/2p}|f_{X^s}(t)|^p \leqslant c(l)\left[(\Lambda_l^{1/2p}|t|^{l/2p}+1)^{-1} + (\Gamma_{1,l}/n)^{\frac{l}{4p}}\right] + c(l,\gamma)(\Lambda_l^{1/l}|t|\Gamma_{1,l}^{1/2}/n^{1/2})^{\gamma/p} + c(3/5)^{n/4}. \tag{1.31}$$

Proof. Assume first that $Y = n^{-1/2} \sum_{1}^{n-2m} \overline{X}_j$. Then

$$f_x(t) = (\mathrm{E}\exp\{itn^{-1/2}(\overline{X}_1, x)\})^{n-2m}.$$

From Lemma 4 of [8] we obtain

$$|f_x(t)| \leqslant |\mathrm{E}\exp\{itn^{-1/2}(X_1,x)\}|^{n-2m}\exp\{2\rho(n-2m)\} + (1/2+\rho)^{n-2m}. \tag{1.32}$$

Here $\rho = \mathrm{var}[P((\overline{X}_1, x) < r) - P((X_1, x) < r)]$. It is easy to see that

$$\rho \leqslant 2/n_1, \tag{1.33}$$

where $n_1 = [n/4]$.

Now, by virtue of Lemma 1.2 we have

$$|\mathbf{E}\exp\{itn^{-1/2}(X_1, x)\}| \leqslant \exp\{-b^2(x)t^2/4n\} \tag{1.34}$$

for $|t| \leq (3nb^2(x)/2\beta_4(x))^{1/2}$, $\beta_4(x) = E(X_1, x)^4$. The inequalities (1.32)-(1.34), taking into account (1.11), imply

$$\mathbf{E}^{1/2}|f_{X^s}(t)| \leqslant c\left[\mathbf{E}^{1/2}\exp\{-b^2(X^s)t^2(n-2m)/4n\} + \mathbf{P}^{1/2}(2\beta_4(x^s)t^2 > 3nb^2(X^s)) + (1/2 + 2/n_1)^{(n-2m)/2}\right] \equiv c[a_1 + a_2 + a_3]. \tag{1.35}$$

From Lemma 1.3 (inequality (1.3)) and Lemma 1.5 (inequality (1.15)) we obtain

$$\mathbf{E}\exp\{-b^2(X^s)t^2\} \leqslant \Lambda_l^{-1}\alpha_\mu^{-l}(|t|^{-l} + \varepsilon^l(l))c(l).$$

Now from the condition (1.11) and from the fact that $a_1 \leq 1$ there follows

$$a_1 \leqslant c(l)\left[(\Lambda_l^{1/2}\alpha_\mu^{l/2}|t|^{l/2} + 1)^{-1} + \alpha_\mu^{-l/2}(\Gamma_{3,l}/n)^{l/4}\right], \tag{1.36}$$

where $\Gamma_{3,l} = \Lambda_l^{-2/l}\sum_1^l \beta_{4j}$.

We estimate a_2. For any t we have

$$\mathbf{P}(\beta_4(X^s)/b^2(X^s) > |t|) \leqslant \mathbf{P}(b_l(X^s) < \varepsilon(l)) + \mathbf{P}(b_l(X^s) \geqslant \varepsilon(l),\ |X^s|^4/b^2(X^s) \geqslant |t|\beta_4^{-1}).$$

From the last inequality, Lemmas 1.5 (inequality (1.16)) and 1.7 there follows that for any $0 < \gamma < \ell/2$ we have

$$\mathbf{P}(\beta_4(X^s)/b^2(X^s) > |t|) \leqslant \Lambda_l^{-1}\alpha_\mu^{-l}\varepsilon^l(l)c(l) + c(l,\gamma)(\alpha_\mu^2\sigma^4\beta_4|t|^{-1}\Lambda_l^{-2/l})^\gamma \leqslant$$
$$\leqslant \alpha_\mu^{-l}(\Gamma_{3,l}/n)^{l/2}c_1(l) + (\alpha_\mu^2\Gamma_{1,l}|t|^{-1})^\gamma\Lambda_l^{2\gamma/l}c(l,\gamma).$$

Returning to (1.35), we obtain

$$a_2 = \mathbf{P}^{1/2}(\beta_4(X^s)/b^2(X^s) > 3n/2t^2) \leqslant \alpha_\mu^{-l/2}(\Gamma_{3,l}/n)^{l/4}c_1(l) + (\alpha_\mu|t|\Gamma_{1,l}^{1/2}/n^{1/2})^\gamma\Lambda_l^{\gamma/l}c(l,\gamma). \tag{1.37}$$

It is easy to see that conditions (1.11) and (1.12) imply

$$a_3 \leqslant (3/5)^{n/4-1}. \tag{1.38}$$

Since $\Gamma_{3,\ell} \leq \Gamma_{1,\ell}$, from (1.35)-(1.38) there follows the estimate (1.30) for $Y = n^{-1/2}\sum_1^{n-2m}\overline{X}_j$. If $Y = n^{-1/2}\sum_1^{n-2m}Y_j$, then the proof of (1.30) becomes simpler.

Estimate (1.31) is proved in a similar manner. The difference consists in the fact that now one has to raise both sides of the inequality (1.32) to the power p, to obtain for $E^{1/2}|f_{X^s}(t)|^p$ an estimate similar to (1.35), and to extract from both sides of this estimate the root of order p. It remains to make use of the inequalities (1.36)-(1.38). The lemma is proved.

For Lemmas 1.9, 1.10 we set

$$n^{-1/2}\sum_1^n \overline{X}_j = Y + V + X_0,$$

where $Y = n^{-1/2}\sum_1^{n-2m}\overline{X}_j$, $\quad V = n^{-1/2}\sum_{n-2m+1}^{n-\mu}\overline{X}_j,\ X_0 = n^{-1/2}\sum_{n-\mu+1}^{n}\overline{X}_j.$

<u>LEMMA 1.9</u>. Let ℓ be a natural number and let $0 < \gamma < \ell/2$. Then

$$|\overline{g}_n(t)| \leqslant \left[(\Lambda_l^{1/2}\alpha^{l/2}|t|^{l/2} + 1)^{-1} + \alpha^{-l/2}(\Gamma_{1,l}/n)^{l/4}\right]c(l) + (\alpha|t|\Gamma_{1,l}^{1/2}/n^{1/2})^\gamma\Lambda_l^{\gamma/l}c(l,\gamma) + \exp\{-m/2\}. \tag{1.39}$$

<u>Proof</u>. We denote

$$g_{\mu;m}(t) = \mathbf{E}\left\{\exp\left\{it\left|n^{-1/2}\sum_1^n \overline{X}_k\right|^2\right\}\Big/\xi_j = 0,\right.$$
$$j = \overline{n-2m+1, n-\mu},\ \xi_k = 1,\ k = \overline{n-\mu+1, n}\},$$

where $\xi_\nu = I(|X_\nu| < \sigma\sqrt{m})$. We have

$$\bar{g}_n(t) = \sum_{\mu=m}^{2m} g_{\mu,m}(t)\, \mathbf{P}\left(\sum_{n-2m+1}^{n} \xi_j = \mu\right) + \mathbf{E}\left(\exp\left\{it\left|n^{-1/2}\sum_{1}^{n} \bar{X}_k\right|^2\right\};\ \sum_{n-2m+1}^{n} \xi_j < m\right). \qquad (1.40)$$

Applying Hoeffding's result on bounded random variables (see, for example, [10]), we obtain

$$\mathbf{P}\left(\sum_{n-2m+1}^{n} \xi_j < m\right) \leqslant \mathbf{P}\left(\sum_{n-2m+1}^{n} (\xi_j - E\xi_j) < 2 - m\right) < e^{-m/2} \qquad (1.41)$$

under the condition (1.12).

We estimate $|g_{\mu,m}(t)|$. We denote

$$A_1 = \{\xi_k = 1,\ k = \overline{n-\mu+1, n}\},\ A_1' = \{\xi_k' = 1,\ k = \overline{n-\mu+1, n}\},$$
$$A_2 = \{\xi_j = 0,\ \ j = \overline{n-2m+1, n-\mu}\}.$$

Obviously, $I(A_1)$ and $I(A_2)$ are measurable functions of X_0 and V, respectively. Therefore, making use of Lemma 1.4, we have

$$|g_{\mu,m}(t)| = |\mathbf{E}\{\exp\{it|X_0 + Y + V|^2/A_1, A_2\}| = \qquad (1.42)$$
$$= |\mathbf{E}\exp\{it|X_0 + Y + V|^2\} I(A_1) I(A_2)|/\mathbf{P}(A_1)\mathbf{P}(A_2) \leqslant \mathbf{E}^{1/2}|f_{X^s}(2t)|\, I(A_1) I(A_1')/\mathbf{P}(A_1) \equiv \mathbf{E}^{1/2}|f_{X^s}(2t)|,$$

where $X = n^{-1/2}\sum_{j=n-\mu+1}^{n} X_j^{(m)}$.

From (1.12) there follows that

$$(3/5)^{(n/4)-1} \leqslant \exp\{-m/2\}. \qquad (1.43)$$

Therefore, (1.40)-(1.42) and Lemma 1.8 (inequality (1.30)) lead to the assertion of Lemma 1.9.

<u>LEMMA 1.10</u>. Let ℓ be a natural number and let $\varepsilon_0 > 0$. Then

$$|\bar{g}_n(t)| \leqslant c_1(l)\Lambda_l^{-1/2}(\max(1, \sigma_1\varepsilon_0 n^{1/2})/|t|)^{1/2} + c_2(l)(\Gamma_{1,l}/n)^{1/4} + c\left(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\right)^{n/4} + \exp\left\{-\frac{m}{2}\right\}.$$

<u>Proof</u>. We consider the random events

$$A_t = \{3nb^2(X^s)/2\beta_4(X^s) \geqslant t^2\},\quad B_{t,\varepsilon_0} = \{2|tX^s(l)|/n^{1/2} \geqslant \varepsilon_0\},$$

where $X = n^{-1/2}\sum_{n-\mu+1}^{n} X_j^{(m)}$. Making use of the inequalities (1.42), (1.32)-(1.34), and also of the conditions (1.11), (1.12), we obtain

$$|g_{\mu,m}(t)| \leqslant \mathbf{E}^{1/2}|f_{X^s}(2t)| \leqslant c\,[\mathbf{E}^{1/2} I(A_t)\exp\{-b^2(X^s)t^2(n-2m)/n\} +$$
$$+ \mathbf{E}^{1/2} I(\bar{A}_t B_{t,\varepsilon_0})\left(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\right)^{(n-2m)/2}] + \mathbf{E}^{1/2} I(\bar{A}_t\bar{B}_{t,\varepsilon_0}) + (1/2 +$$
$$+ 2/n_1)^{(n-2m)/2} \leqslant c\,\mathbf{E}^{1/2}\exp\{-b^2(X^s)t^2/2\} + \left(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\right)^{n/4} + (3/5)^{n/4-1} + \mathbf{P}^{1/2}(\bar{B}_{t,\varepsilon_0}). \qquad (1.44)$$

Then, we have

$$\mathbf{P}(\bar{B}_{t,\varepsilon_0}) = \mathbf{P}(|X^s(l)| < \varepsilon_0 n^{1/2}/2|t|) \leqslant \mathbf{P}(b_l(X^s) < \varepsilon_0\sigma_1 n^{1/2}/2|t|).$$

Therefore, from Lemma 1.5 there follows that

$$\mathbf{P}^{1/2}(\bar{B}_{t,\varepsilon_0}) \leqslant \Lambda_l^{-1/2}\alpha_\mu^{-l/2}((\varepsilon_0\sigma_1 n^{1/2}/2|t|)^{l/2} + \varepsilon^{l/2}(l))\,c(l). \qquad (1.45)$$

We select $m = m_n$. Now (1.44), (1.45), and (1.36) lead to

$$|g_{\mu,m}(t)| \leqslant c_1\Lambda_l^{-1/2}(\max(1, \varepsilon_0\sigma_1 n^{1/2})/|t|)^{l/2} + c_2(l)(\Gamma_{3,l}^{1/2}/n^{1/2})^{l/2} + \left(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\right)^{n/4} + \left(\frac{3}{5}\right)^{\frac{n}{4}-1}. \qquad (1.46)$$

Since $\Gamma_{3,\ell} \leq \Gamma_{1,\ell}$, the assertion of Lemma 1.10 follows from (1.40), (1.41), (1.46), and (1.43).

2. Auxiliary Statements. An Estimate of $|\bar{g}_n(t) - g(t)|$

LEMMA 2.1. Let Z = X + Y, where the random variables X and Y are independent, and let Ψ be a measurable mapping $H \times H \to H$. Then for any $x \in H$, $y \in H$, natural numbers j, k, and real numbers $p > 0$, $q > 0$, $1/p + 1/q = 1$, we have the estimate

$$|\mathrm{E}(Z, x)^j (Z, y)^k \exp\{it((Z, \Psi(x, y)) + |Z|^2)\}| \leqslant c(j, k)|x|^j |y|^k \sum_{\nu=0}^{k+j} \mathrm{E}^{1/2}|Y|^{2\nu} \mathrm{E}^{1/q}|X|^{(k+j-\nu)q} \mathrm{E}^{1/2p}|f_{X^s}(2t)|^p, \quad (2.1)$$

where $f_x(t) = \mathrm{E}\exp\{it(Y, x)\}$.

Proof. We set

$$\mathrm{E}_{jk}(x, y) = \mathrm{E}(Z, x)^j (Z, y)^k \exp\{it((Z, \Psi(x, y)) + |Z|^2)\},$$

$$\mathrm{E}(j_1, j_2, j_3, j_4, x, y) = \mathrm{E}(X, x)^{j_1} (Y, x)^{j_2} (X, y)^{j_3} (Y, y)^{j_4} \exp\{it((Z, \Psi(x, y)) + |Z|^2)\}.$$

It is easy to see that

$$\mathrm{E}_{jk}(x, y) = \sum_{\substack{j_1+j_2=j,\\ j_3+j_4=k}} \mathrm{E}(j_1, j_2, j_3, j_4, x, y) C_j^{j_1} C_k^{j_3}. \quad (2.2)$$

Making use of Hölder's inequality and the properties of conditional mathematical expectations, we have the following chain of inequalities:

$$|\mathrm{E}(j_1, j_2, j_3, j_4, x, y)| = |\mathrm{E}(Y, x)^{j_2} (Y, y)^{j_4} \exp\{it((Y, \Psi(x, y)) +$$
$$+ |Y|^2)\} \mathrm{E}\{(X, x)^{j_1} (X, y)^{j_3} \exp\{it((X, \Psi(x, y)) + 2(X, Y) + |X|^2)\}/Y\}| \leqslant$$
$$\leqslant \mathrm{E}^{1/2}(Y, x)^{2j_2} (Y, y)^{2j_4} \mathrm{E}^{1/2}|\mathrm{E}\{(X, x)^{j_1} (X, y)^{j_3} \exp\{it((X, \Psi(x, y)) +$$
$$+ 2(X, Y) + |X|^2)\}/Y\}|^2 \leqslant |x|^{j_2} |y|^{j_4} \mathrm{E}^{1/2}|Y|^{2(j_2+j_4)} \mathrm{E}^{1/2}\mathrm{E}\{(X, x)^{j_1} \times$$
$$\times (X', x)^{j_1} (X, y)^{j_3} (X', y)^{j_3} \exp\{it((X^s, \Psi(x, y)) + 2(X^s, Y) + |X|^2 -$$
$$- |X'|^2)\}/Y\} = |x|^{j_2} |y|^{j_4} \mathrm{E}^{1/2}|Y|^{2(j_2+j_4)} \mathrm{E}^{1/2}(X, x)^{j_1} (X', x)^{j_1} (X, y)^{j_3} \times$$
$$\times (X', y)^{j_3} \exp\{it((X^s, \Psi(x, y)) + |X|^2 - |X'|^2)\} \times$$
$$\times \mathrm{E}\{\exp\{2it(X^s, Y)\}/X, X'\} \leqslant |x|^j |y|^k \mathrm{E}^{1/2}|Y|^{2(j_2+j_4)} \mathrm{E}^{1/2}|f_{X^s}(2t)| \times$$
$$\times |X|^{j_1+j_3} |X'|^{j_1+j_3} \leqslant |x|^j |y|^k \mathrm{E}^{1/2}|Y|^{2(j_2+j_4)} \mathrm{E}^{1/q}|X|^{(j_1+j_3)q} \mathrm{E}^{1/2p}|f_{X^s}(2t)|^p. \quad (2.3)$$

From (2.2) and (2.3) there follows (2.1). The lemma is proved.

We set

$$e(\cdot, x, t) = \exp\{it((\cdot, x) + |x|^2)\}, \quad J(\cdot, t) = \int e(\cdot, x, t) Q^{2*}(dx).$$

Here $\int \equiv \int_H$, Q is a generalized measure on H with Q(H) = 0, $Q^{2*} = Q * Q$ is the convolution of measures.

We shall use the notations

$$\mathrm{E}f(U) = \int f(x) Q(dx),$$
$$\mathrm{E}f(U, V) = \int\int f(x, y) Q(dx) Q(dy),$$

where f(x) and f(x, y) are arbitrary Borel functions.

In the sequel it is assumed that the summation indices are nonnegative.

LEMMA 2.2. We have the expansion

$$J(\cdot, t) = \mathrm{E}^2 e(\cdot, U, t) + 2it\mathrm{E}[\sum_{j<1, k<1} (it)^{j+k} (\cdot, U)^j (\cdot, V)^k +$$
$$+ 2(1 + it(\cdot, U)) r_1(\cdot, V, t) + r_1(\cdot, U, t) r_1(\cdot, V, t)](U, V) + 2(it)^2 \mathrm{E}[1 +$$
$$+ 2r_2(\cdot, V, t) + r_2(\cdot, U, t) r_2(\cdot, V, t)](U, V)^2 + 4(it)^3 \mathrm{E}e(\cdot, U, t) e(\cdot, V, t) r_3(U, V, t), \quad (2.4)$$

where

$$r_1(\cdot, x, t) = e(\cdot, x, t) - 1 - it(\cdot, x) = -t^2(\cdot, x)^2 \int_0^1 \exp\{it\lambda(\cdot, x)\}(1-\lambda)\,d\lambda + it\exp\{it(\cdot, x)\}|x|^2 \int_0^1 \exp\{it\lambda|x|^2\}\,d\lambda,$$

$$r_2(\cdot, x, t) = e(\cdot, x, t) - 1 = it(\cdot, x)\int_0^1 \exp\{it\lambda(x, \cdot)\}\,d\lambda + O(|t|^{1/2}|x|),$$

$$r_3(x, y, t) = (x, y)^3 \int_0^1 \exp\{2it\lambda(x, y)\}(1-\lambda)^2\,d\lambda.$$

Proof. Making use of the identity

$$e(\cdot, x+y, t) = e(\cdot, x, t)e(\cdot, y, t)\exp\{2it(x, y)\}$$

and the expansion

$$\exp\{it(x, y)\} = 1 + it(x, y) - \frac{t^2}{2}(x, y)^2 + \frac{(it)^3}{2}r_3(x, y, t/2),$$

we obtain

$$J(\cdot, t) = \mathbf{E}e(\cdot, U+V, t) = \mathbf{E}^2 e(\cdot, U, t) + 2it\mathbf{E}e(\cdot, U, t)e(\cdot, V, t)(U, V) -$$

$$- 2t^2\mathbf{E}e(\cdot, U, t)e(\cdot, V, t)(U, V)^2 + 4(it)^3\mathbf{E}r_3(U, V, t)e(\cdot, U, t)e(\cdot, V, t) \equiv \sum_1^4 A_j. \tag{2.5}$$

Obviously,

$$e(\cdot, x, t) = (1 + it|x|^2 \int_0^1 \exp\{it\lambda|x|^2\}\,d\lambda)\,\exp\{it(\cdot, x)\} = 1 + it(\cdot, x) + r_1(\cdot, x, t),$$

From here

$$A_2 = 2it\mathbf{E}\Big[\sum_{j<1, k<1}(it)^{j+k}(\cdot, U)^j(\cdot, V)^k + 2(1 + it(\cdot, U))r_1(\cdot, V, t) + r_1(\cdot, U, t)r_1(\cdot, V, t)\Big](U, V). \tag{2.6}$$

Similarly,

$$A_3 = -2t^2\mathbf{E}[1 + 2r_2(\cdot, V, t) + r_2(\cdot, U, t)r_2(\cdot, V, t)](U, V)^2. \tag{2.7}$$

Taking into account that

$$e(\cdot, x, t) = (1 + O(|t|^{1/2}|x|))\exp\{it(\cdot, x)\} = 1 + it(\cdot, x)\int_0^1 \exp\{it\lambda(\cdot, x)\}\,d\lambda + O(|t|^{1/2}|x|),$$

from (2.5)-(2.7) we obtain the assertion of the lemma.

LEMMA 2.3. We have the representation

$$\mathbf{E}e(\cdot, U, t) = \mathbf{E}\Big[1 + it((\cdot, U) + |U|^2) + \frac{(it)^2}{2}(\cdot, U)^2 + \frac{(it)^3}{2}(\cdot, U)^3\int_0^1 \exp\{it\lambda(\cdot, U)\} \times$$

$$\times (1-\lambda)^2\,d\lambda - t^2|U|^2(\cdot, U)\int_0^1 \exp\{it\lambda(\cdot, U)\}\,d\lambda + r_4(U, t)\exp\{it(\cdot, U)\}\Big], \tag{2.8}$$

where $r_4(x, t) = O(|t|^{3/2}|x|^3)$.

Proof. Obviously,

$$\exp\{it|x|^2\} = 1 + it|x|^2 + O(|t|^{3/2}|x|^3). \tag{2.9}$$

From here

$$e(\cdot, x, t) = 1 + it\Big((\cdot, x) + |x|^2\Big) - \frac{t^2}{2}(\cdot, x)^2 +$$

$$+ \frac{(it)^3}{2}(\cdot, x)^3\int_0^1 \exp\{it\lambda(\cdot, x)\}(1-\lambda)^2\,d\lambda - t^2|x|^2(\cdot, x)\int_0^1 \exp\{it\lambda(\cdot, x)\}\,d\lambda + \exp\{it(\cdot, x)\}O(|t|^{3/2}|x|^3).$$

It remains to take the mathematical expectation of both parts of this equality.

LEMMA 2.4. We have the expansion

$$\mathrm{E}\exp\{it|Y_1+y|^2\}=g(t)\left[1+it\sum_1^\infty\frac{(y,e_j)^2}{1-2it\sigma_j^2}+O(t^2|y|^4)\right].$$

Proof. The assertion of the lemma follows from the known equality

$$g(t,y)\equiv\mathrm{E}\exp\{it|Y_1+y|^2\}=g(t)\exp\left\{it\sum_1^\infty\frac{(y,e_j)^2}{1-2it\sigma_j^2}\right\} \tag{2.10}$$

(see, for example, [11]) and from the estimate

$$\left|it\sum_1^\infty(y,e_j)^2/(1-2it\sigma_j^2)\right|^2\leqslant t^2|y|^4.$$

We note that in [12] one derives a representation of $g(t, y)$, distinct from (2.10). However, it is less convenient for our purpose.

LEMMA 2.5. For any $p > 1$, natural number ℓ, $0 < \gamma < \ell/2$, we have

$$|\bar g_n(t)-g(t)|\leqslant n^{-1}\Big\{c\frac{\beta_4}{\sigma_4}(\sigma^4t^2+\sigma^2|t|)\prod_{j=1}^\infty(1+t^2\sigma_j^4)^{-1/4}+$$
$$+c(p)\frac{\beta_3^2}{\sigma^6}(\sigma^{12}t^6+\sigma^2|t|)\Big[c(l)\big((\Lambda_l^{1/2p}|t|^{l/2p}+1)^{-1}+(\Gamma_{1,l}/n)^{l/4p}\big)+c(l,\gamma)\big(|t|\Gamma_{1,l}^{1/2}/n^{1/2}\big)^{\gamma/p}\Lambda_l^{\gamma/lp}+\left(\frac{3}{5}\right)^{\frac{n}{4}}\Big]\Big\}.$$

Proof. Let F, $\bar F$, Φ be the distributions of the random variables X_1, $\bar X_1$, Y_1, respectively, let $\bar G=\bar F-\Phi$.

From the elementary identity

$$a^n-b^n=n(a-b)b^{n-1}+(a-b)^2\sum_{l=0}^{n-2}(l+1)b^la^{n-l-2}$$

there follows that for any Borel set $A\subset H$ we have

$$\mathbf{P}\left(n^{-1/2}\sum_1^n\bar X_k\in A\right)-\mathbf{P}\left(n^{-1/2}\sum_1^nY_k\in A\right)$$
$$=n\int\mathbf{P}\left(n^{-1/2}\left(\sum_1^{n-1}Y_k+y\right)\in A\right)\bar G(dy)+\sum_{l=0}^{n-2}(l+1)\int\mathbf{P}\left(Z_l+n^{-1/2}y\in A\right)\bar G^{2*}(dy),$$

where $Z_\ell=n^{-1/2}\left(\sum_1^lY_j+\sum_{l+1}^{n-2}\bar X_j\right)$, $l=\overline{0,n-2}$. From here,

$$\bar g_n(t)-g(t)=b_n^{-1}n\mathrm{E}\int\exp\left\{it\left|n^{-1/2}\left(\sum_1^{n-1}Y_j+y\right)\right|^2\right\}\bar G_1(dy)+b_n^{-2}\sum_{l=0}^{n-2}(l+1)\mathrm{E}\int\exp\{it|Z_l+n^{-1/2}y|^2\}\bar G_1^{2*}(dy). \tag{2.11}$$

Here $b_n=\mathbf{P}(|X_1|<\sigma\sqrt{m_n})$, $\bar G_1=b_n\bar G$.

First we estimate

$$\mathrm{E}_l\equiv\mathrm{E}\int\exp\{it|Z_l+n^{-1/2}y|^2\}\bar G_1^{2*}(dy)=\mathrm{E}\exp\{it|Z_l|^2\}\int\exp\{it((n^{-1/2}y,2Z_l)+$$
$$+|n^{-1/2}y|^2)\}\bar G_1^{2*}(dy)=\mathrm{E}\exp\{it|Z_l|^2\}J(2Z_l,t),$$

$Q(A)=\bar G_1(\sqrt{n}A)$. To this end we make use of the representation (2.4) and we shall estimate the terms occurring in this sum.

It is easy to see that

$$\mathrm{E}_l=\Omega_1+2n^{-1/2}\sum_{j<1,k<1}\mathrm{E}(j,k)(it/\sqrt{n})^{j+k+1}+\sum_1^5R_j+\Omega_2, \tag{2.12}$$

where

$$\Omega_1=\mathrm{E}\exp\{it|Z_l|^2\}\left(\int e(2Z_l,n^{-1/2}x,t)\bar G_1(dx)\right)^2,$$

$$\Omega_2 = -\frac{2t^2}{n}\mathrm{E}\exp\{it|Z_l|^2\}\int\int (x, y)^2 \overline{G}_1(dx)\overline{G}_1(dy),$$

$$\mathrm{E}(j, k) = \mathrm{E}\exp\{it|Z_l|^2\}\int\int (2Z_l, x)^j (2Z_l, y)^k (x, y)\overline{G}_1(dx)\overline{G}_1(dy),$$

$$R_1 = \frac{4it}{n}\mathrm{E}\exp\{it|Z_l|^2\}\int\int (itn^{-1/2}(2Z_l, x) + 1)(x, y) r_1(2Z_l, n^{-1/2}y, t)\overline{G}_1(dx)\overline{G}_1(dy),$$

$$R_2 = \frac{2it}{n}\mathrm{E}\exp\{it|Z_l|^2\}\int\int r_1(2Z_l, n^{-1/2}x, t)\, r_1(2Z_l, n^{-1/2}y, t)(x, y)\overline{G}_1(dx)\overline{G}_1(dy),$$

$$R_3 = -\frac{4t^2}{n}\mathrm{E}\exp\{it|Z_l|^2\}\int\int (x, y)^2 r_2(2Z_l, n^{-1/2}x, t)\overline{G}_1(dx)\overline{G}_1(dy),$$

$$R_4 = -\frac{2t^2}{n}\mathrm{E}\exp\{it|Z_l|^2\}\int\int r_2(2Z_l, n^{-1/2}x, t)\, r_2(2Z_l, n^{-1/2}y, t)(x, y)^2\overline{G}_1(dx)\overline{G}_1(dy),$$

$$R_5 = 4(it)^3\mathrm{E}\exp\{it|Z_l|^2\}\int\int e(2Z_l, n^{-1/2}x, t)\, e(2Z_l, n^{-1/2}y, t)\, r_3(n^{-1/2}x, n^{-1/2}y, t)\overline{G}_1(dx)\overline{G}_1(dy).$$

It is easy to see that for $0 \le j \le 1$, $0 \le k \le 1$ we have

$$\int (x, y)^j (\cdot, y)^k \overline{G}_1(dy) = \int (x, y)^j (\cdot, y)^k [-I_y F + (1 - b_n)\Phi](dy), \tag{2.13}$$

where $I_y = I(|y| \ge \sigma\sqrt{m_n})$. From here there follows that for $0 \le j \le 1$, $0 \le k \le 1$ we have

$$\int\int (x, y)(\cdot, x)^j (\cdot, y)^k \overline{G}_1(dx)\overline{G}_1(dy) =$$
$$= \int\int (x, y)(\cdot, x)^j (\cdot, y)^k [I_x I_y F(dx) F(dy) - I_x\cdot(1 - b_n)\times$$
$$\times F(dx)\Phi(dy) - I_y(1 - b_n)F(dy)\Phi(dx) + (1 - b_n)^2\Phi(dx)\Phi(dy)]. \tag{2.14}$$

In Lemma 2.1 we set now $Z = Z_\ell$, $Y \equiv Y_{(\ell)} = n^{-1/2}\sum_{1}^{n-2m_n} Y_j$ for $\ell \ge n - 2m_n$, $Y \equiv Y_{(\ell)} = n^{-1/2}\sum_{2m_n-1}^{n-2} \overline{X}_j$ for $\ell < n - 2m_n$, $X \equiv X_{(\ell)} = Z - Y$, $\Psi(x, y) \equiv 0$. Then, we denote $N_p(t) = \max_{0<l<n-2} E^{1/2p}|f_{X^s}(2t)|^p$, $p > 1$.

Applying Lemmas 2.1 and 1.6, we obtain

$$|\mathrm{E}\exp\{it|Z_l|^2\}\int\int (x, y)(\cdot, x)^j(\cdot, y)^k I_x I_y F(dx)F(dy)| \leqslant$$
$$\leqslant c(p)N_p(t)\sigma^{j+k}\mathrm{E}(|X_1|^{j+1}; |X_1| \geqslant \sigma\sqrt{m_n})\, E(|X_1|^{k+1};$$
$$|X_1| \geqslant \sigma\sqrt{m_n}) \leqslant c_1(p)N_p(t)\beta_3^2\sigma^{2(j+k)-4}n^{\frac{j+k}{2}-2}. \tag{2.15}$$

It is easy to see that for any $\nu \ge 0$ we have

$$1 - b_n \leqslant \mathrm{E}|X_1|^\nu/(\sigma\sqrt{m_n})^\nu. \tag{2.16}$$

In (2.16) we set $\nu = 3$. Then, similarly to (2.15),

$$(1 - b_n)|\mathrm{E}\exp\{it|Z_l|^2\}\int\int (x, y)(Z_l, x)^j (Z_l, y)^k I_x F(dx)\Phi(dy)| \leqslant c(p)N_p(t)\beta_3^2\sigma^{2(j+k)-4}n^{(j-5)/2}. \tag{2.17}$$

For $\nu = 2$ from (2.16) we obtain

$$b_n \geqslant 1 - 1/m_n \geqslant 20/21. \tag{2.18}$$

From (2.14)-(2.18) there follows that

$$|E(j, k)| \leqslant c(p)N_p(t)\beta_3^2\sigma^{2(j+k)-4}n^{\frac{j+k}{2}-2}. \tag{2.19}$$

Similarly, setting this time $\Psi(x, y) = \lambda y n^{-1/2}$ and using instead of (2.14) the equality

$$\int\int (x, y)(\cdot, x)^j(\cdot, y)^2\overline{G}_1(dx)\overline{G}_1(dy) = \int\int (x, y)(\cdot, x)^j(\cdot, y)^2[-I_x F + (1 - b_n)\Phi](dx)\overline{G}_1(dy), \; j \leqslant 1, \tag{2.20}$$

we find

$$|R_1| \leqslant K_p(n, t)(\sigma^8 t^4 + \sigma^4 t^2), \tag{2.21}$$

where $K_p(n, t) = c(p)N_p(t)\beta_3^2/\sigma^6 n^3$.

Setting $\Psi(x, y) = (\lambda_1 x + \lambda_2 y)n^{-1/2}$ and making use of equalities (2.14), (2.20), and also of the estimate

$$\left|\int\int (x, y)(\cdot, x)^2(\cdot, y)^2 \bar{G}_1(dx)\bar{G}_1(dy)\right| \leqslant c|\cdot|^4\beta_3^2, \tag{2.22}$$

we conclude that

$$|R_2| \leqslant K_p(n, t)(\sigma^{10}|t|^5 + \sigma^6|t|^3). \tag{2.23}$$

Now, applying, according to the circumstances, one of the relations (2.14), (2.20), (2.22), we obtain

$$\begin{aligned} &|R_3| \leqslant K_p(n, t)(\sigma^6|t|^3 + \sigma^5|t|^{5/2}),\\ &|R_4| \leqslant K_p(n, t)(\sigma^8 t^4 + \sigma^6|t|^3),\\ &|R_5| \leqslant K_p(n, t)\sigma^6|t|^3,\\ &|\Omega_2| \leqslant K_p(n, t)\sigma^4 t^2. \end{aligned} \tag{2.24}$$

For the estimation of Ω_1 one has to use first the representation (2.8). After that, proceeding just as before, we find the estimate

$$|\Omega_1| \leqslant K_p(n, t)(\sigma^{12}t^6 + (\sigma^2 t)^2). \tag{2.25}$$

Collecting now the estimates (2.19), (2.21), (2.23)-(2.25), we obtain as a consequence of (2.12)

$$|\mathrm{E}_t| \leqslant K_p(n, t)(\sigma^{12}t^6 + \sigma^2|t|). \tag{2.26}$$

We consider now the quantity

$$D \equiv \mathbf{E}\int \exp\left\{it\left|n^{-1/2}\left(\sum_1^{n-1} Y_j + y\right)\right|^2\right\}\bar{G}_1(dy).$$

By virtue of Lemma 2.4 we have

$$D = g(a_n t)\int\left(1 + it\sum_1^\infty (n^{-1/2}y, e_j)^2/(1 - 2ita_n\sigma_j^2) + O(t^2|n^{-1/2}y|^4)\right)\bar{G}_1(dy),$$

where $a_n = (n - 1)/n$.

Similarly to (2.13) we have

$$\int (y, e_j)^2\bar{G}_1(dy) = \int (y, e_j)^2[-I_y F + (1 - b_n)\Phi](dy).$$

From here, taking into account (2.16),

$$\left|\int\sum_1^\infty \frac{(n^{-1/2}y, e_j)^2}{1 - 2ita_n\sigma_j^2}\bar{G}_1(dy)\right| = \left|\int\sum_1^\infty \frac{(n^{-1/2}y, e_j)^2}{1 - 2ita_n\sigma_j^2}[-I_y F + (1 - b_n)\Phi](dy)\right| \leqslant c\beta_4/\sigma^2 n^2.$$

As a result we obtain

$$|D| \leqslant |g(a_n t)|(\sigma^4 t^2 + \sigma^2|t|)\beta_4/\sigma^4 n^2. \tag{2.27}$$

From (2.11), (2.26), (2.27), (2.18), and (1.30) there follows the assertion of Lemma 2.5.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We set

$$\bar{\Delta}_n = \sup_{r>0}\left|\mathbf{P}\left(\left|n^{-1/2}\sum_1^n \bar{X}_j\right| < r\right) - \mathbf{P}(|Y_1| < r)\right|.$$

It is easy to see that

$$\Delta_n \leqslant \bar{\Delta}_n + c\beta_4/\sigma^4 n. \tag{3.1}$$

For any $\tau > 0$, by Esseen's inequality and Lemma 8 of [8] we have

$$\bar{\Delta}_n \leqslant c\left[\int_{|t|<\tau}|t^{-1}[\bar{g}_n(t) - g(t)]|dt + (\sigma_1\sigma_2\tau)^{-1}\right]. \tag{3.2}$$

We select τ from the condition

$$(n/\Gamma_{1,l})^{\eta}\left(\Lambda_l^{1/l}\tau\Gamma_{1,l}/n\right)^{\gamma}=(\sigma_1\sigma_2\tau)^{-1}, \tag{3.3}$$

where γ and η are positive constants, satisfying the condition $2\eta < \gamma < \ell/2$; more precisely, the relation between η, γ and ℓ will be determined in (3.21).

We set

$$\tau_1=\left(\sqrt{n}/\Gamma_{2,l}\right)^{1/3}\Lambda_l^{-1/l},\ \tau_2=2\,(n/\Gamma_{1,l})^{\frac{\gamma}{2\gamma+l}}\Lambda_l^{-1/l},$$
$$\tau_3=(n/\Gamma_{1,l})^{1/2-\eta/\gamma}\,\tau_2/2.$$

The cases $\tau_2 \geq \tau_1$ and $\tau_2 < \tau_1$ are possible. We shall restrict ourselves to the case $\tau_2 \geq \tau_1$.

We define a function $\alpha(t)$ by the relation

$$(t\alpha(t))^{-l/2}\Lambda_l^{-1/2}=\left(\Lambda_l^{1/l}t\alpha(t)\,\Gamma_{1,l}^{1/2}/n^{1/2}\right)^{\gamma},\ t>0. \tag{3.4}$$

It is easy to see that

$$t\alpha(t)=\Lambda_l^{-1/l}(n/\Gamma_{1,l})^{\gamma/(2\gamma+l)}. \tag{3.5}$$

Bounds of τ_2 and τ_3 are easily described in terms of $\alpha(t)$; namely:

$$\alpha(\tau_2)=1/2,\quad \alpha(\tau_3)=(\Gamma_{1,l}/n)^{1/2-\eta/\gamma}. \tag{3.6}$$

From (3.3) we obtain

$$\tau=(n/\Gamma_{1,l})^{\frac{\gamma-\eta}{\gamma+1}}\left(\Lambda_l^{\gamma/l}\sigma_1\sigma_2\right)^{-\frac{1}{\gamma+1}}. \tag{3.7}$$

Simple computations show that

$$\tau/\tau_3=(n/\Gamma_{1,l})^{\frac{\gamma^2+\eta}{\gamma(\gamma+1)}-\frac{1}{2}-\frac{\gamma}{2\gamma+l}}\left(\Lambda_l^{1/l}/\sigma_1\sigma_2\right)^{\frac{1}{\gamma+1}}. \tag{3.8}$$

If $\gamma > 1$, $\ell > 4$, then by virtue of (2) and (3.8) we have $\tau > \tau_3$. It is easy to see that

$$\int\limits_{|t|<\tau}\left|t^{-1}(\bar g_n(t)-g(t))\right|dt\leqslant\int\limits_{|t|<\tau_1}\left|t^{-1}(\bar g_n(t)-g(t))\right|dt+\sum_{k=1}^{3}\int\limits_{\tau_k<|t|<\tau_{k+1}}\left|\frac{\bar g_n(t)}{t}\right|dt+\int\limits_{|t|>\tau_1}\left|\frac{g(t)}{t}\right|dt\equiv\sum_{k=1}^{5}J_k, \tag{3.9}$$

where $\tau_4 = \tau$.

We estimate J_1. As a consequence of (3.4) and (3.6), for any $0 < \tau < \tau_2$ we have

$$(t/2)^{-l/2}\Lambda_l^{-1/2}>\left(\Lambda_l^{1/l}t\Gamma_{1,l}^{1/2}/n^{1/2}2\right)^{\gamma}. \tag{3.10}$$

Making use of Lemma 2.5 and taking into account (3.10), we obtain for $\ell \leq 12p$, $p > 1$, the estimate

$$J_1\leqslant c\beta_4/n\Lambda_l^{2/l}+\left[c(l,\gamma)\left(\Gamma_{2,l}/\sqrt{n}\right)^{l/6p}\Psi_{l/2p}(n)\ c(l)(\Gamma_{1,l}/n)^{l/4p}+(3/5)^{n/4}\right]c(p), \tag{3.11}$$

where $\Psi_k(n) = \ln(n/\Gamma_{2,\ell}^2)$ for $k = 6$, $\Psi_k(n) = 1$ for $k \neq 6$. We have also applied here the equality $\tau_1^6\beta_3^2\sigma^6/n = 1$.

In the sequel we shall make use of (2) without any special mention.

It is known [13] that $\Delta_n \leq \beta_3 A n^{-1/2}$ for $\beta_3 < \infty$, where A depends on the operator T (see also [8]). Therefore, in the inequality (3.11) it makes sense to consider only $\ell \geq 7$. In the sequel we shall have in view only such ℓ.

Setting $p = 14/13$, we obtain the estimate

$$J_1\leqslant c(l,\gamma)\,(\Gamma_{2,l}/\sqrt{n})^{2l/13}+c(l)\,\Gamma_{1,l}/n \tag{3.12}$$

for $7 \leq \ell \leq 12$. For the sake of the simplicity of the formulation of the result, we disregard the possibility of writing down a somewhat sharper estimate than (3.12).

If $\ell \geq 13$, then by the same method one obtains the estimate

$$J_1\leqslant\left(c(l,\gamma)\,\Gamma_{2,l}^2+c(l)\,\Gamma_{1,l}\right)/n. \tag{3.13}$$

The integrals J_2 - J_4 will be estimated with the aid of Lemma 1.9, selecting α in an appropriate manner.

First we consider the integral J_2. We set $\alpha = 1/2$. Taking into account (3.10), we have

$$J_2 \leqslant c(l)\{\Lambda_l^{-1/2}\tau_1^{-l/2} + [(\Gamma_{1,l}/n)^{l/4} + c\cdot\exp\{-n/8\}]\ln(\tau_2/\tau_1)\} \equiv \sum_1^3 J_{2j}.$$

Making use of the fact that $\tau_2/\tau_1 \leq \tau_2\Lambda_\ell^{1/\ell}$, we obtain

$$J_{2j} \leqslant c(l)(\Gamma_{1,l}/n)^{l/5}, \quad j = 2, 3.$$

As a result,

$$J_2 \leqslant c(l)[(\Gamma_{2,l}/\sqrt{n})^{l/6} + (\Gamma_{1,l}/n)^{l/5}]. \tag{3.14}$$

We define a natural number $m(t)$ by the equality

$$|m(t)/n - \alpha^2(t)| = \min_m |m/n - \alpha^2(t)|.$$

We estimate J_3. We set $m = m(t)$. Then $\alpha = \sqrt{m(t)/n}$ and, as one can easily see,

$$|\alpha^2 - \alpha^2(t)| \leqslant 1/n. \tag{3.15}$$

By virtue of (3.4),(3.5), and (3.15), we have

$$J_3 \leqslant c(l,\gamma)(\Gamma_{1,l}/n)^{\frac{\gamma l}{2(2\gamma+l)}}\ln\frac{\tau_3}{\tau_2} + c(l)(\Gamma_{1,l}/n)^{l/4}\int_{\tau_2}^{\tau_3} t^{-1}\alpha^{-l/2}(t)\,dt + c\cdot\exp\left\{-\frac{1}{4}(n/\Gamma_{1,l})^{\frac{2\eta}{\gamma}}\right\}\ln\frac{\tau_3}{\tau_2}.$$

Since $\alpha(t) = ht^{-1}$, where $h = \Lambda_\ell^{-1/\ell}(n/\Gamma_{1,\ell})^{\gamma/(2\gamma+\ell)}$, we have

$$\int_{\tau_2}^{\tau_3} t^{-1}\alpha^{-l/2}(t)\,dt = (th^{-1})^{l/2}\frac{2}{l}\Big|_{\tau_2}^{\tau_3} \leqslant \alpha^{-l/2}(\tau_3) = (\Gamma_{1,l}/n)^{-l/4+l\eta/2\gamma}.$$

Consequently,

$$J_3 \leqslant \left[c(l,\gamma)(\Gamma_{1,l}/n)^{\frac{\gamma l}{2(2\gamma+l)}} + c\exp\left\{-\frac{1}{4}(n/\Gamma_{1,l})^{\frac{2\eta}{\gamma}}\right\}\right]\ln(\tau_3/\tau_2) + c(l)(\Gamma_{1,l}/n)^{\eta l/2\gamma}. \tag{3.16}$$

For the estimation of J_4, we set $m = m(\tau_3)$. By virtue of (3.4) and (3.6), for any $t > \tau^3$ we have

$$[t(n/\Gamma_{1,l})^{\eta/\gamma-1/2}]^{-l/2}\Lambda_l^{-1/2} < [\Lambda_l^{-1/l}t(\Gamma_{1,l}/n)^{1-\eta/\gamma}]^{\gamma}.$$

From here, taking into account (3.15), we obtain

$$J_4 \leqslant c(l,\gamma)[\Lambda_l^{1/l}\tau(\Gamma_{1,l}/n)^{1-\eta/\gamma}]^{\gamma} + \left[c(l)(\Gamma_{1,l})^{\eta l/2\gamma} + \exp\left\{-\frac{1}{4}(n/\Gamma_{1,l})^{\frac{2\eta}{\gamma}}\right\}\right]\ln(\tau/\tau_3) \equiv \sum_1^3 J_{4k}. \tag{3.17}$$

From (3.3) and (3.7), taking into account the inequality $\Lambda_\ell^{1/\ell} \leq \sigma_1\sigma_2$, there follows

$$J_{41} \leqslant c(l,\gamma)(\Gamma_{1,l}/n)^{\frac{\gamma-\eta}{\gamma+1}}. \tag{3.18}$$

Obviously,

$$J_{43} \leqslant c(l,\gamma,\eta)\Gamma_{1,l}/n. \tag{3.19}$$

Let $0 < \varepsilon < \ell/2$. We find

$$a_\varepsilon := \max_{0<2\eta<\gamma\leqslant l/2-\varepsilon}\min\left(\frac{l\eta}{2\gamma}, \frac{\gamma-\eta}{\gamma+1}\right).$$

We perform the substitution $u = \eta/\gamma$. Since $\gamma/(\gamma + 1)$ is an increasing function, we have

$$a_\varepsilon = \max_{\substack{0<u<1/2\\ \gamma\geqslant l-\varepsilon}}\min\left(\frac{l}{2}u, \frac{(1-u)\gamma}{\gamma+1}\right) = \max_{0<u<1/2}\min\left(\frac{l}{2}u, \frac{(1-u)(l-2\varepsilon)}{l+2-2\varepsilon}\right).$$

Now we make use of the following simple fact. If $f_1(u)$ is an increasing while $f_2(u)$ is a decreasing continuous function, $f_1(u_0) = f_2(u_0)$, $u_0 \in [a, b]$, then

$$\max_{a\leqslant u\leqslant b}\min(f_1(u), f_2(u)) = f_1(u_0).$$

In the considered case $u_0 = 2\Big/\left(l + 4 + \frac{4\varepsilon}{l-2\varepsilon}\right)$. Consequently,

$$a_\varepsilon = l\Big/\left(l + 4 + \frac{4\varepsilon}{l-2\varepsilon}\right). \tag{3.20}$$

We set

$$\gamma = \frac{l}{2+\delta},\ \eta = \gamma\cdot u_0 = \frac{2}{2+\delta}\cdot\frac{l}{l+4+\delta}, \tag{3.21}$$

where $\varepsilon = \delta l/(4 + 2\delta)$. Then from (3.16)-(3.20) we obtain that for each $\delta > 0$ one has

$$J_3 + J_4 \leqslant c(l, \delta)(\Gamma_{1,l}/n)^{l/(l+4+\delta)}. \tag{3.22}$$

It remains to estimate J_5. It is easy to see that

$$J_5 \leqslant c\int_{t\geqslant\tau_1} t^{-l/2-1}dt\Lambda_l^{-1/2} \leqslant c\tau_1^{-l/2}\Lambda_l^{-1/2}.$$

From here

$$J_5 \leqslant c(\Gamma_{2,l}/\sqrt{n})^{l/6}. \tag{3.23}$$

From (3.1), (3.2), (3.7), (3.9), (3.12)-(3.14), (3.21), (3.22), and (3.23) there follows the assertion of Theorem 1.

Proof of Theorem 2. We select $\tau = \tau_4$ from the condition $\Lambda_l^{-1/2}(n^{1/2}/\Gamma_{1,l}^{1/2}\tau_3)^{l/2} = (\sigma_1\sigma_2\tau)^{-1}$. This is equivalent to the requirement

$$(\Gamma_{1,l}/n)^{\left(\frac{\gamma}{2\gamma+l}-\frac{\eta}{\gamma}\right)\frac{l}{2}} = (\sigma_1\sigma_2\tau)^{-1}. \tag{3.24}$$

From here

$$\tau = (\sigma_1\sigma_2)^{-1}(n/\Gamma_{1,l})^{\frac{l}{2}\left(\frac{\gamma}{2\gamma+l}-\frac{\eta}{\gamma}\right)}, \tag{3.25}$$

For the estimation of the integral J_4 in (3.9), we apply Lemma 1.10. By virtue of (3.24) and (3.25) we have

$$J_4 \leqslant c_1(l)\max\left(1, (\varepsilon_0\sigma_1\Gamma_{1,l}^{1/2})^{\frac{l}{2}}\right)(\Gamma_{1,l}/n)^{\frac{l}{2}\left(\frac{\gamma}{2\gamma+l}-\frac{\eta}{\gamma}\right)} + \left[c_2(l)(\Gamma_{1,l}/n)^{l/4} + \left(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\right)^{n/4} + \exp\left\{-\frac{n}{8}\right\}\right]\ln\frac{\tau}{\tau_3}. \tag{3.26}$$

We set $\varepsilon > 0$,

$$\eta/\gamma = \gamma/(2\gamma + l) - \eta/\gamma,\ \ \gamma = 4l/(8+\varepsilon). \tag{3.27}$$

Then $\eta l/2\gamma = l/(16 + \varepsilon)$. From (3.16), (3.26), and (3.27) there follows that for any $\delta > 0$ we have

$$J_3 + J_4 \leqslant c(l, \delta)\max\left(1, (\varepsilon_0\sigma_1\Gamma_{1,l}^{1/2})^{l/2}\right)(\Gamma_{1,l}/n)^{l/(16+\delta)} + c(l)\left(\sup_{|x(l)|\geqslant\varepsilon_0}|v(x)|\right)^{n/4}\ln(n/\Gamma_{1,l}). \tag{3.28}$$

From (3.1), (3.2), (3.9), (3.13), (3.14), (3.25), (3.27), (3.28), and (3.23) there follows the assertion of Theorem 2.

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THE BERTRAND PROBLEM

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1. In 1852 the French mathematician and mechanician Bertrand began the study of the problem whether a mechanical system has a integral of motion that is quadratic in velocity and is independent of the total energy (see [1] or [2, Secs. 151 and 152]). In 1901 Darboux derived an equation which must be satisfied, in the case of systems with two degrees of freedom, by the potential of a conservative system that has an additional integral of motion that is quadratic in velocity. Later the two-dimensional case was investigated quite completely in [3].

In this article we consider mechanical systems that admit an invariant of motion that is polynomial in velocity and is of order at most two. We obtain conditions, which the potential of such a system (generalizing the Darboux equation) must satisfy, and a formula that expresses the force function of the system in terms of the kinetic energy and the invariant in case the invariant is regular.

Let (M, T, π) be a mechanical system with the configuration space M, kinetic energy T (T is a function of class C^∞ on the tangent bundle T(M) of the C^∞-manifold M that is a positive-definite quadratic form on each fiber), and the force function π (π is a horizontal 1-form on the manifold T(M); if the system has potential V, then $\pi = dV$). All the definitions and notation have been taken from [4] or [5]. The quantities T and π can depend on the time t.

Let Ω denote a standard symplectic 2-form on the cotangent bundle $T^*(M)$ of the manifold M. If f is a regular smooth function on the manifold T(M), then by Ω_f we denote the 2-form of the symplectic structure on T_M induced by the function f [5, p. 132].

We will use local coordinates $q = (q_1, \ldots, q_n)$ in the neighborhood U of a point $x \in M$ $(q(x) = 0)$ and the frame $\{\partial_{q_i} \in T_q(M),\ i = 1, \ldots, n\}$, corresponding to them, in the domain $T(U) \subset T(M)$ and the coframe $\{dq_i \in T^*_q(M),\ i = 1, \ldots, n\}$ in the domain $T^*(U) \subset T^*(M)$. In particular, the coordinates in the domain T(U) of the tangent bundle T(M) will be denoted by $(q_1, \ldots, q_n; \dot q_1, \ldots, \dot q_n)$.

For a function $f \in C^\infty(T(M))$ we introduct a matrix-valued function M_f that is defined in T(U): M_f: $(q, \dot q) \to (\partial_{\dot q_i}\partial_{\dot q_j} f)(q, \dot q)$. In the domain T(U) the kinetic energy can be expressed in the form $T = \sum_{i,j=1}^{n} \frac{1}{2}(M_T)_{ij}(q, t)\, \dot q_i \dot q_j$, and the force function can be expressed in the form $\pi = \sum_{i=1}^{n} \pi_i(q, \dot q, t)\, dq_i$.

Let p denote the projection of $T(M) \times \mathbf{R}$ on T(M). For $h \in C^\infty(T(M))$ we define the vector field $Y_h = X_h + \partial_t$ on $T(M) \times \mathbf{R}$ by the condition

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