A NON-UNIFORM BOUND OF THE REMAINDER TERM IN THE CENTRAL LIMIT THEOREM FOR BERNOULLI RANDOM VARIABLES

S. V. Nagaev[1], V. I. Chebotarev[2], and A. Ya. Zolotukhin[3]

A bound for the remainder in the Esseen expansion is obtained in the case of Bernoulli random variables. The bound consists of two parts, uniform and non-uniform. The uniform part depends only on n and p, and the non-uniform part depends also on x. This bound is compared with other known bounds. It is shown how this result can be applied to the problem of the absolute constant in the Berry–Esseen inequality.

1. Introduction

Let Z be a two-point random variable with the distribution

$$\mathbf{P}(Z=1)=p, \quad \mathbf{P}(Z=0)=q, \; p+q=1, \tag{1}$$

and $Z_1, Z_2, \ldots, Z_n$ be independent copies of Z. Denote

$$X_j=\frac{Z_j-p}{\sqrt{pq}}, \quad j=1, \ldots, n, \quad \beta_k(p)=\mathbf{E}|X_1|^k, \quad l_n(p)=\frac{\beta_3(p)}{\sqrt{n}}, \quad \alpha_k(p)=\mathbf{E}X_1^k, \tag{2}$$

$$F_n(x)=\mathbf{P}\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^{n}X_j<x\Big), \; \varphi(t)=\frac{1}{\sqrt{2\pi}}e^{-t^2/2}, \; \Phi(x)=\int_{-\infty}^{x}\varphi(t)\,dt, \; \Delta_n(p)=\sup_x|F_n(x)-\Phi(x)|. \tag{3}$$

It was proved in [1] that for all $n\geqslant 1$ and $0<p\leqslant 0.5$

$$\Delta_n(p)\leqslant 0.4215\, l_n(p).$$

As is well known [2], the constant in the Berry–Esseen inequality is no less than $C_E\equiv\frac{3+\sqrt{10}}{6\sqrt{2\pi}}= 0.409732\ \ldots$. To date, the following bound is obtained,

$$\Delta_n(p)<0.4099539\, l_n(p). \tag{4}$$

The proof of this bound required, in particular, rather time-consuming computations, and the authors thank the Department of Mathematical Statistics of the Faculty of Computational Mathematics and Cybernetics, Lomonosov Moscow State University, for support in the organization of their access to the supercomputers of Moscow State University.

The aim of the present paper is to obtain the non-uniform bound for $F_n(x)-\Phi(x)$. Using the results, methods, and techniques from [1], we deduce the bound of the type

$$|F_n(x)-\Phi(x)|\leqslant l_n(p)\,G_0(p,x)+l_n^2(p)\Big(G_1(p,x)+c+l_n(p)\,G_2(p,x)\Big), \tag{5}$$

[1] Sobolev Institute of Mathematics SB RAN, Novosibirsk, Russia, e-mail: nagaev@math.nsc.ru
[2] Computing Center, Far Eastern Branch of RAS, Khabarovsk, Russia, e-mail: chebotarev@as.khb.ru
[3] Tula State University, Tula, Russia, e-mail: zolot_aj@mail.ru

Proceedings of the XXXII International Seminar on Stability Problems for Stochastic Models, Trondheim, Norway, June 16–21, 2014.

1072-3374/16/2141-0083 © 2016 Springer Science+Business Media New York

where $G_0(p,x)$, $G_1(p,x)$, and $G_2(p,x)$ are functions which for every fixed p exponentially decrease in x, and c is a sufficiently small constant.

The starting point of our reasoning is the statement that can be called *the smoothing equality.* For arbitrary distribution function $\Psi(x)$ and Borel bounded function $f(x)$, denote

$$(\Psi * f)(x) = \int_{-\infty}^{\infty} f(x-y)\, d\Psi(y). \tag{6}$$

Let $G(x)$ be a left-continuous non-decreasing step function, $h > 0$ be the least distance between the discontinuity points of G, $P(x)$ be the uniform distribution on $[-h/2, h/2]$, and $G_0(x)$ be a continuous function. Denote

$$\delta(x) = G(x) - G_0(x), \quad B_0^{\pm}(h,x) = \frac{1}{h}\int_0^h [G_0(x \pm s) - G_0(x)]\, ds.$$

Lemma 1. *Let x_0 be a discontinuity point of the function G. Then*

$$\delta(x_0\pm) = (P * \delta)(x_0 \pm h/2) + B_0^{\pm}(h, x_0). \tag{7}$$

Remark 1. Note that the first summand in (7) is a continuous function in contrast to the initial difference $\delta(x)$. The second summand in (7) plays the role of the rate of smoothing. The simplicity of this exact result is due to the fact that just the uniform distribution is used as the smoothing one. Also note that, as a rule, the known statements associated with the smoothing distributions take the form of inequalities (see, for instance, [3–6]). All these claims require much more complicated proofs than that of Lemma 1; see Section 2.

In what follows we denote the discontinuity points of the function F_n by the symbol x_k, $k = 0, 1, \ldots, n$. Obviously,

$$x_k = \frac{k - np}{\sqrt{npq}}, \quad k = 0, 1, \ldots, n,$$

and the distance between the neighboring points is equal to

$$h_n = \frac{1}{\sqrt{npq}}.$$

Introduce the following notation:

$$\begin{aligned} &Q_n(p,x) = -\frac{\alpha_3(p)}{3!\sqrt{n}}\,\varphi''(x), \quad c_n = \sqrt{\frac{n}{n-1}}, \quad \widetilde{Q}_n(p,x) = c_n^3 Q_n(p, x c_n), \\ &\Phi_n(p,x) = \Phi(x) + \widetilde{Q}_n(p,x), \quad \delta_n(p,x) = F_n(x) - \Phi_n(p,x), \\ &P_n(x) = \mathbf{P}(\zeta_n < x), \end{aligned} \tag{8}$$

where ζ_n has the uniform distribution on $[-h_n/2, h_n/2]$.

Using the method and the results of [1], in Section 3 we prove the following statement.

Lemma 2. *Under the conditions*

$$0.02 \leqslant p \leqslant 0.5, \quad n \geqslant 200, \tag{9}$$

the following bound holds:

$$|(P_n * \delta_n)(x)| \leqslant T(p,n) \equiv \sum_{j=1}^{4} T_j(p,n), \tag{10}$$

where $T_1(p,n)$ *and* $T_4(p,n)$ *are the functions defined by formulas* (64) *and* (67) *respectively, and*

$$T_2(p,n) = K_2(p,n),\ T_3(p,n) = K_3(p,n)$$

are the functions defined in [1, pp. 217, 218]. *Moreover,* $nT(p,n)$ *decreases in* n *for every* p,

$$T(p,n) < 0.246\, l_n^2(p), \tag{11}$$

and if $n \geqslant 500$, *then*

$$T(p,n) < 0.1602\, l_n^2(p). \tag{12}$$

Remark 2. Some comment on the behavior of $T(p,n)$. Using the explicit form of the functions included in $T(p,n)$ it is not difficult to verify that

$$\lim_{n\to\infty} \frac{T(p,n)}{l_n^2(p)} = W(p) \equiv \frac{3|1-6pq| + 4(1-2p)^2 + 3}{36\pi(1-2pq)^2}.$$

Indeed, this formula is a consequence of the equalities

$$\lim_{n\to\infty} \frac{T_1(p,n)}{l_n^2(p)} = \frac{\omega_4(p)}{12\pi(1-2pq)^2} = \frac{|1-6pq|}{12\pi(1-2pq)^2}, \qquad \lim_{n\to\infty} \frac{K_3(p,n)}{l_n^2(p)} = \frac{1}{12\pi(1-2pq)^2},$$

$$\lim_{n\to\infty} \frac{K_2(p,n)}{l_n^2(p)} = \frac{\omega_3^2(p)}{9\pi(1-2pq)^2} = \frac{(1-2p)^2}{9\pi(1-2pq)^2}, \qquad \lim_{n\to\infty} \frac{T_4(p,n)}{l_n^2(p)} = 0.$$

Note that $\max\limits_{p\in[0.02,0.5]} \frac{T(p,n)}{l_n^2(p)} < 0.1602$ if $n \geqslant 500$ while $\max\limits_{p\in[0,0.5]} W(p) = W(0.5) = \frac{1}{2\pi} = 0.159154\ldots$. Graphs of the functions $\frac{T(p,n)}{l_n^2(p)}\Big|_{n=200}$, $\frac{T(p,n)}{l_n^2(p)}\Big|_{n=500}$ and $W(p)$ for $p \in [0.02, 0.5]$ are shown in Fig. 1. Note that in this figure the origin is shifted to the point $(0.02, 0)$.

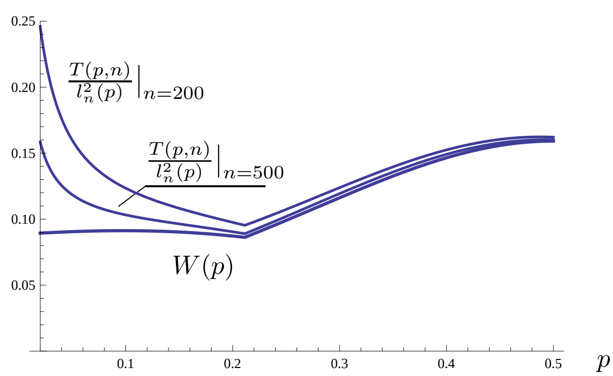

Fig. 1.

Let us introduce the function

$$S(x) = \lceil x \rceil - x - \frac{1}{2}, \tag{13}$$

where $\lceil x \rceil$ is the least integer that is no less than x.

The main statement of this work is

Theorem 1. *Let the condition* (9) *be fulfilled. Then for every point* x: $x_{k-1} < x \leqslant x_k$, $k = 1, \ldots, n$ *we have*

$$F_n(x) - \Phi(x) = \widetilde{Q}_n(p,x) + \frac{1}{\sqrt{npq}} S(np + x\sqrt{npq})\, \varphi(x) + R_1(p,n,x), \tag{14}$$

where

$$|R_1(p,n,x)| \leqslant T(p,n) + \frac{1}{npq}\Big(\frac{7}{6}\max_{x_{k-1}\leqslant y\leqslant x_k}|\varphi'(y)| + \frac{1-2p}{4}\max_{x_{k-1}\leqslant y\leqslant x_k}|\varphi^{(3)}(y)|\Big); \tag{15}$$

the function $T(p,n)$ *is defined in Lemma* 2 *and satisfies the bounds* (11) *and* (12).

Note that (14) is similar to the known result by C.-G. Esseen for lattice distributions [7]. To formulate it we introduce the following notations: X, X_1, X_2, … , X_n is a sequence of i.i.d. random variables with zero mean, unit variance, and finite third moment,

$$G_n(x) = \mathbf{P}\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} X_i < x\Big), \quad \alpha_k = \mathbf{E}X^k, \quad Q_n(x) = -\frac{\alpha_3}{6\sqrt{n}}\,\varphi''(x). \tag{16}$$

It was proved in [7, p. 56] (see also [8, p. 228]) that if X is a lattice random variable taking values of the form $w + kh$, $k = 0, \pm 1, \pm 2, \dots$, then

$$G_n(x) - \Phi(x) = Q_n(x) + \frac{h}{\sqrt{n}}\,\psi_n(x) + o\Big(\frac{1}{\sqrt{n}}\Big) \tag{17}$$

uniformly in x, where $\psi_n(x) = S_1\big(\frac{x\sqrt{n}-nw}{h}\big)\,\varphi(x)$, and $S_1(x) = [x] - x + \frac{1}{2}$, $[x]$ is the integer part of x. Note that in his work Esseen considers distribution functions that are defined in discontinuity points as the half-sum of the left and right limits (see [7, p. 9]). In our work, the distribution functions are left-continuous by definition. Therefore, Eq. (17) is not true in discontinuity points of such functions G_n. However, the problem disappears if the function ψ_n is slightly changed, namely, if $S_1(\cdot)$ is replaced by the function $S(\cdot)$ defined in (13). In what follows we will consider (17) with such a correction.

We can see that (14) has the form of (17) in the particular case of Bernoulli random variables with the replacement of $o\big(\frac{1}{\sqrt{n}}\big)$ by $R_1(p,n,x)$. Moreover, the difference is that instead of $Q_n(x)$ in (17) we have $\widetilde{Q}_n(p,x)$ in (14). This is due to the fact that, in particular, we use the Bergström method (compare (51) and [9, pp. 2,3]). One can say that Theorem 1 gives the bound of the remainder in the so-called short Bergström expansion. In the following statement we pass over to the function $Q_n(p,x)$ and get the bound in the short Edgeworth expansion. We shall use the notation $\omega(p) = p^2 + q^2$.

Corollary 1. *Let*

$$0.02 \leqslant p \leqslant 0.5, \quad n \geqslant 500. \tag{18}$$

Then for every point $x : x_{k-1} < x \leqslant x_k$, $k = 1, \dots, n$, *we have*

$$F_n(x) - \Phi(x) = Q_n(p,x) + \frac{1}{\sqrt{npq}}S(np + x\sqrt{npq})\,\varphi(x) + R_2(p,n,x), \tag{19}$$

where

$$|R_2(p,n,x)| \leqslant l_n^2(p)\Big[\frac{T(p,n)}{l_n^2(p)} + \frac{1}{\omega^2(p)}\Big(\frac{7}{6}\max_{x_{k-1}\leqslant y\leqslant x_k}|\varphi'(y)| + \frac{1-2p}{4}\max_{x_{k-1}\leqslant y\leqslant x_k}|\varphi^{(3)}(y)|\Big)\Big] + \\ + l_n^3(p)\,\frac{pq(1-2p)}{12\omega^3(p)}\Big(|x|c_n^2\max_{|x|\leqslant|y|\leqslant|x|c_n}|\varphi^{(3)}(y)| + 3c_n^3|\varphi''(xc_n)|\Big), \tag{20}$$

$T(p,n)$ *satisfies the bound* (12).

Introduce the special notations for the intervals of two types:

$$J^+(h,x) = \big(x, x+h\big), \quad J^-(h,x) = \big(x-h, x\big).$$

Theorem 1 implies

Corollary 2. *Let condition* (18) *be fulfilled. Then for every* $k = 0, 1, \ldots, n$, *we have*

$$F_n(x_k\pm) - \Phi(x_k) = \widetilde{Q}_n(p, x_k) \pm \frac{1}{2\sqrt{npq}}\,\varphi(x_k) + R_1^{\pm}(p, n, x_k), \tag{21}$$

$$F_n(x_k\pm) - \Phi(x_k) = Q_n(p, x_k) \pm \frac{1}{2\sqrt{npq}}\,\varphi(x_k) + R_2^{\pm}(p, n, x_k), \tag{22}$$

where $R_1^{\pm}(p, n, x_k)$ *and* $R_2^{\pm}(p, n, x_k)$ *satisfy the inequalities*

$$|R_1^{\pm}(p, n, x_k)| \leqslant l_n^2(p)\left[0.1602 + \frac{1}{\omega^2(p)}\Big(\frac{7}{6}\sup_{y\in J^{\pm}(h_n, x_k)}|\varphi'(y)| + \frac{1-2p}{4}\sup_{y\in J^{\pm}(h_n, x_k)}|\varphi^{(3)}(y)|\Big)\right] \equiv$$
$$\equiv \widetilde{R}_1^{\pm}(p, n, x_k), \tag{23}$$

$$|R_2^{\pm}(p, n, x_k)| \leqslant \widetilde{R}_1^{\pm}(p, n, x_k) + l_n^3(p)\,\frac{pq(1-2p)}{12\omega^3(p)}\Big(|x_k| c_n^2 \max_{|x_k|\leqslant|y|\leqslant|x_k|c_n}|\varphi^{(3)}(y)| + 3c_n^3|\varphi''(x_k c_n)|\Big). \tag{24}$$

Estimating the right-hand side of the inequality (24) we obtain the following simpler version of Corollary 2. Denote the greatest root of the equation $\varphi^{(4)}(y) = 0$ by

$$y_0 = \sqrt{3 + \sqrt{6}} = 2.334\ \ldots\ .$$

We shall also use the functions

$$G_1(p, x) = |x|\Big(\frac{x^2}{4} + \frac{5}{12}\Big)\frac{1}{\omega^2(p)}\,\varphi(x), \quad G_2(p, x) = 0.084(x^4 + 0.0031x^2 - 3)\,\frac{pq(1-2p)}{\omega^3(p)}\,\varphi(x),$$

$$G_1^{+}(p, x) = \begin{cases} G_1(p, x), & \text{if } x > 0,\\ G_1(p, |x| - h_n), & \text{if } x < 0, \end{cases} \qquad G_1^{-}(p, x) = \begin{cases} G_1(p, x - h_n), & \text{if } x > 0,\\ G_1(p, x), & \text{if } x < 0. \end{cases}$$

Corollary 3. *Let* $0.02 \leqslant p \leqslant 0.5$, $n \geqslant 500$, x *be a discontinuity point of the function* F_n. *If* $|x| > y_0 + h_n$, *then*

$$F_n(x\pm) - \Phi(x) = Q_n(p, x) \pm \frac{1}{2\sqrt{npq}}\,\varphi(x) + R_2^{\pm}(p, n, x), \tag{25}$$

where

$$|R_2^{\pm}(p, n, x)| \leqslant l_n^2(p)\Big(G_1^{\pm}(p, x) + 0.1602 + l_n(p)\,G_2(p, x)\Big).$$

Proof. Note that under the condition $|x| > y_0$, the function $|\varphi^{(3)}(x)|$ decreases in $|x|$. Considering, for instance, the case $x > y_0 + h_n$, we have

$$\frac{7}{6}\sup_{y\in J^{+}(h_n, x)}|\varphi'(y)| + \frac{1-2p}{4}\sup_{y\in J^{+}(h_n, x)}|\varphi^{(3)}(y)| \leqslant \frac{7}{6}|\varphi'(x)| + \frac{1}{4}|\varphi^{(3)}(x)| =$$
$$= \Big(\frac{7}{6}|x| + \frac{|x^3 - 3x|}{4}\Big)\varphi(x) = |x|\Big(\frac{x^2}{4} + \frac{5}{12}\Big)\varphi(x),$$

$$\frac{1}{12}\Big(c_n^2|x|\,|\varphi^{(3)}(x)| + 3c_n^3|\varphi''(x)|\Big) = \frac{c_n^2}{12}\Big(x^4 + 3(c_n - 1)x^2 - 3c_n\Big)\varphi(x) \leqslant 0.084(x^4 + 0.0031x^2 - 3)\,\varphi(x).$$

The last inequality holds because $n \geqslant 500$. The statement of the corollary in this case follows from these bounds and (22)–(24). In other cases, the proof is similar. Thus, for example, if $x + h_n < -y_0$, then

$$\sup_{y\in J^{+}(h_n, x)}|\varphi^{(3)}(y)| = |\varphi^{(3)}(x + h_n)| = |\varphi^{(3)}(|x| - h_n)|,$$

$$\max_{|x|\leqslant|y|\leqslant|x|c_n} |\varphi^{(3)}(y)| = |\varphi^{(3)}(x)|.$$

Remark 3. Note that the following simpler bound is true: for $|x| > y_0 + h_n$

$$|R_2^{\pm}(p,n,x)| \leqslant l_n^2(p)\Big(G_1(p,|x|-h_n) + 0.1602 + l_n(p)\,G_2(p,x)\Big). \tag{26}$$

Denote

$$\mathcal{E}(p) = \frac{2-p}{3\sqrt{2\pi}\,[p^2+(1-p)^2]}, \quad \mathcal{E}_1(p) = \frac{1-2p}{6(p^2+q^2)\sqrt{2\pi}}, \quad \mathcal{E}_2(p) = \frac{1}{2\sqrt{2\pi}(p^2+q^2)}.$$

The Esseen function $\mathcal{E}(p)$ is connected with the problem of finding the absolute constant in the Berry–Esseen inequality (see [1]). It is easy to see that

$$\mathcal{E}(p) = \mathcal{E}_1(p) + \mathcal{E}_2(p), \quad \max_p \mathcal{E}(p) = C_E. \tag{27}$$

It is not hard to verify that

$$\frac{1}{l_n(p)}\,Q_n(p,x) = e^{-x^2/2}(1-x^2)\,\mathcal{E}_1(p), \quad \frac{\varphi(x)}{\beta_3(p)2\sqrt{pq}} = e^{-x^2/2}\mathcal{E}_2(p),$$
$$\frac{1}{l_n(p)}\,\widetilde{Q}_n(p,x) = e^{-x^2c_n^2/2}\,c_n^3(1-x^2c_n^2)\,\mathcal{E}_1(p).$$

Using these equalities, from Theorem 1 we obtain

Corollary 4. *Let condition* (18) *be fulfilled. Then for every* $k = 0, 1, \dots, n$

$$F_n(x_k\pm) - \Phi(x_k) = l_n(p)\left(e^{-x_k^2c_n^2/2}\,c_n^3(1-x_k^2c_n^2)\,\mathcal{E}_1(p) \pm e^{-x_k^2/2}\mathcal{E}_2(p)\right) + R_1^{\pm}(p,n,x_k) = \tag{28}$$

$$= l_n(p)\,e^{-x_k^2/2}\Big((1-x_k^2)\,\mathcal{E}_1(p) \pm \mathcal{E}_2(p)\Big) + R_2^{\pm}(p,n,x_k), \tag{29}$$

where $R_1^{\pm}(p,n,x_k)$ *and* $R_2^{\pm}(p,n,x_k)$ *satisfy inequalities* (23) *and* (24) *respectively.*

It is known that there exists a discontinuity point x_k of the function F_n such that

$$\Delta_n(p) = F_n(x_k+) - \Phi(x_k) \ \text{ or } \ \Delta_n(p) = \Phi(x_k) - F_n(x_k).$$

Corollary 4 implies the following representation of the quantity $\frac{1}{l_n(p)}\,\Delta_n(p)$ as the sum of the main term and the remainder, while Theorem 1.1 from [1] gives only the inequality for this quantity.

Corollary 5. *Let condition* (18) *be fulfilled. Then for some* $k = 0, 1, \dots, n$ *either the equality*

$$\Delta_n(p) = l_n(p)\,e^{-x_k^2/2}\Big((1-x_k^2)\,\mathcal{E}_1(p) + \mathcal{E}_2(p)\Big) + R_2^{+}(p,n,x_k) \tag{30}$$

or the equality

$$\Delta_n(p) = l_n(p)\,e^{-x_k^2/2}\Big(-(1-x_k^2)\,\mathcal{E}_1(p) + \mathcal{E}_2(p)\Big) - R_2^{-}(p,n,x_k) \tag{31}$$

holds, where $R_1^{\pm}(p,n,x_k)$ *and* $R_2^{\pm}(p,n,x_k)$ *satisfy the inequalities* (23) *and* (24) *respectively.*

In addition, as will be shown below, Corollary 4 can be used to optimize the algorithm for computing the quantity $\frac{1}{l_n(p)}\,\Delta_n(p)$ when the parameter n is large.

It follows from [1, Theorem 1.1] that for every $\varepsilon > 0$ there exists N such that for $n \geqslant N \geqslant 200$

$$\max_{0.02\leqslant p\leqslant 0.5} \frac{1}{l_n(p)}\,\Delta_n(p) \leqslant C_E + \varepsilon. \tag{32}$$

To obtain the bound (32) for all n, it suffices to calculate $\max\limits_{0.02\leqslant p\leqslant 0.5} \frac{1}{l_n(p)}\,\Delta_n(p)$ for $1 \leqslant n \leqslant N-1$. However, if ε is small enough, the number N may be rather large, and this can lead to long calculations. For example, if $\varepsilon \leqslant 0.00022$, i.e., $C_E+\varepsilon < 0.409953$, then $N \geqslant 5\cdot 10^5$. And when n takes values, for instance, in the range from 145000 to 181000, it took about 30 hours to calculate quantities $\max\limits_p \frac{1}{l_n(p)}\,\Delta_n(p)$ on one of the supercomputers of Moscow State University.

For fixed n and p the procedure for computing the quantity $\frac{1}{l_n(p)}\,\Delta_n(p)$ is reduced to the computation of $\frac{1}{l_n(p)}\,|F_n(x_k\pm)-\Phi(x_k)|$ for $k\in I(n)$, where $I(n)$ is the set of all integers from the interval $[0,n]$. The non-uniform nature of the bounds obtained in Corollary 4, instead of $I(n)$ allows one to use a more narrow interval (see Corollaries 6 and 7).

Corollary 6. *Let the condition* (9) *be fulfilled, and x be the discontinuity point of the function F_n. If $x\in A=\{x:\, x\geqslant y_0+h_n$ or $x\leqslant -y_0-h_n\}$, then*

$$\frac{1}{l_n(p)}\,|F_n(x\pm)-\Phi(x)| < 0.222. \tag{33}$$

Proof. Denote

$$g_1^{\pm}(p,n,x) = e^{-x^2c_n^2/2}\,c_n^3\Big(1-x^2c_n^2\Big)\,\mathcal{E}_1(p) \pm e^{-x^2/2}\,\mathcal{E}_2(p), \quad g_2(p,n)=0.246\,l_n(p),$$
$$g_3^{\pm}(p,n,x) = \frac{1}{\sqrt{npq}\,(p^2+q^2)}\left[\frac{7}{6}\sup_{y\in J^{\pm}(h_n,x)}|\varphi'(y)| + \frac{1-2p}{4}\sup_{y\in J^{\pm}(h_n,x)}|\varphi^{(3)}(y)|\right].$$

Note that under condition (9) the inequality holds which differs from (23) only by replacement of 0.1602 by $\frac{T(p,n)}{l_n^2(p)}$. Using this inequality and (11), (28) as well, we get:

$$\frac{1}{l_n(p)}\,|F_n(x\pm)-\Phi(x)| \leqslant |g_1^{\pm}(p,n,x)| + g_2(p,n) + g_3^{\pm}(p,n,x). \tag{34}$$

It is easily seen that

$$g_2(p,n) \leqslant \frac{1.69}{\sqrt{n}} \leqslant \frac{1.69}{\sqrt{200}} < 0.1196. \tag{35}$$

Moreover, for $|x|\geqslant\sqrt{3}$

$$|g_1^{\pm}(p,n,x)| \leqslant c_n^3 e^{-x^2/2}\big(x^2-1\big)\mathcal{E}_1(p) + e^{-x^2/2}\mathcal{E}_2(p) \leqslant$$
$$\leqslant e^{-x^2/2}\left(\left(\frac{200}{199}\right)^{3/2}(x^2-1)\mathcal{E}_1(0.02)+\mathcal{E}_2(0.5)\right) \equiv v_1(x).$$

The function $v_1(x)$ decreases for $x>0$. Consequently,

$$|g_1^{\pm}(p,n,x)| \leqslant v_1(y_0+h_n) < v_1(y_0) < 0.046. \tag{36}$$

Let $j=1$ or 3. Since

$$\sup_{y\in J^{\pm}(h_n,x)}|\varphi^{(j)}(y)| \leqslant |\varphi^{(j)}(y_0)| \ \text{ if } \ x\in A,$$

using the bounds $|\varphi'(y_0)|<0.06106$, $|\varphi^{(3)}(y_0)|<0.1496$, we obtain

$$g_3^{\pm}(p,n,x) < \frac{1}{\sqrt{npq}\,(p^2+q^2)}\left[\frac{7}{6}\,0.06106 + \frac{1-2p}{4}\,0.1496\right].$$

The right-hand side of this inequality decreases in $p\in(0.02,0.5]$. Calculating it for $p=0.02$, we obtain

$$g_3^{\pm}(p,n,x) < \frac{0.797}{\sqrt{n}} < 0.0564. \tag{37}$$

It follows from (34)–(37) that under the condition (9) and $x \in A$, the bound

$$\frac{1}{l_n(p)}\,|F_n(x\pm) - \Phi(x)| < 0.1196 + 0.046 + 0.0564 = 0.222$$

holds. Thus, Corollary 6 is proved.

Denote

$$D(p,n) = \max_{k\in I(n)} \frac{1}{l_n(p)}\,|F_n(x_k\pm) - \Phi(x_k)|.$$

Corollary 7. *The equality*

$$\sup_{0.02\leqslant p\leqslant 0.5,n} D(p,n) = \sup_{0.02\leqslant p\leqslant 0.5,n}\ \max_{k\in I(p,n)} \frac{1}{l_p(n)}\,|F_n(x_k\pm) - \Phi(x_k)|$$

is valid, where

$$I(p,n) = \Big\{k:\ np - y_0\sqrt{npq} - 1 < k < np + y_0\sqrt{npq} + 1\Big\}.$$

Proof. Since $\sup\limits_{p,n} D(p,n) \geqslant C_E$, the statement of Corollary 7 follows from Corollary 6.

Example 1. Let $n = 5\cdot 10^5$, $p = 0.4$. It follows from Corollary 7 that, when finding $D(p,n)$, we can use only 1618 values of k instead of all $k \in I(5\cdot 10^5)$: $199191 \leqslant k \leqslant 200809$.

Comparison with the known bounds. 1. Although the limiting properties of the binomial distribution have been studied for more than three centuries, until now this field still attracts the attention of specialists in probability theory. For instance, in 2012 A. M. Zubkov and A. A. Serov [10] obtained very precise lower and upper bounds for $F_n(x_k+)$, simplifying a result obtained in 1984 by D. Alfers and H. Dinges [11]. Introduce the notation

$$H(x,p) = x\,\ln\frac{x}{p} + (1-x)\,\ln\frac{1-x}{1-p},$$
$$W_n(p,x) = I(x=0)\,q^n + I(0<x<n)\Phi\Big(\mathrm{sgn}\,\big(\frac{x}{n}-p\big)\sqrt{2nH\big(\frac{x}{n},p\big)}\,\Big) + I(x=n)\,(1-p^n).$$

Here $I(x\in A) = 1$, if $x \in A$, and $I(x \in A) = 0$ otherwise.

Theorem A [10]. *For every* $k = 0, 1, \ \dots\ , n-1$ *and* $p \in (0,1)$ *the following inequality is true*:

$$W_n(p,k) \leqslant F_n(x_k+) \leqslant W_n(p,k+1). \tag{38}$$

Consider, for example, the case $k > np$. Let $n \to \infty$, where $c_1 < \frac{k-np}{\sqrt{npq}} < c_2$ for some constants c_1, c_2. Then one can show that

$$W_n(p,k) = \Phi\Big(\sqrt{2nH\Big(\frac{k}{n},p\Big)}\,\Big) \sim \Phi(x_k) - \frac{\alpha_3(p)}{6\sqrt{n}}\,x_k^2\,\varphi(x_k).$$

However, in view of (22),

$$F_n(x_k+) = \Phi(x_k) + \frac{\alpha_3(p)}{6\sqrt{n}}\,(1-x_k^2)\,\varphi(x_k) + \frac{1}{2\sqrt{npq}}\,\varphi(x_k) + R_2^+(p,n,x_k), \tag{39}$$

where according to (24) we have $R_2^+(p,n,x_k) = O\big(\frac{1}{n}\big)$ with the bound in the explicit form. Consequently, Theorem A gives an approximation to $F_n(x_k+)$ of the form $O\big(\frac{1}{\sqrt{n}}\big)$ while (39) is a more accurate approximation of the form $O\big(\frac{1}{n}\big)$.

2. Let us compare Corollary 1 with the following result from the monography by J. V. Uspensky published in 1937 [12]. We formulate it using our notations and with more accurate constants.

Theorem B [12]. *The quantity U from the representation*

$$F_n(x) - \Phi(x) = Q_n(p,x) + \frac{1}{\sqrt{npq}} S(np + x\sqrt{npq})\,\varphi(x) + U \tag{40}$$

satisfies the inequality

$$|U| \leqslant \frac{0.1809 + 0.2694|p-q|}{npq} + 0.9742\,e^{-3\sqrt{npq}/2} \tag{41}$$

provided that $npq \geqslant 25$.

Remark 4. Note that instead of (41) the bound

$$|U| \leqslant \frac{0.2 + 0.25|p-q|}{npq} + e^{-3\sqrt{npq}/2} \tag{42}$$

is in [12, p. 130]. However, there is a mistake in [12] when deducing the factor 0.25.

Let us compare the summand $T(p,n)$ from the right-hand side of (20) with the right-hand side of (42). Show that under the conditions

$$0.02 \leqslant p \leqslant 0.5, \quad n \geqslant 200, \quad npq \geqslant 25$$

the following bound is valid:

$$npq\,T(p,n) < 0.1809 + 0.2694\,|p-q| + 0.9742\,npq\,e^{-3\sqrt{npq}/2}. \tag{43}$$

Using the notation and formulas (63), (64), (67) and [1, p. 219], as a result of elementary but rather bulky calculations (which we omit here), we obtain the bound

$$\sigma_n^2\,T(p,n) < T(p) + 0.00462\,\sigma_n^2\,e^{-3\sigma_n/2}, \tag{44}$$

where

$$T(p) = \frac{\omega_4(p)}{12\pi}\left(\frac{200}{199}\right)^2 + \frac{\omega_5(p)}{40\sqrt{2\pi}\,5}\left(\frac{200}{199}\right)^{5/2} + \frac{\omega_6(p)}{90\pi\,25}\left(\frac{200}{199}\right)^3 + 0.01223\,\omega^2(p) + 0.03006+$$
$$+\,0.00533\frac{\omega(p)\omega_3(p)\sqrt{2}}{36\pi}\left(\frac{200}{199}\right)^3 + \frac{1}{\pi}\sum_{j=1}^{5}\frac{\gamma_{j+5}\,A_{j+5}(200)\,V_{j+5}(p)}{5^{j-1}}\left[1 + \frac{2\widetilde{\gamma}_{j+5}}{99}\,\exp\left\{\frac{6^{1/3}}{25^{1/3}\,4\,\omega^{4/3}(p)}\right\}\right].$$

Comparing the right-hand side of inequality (44) with the right-hand side of (43), we can see that the summand with the exponential dependence on σ_n is included in (44) with the coefficient which is almost 60 times less than the factor on the right-hand side of (43). Moreover, in Fig. 2 one can see that the graph of $T(p)$ is significantly lower than the straight line $u(p) = 0.1809 + 0.2694\,|p-q|$.

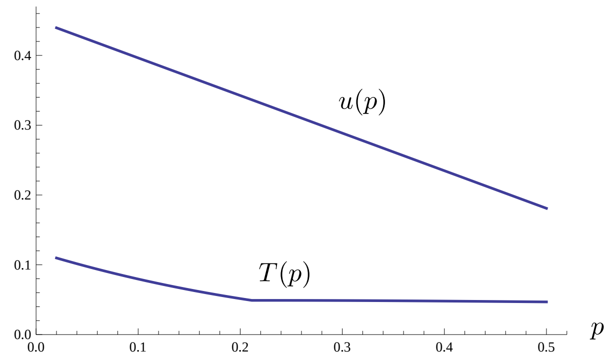

Fig. 2. Graphs of the functions $T(p)$ and $u(p)$.

3. Compare Corollary 3 with the current best non-uniform bound of a general type obtained by I. G. Shevtsova in [13]. In the particular case of Bernoulli random variables considered here the bound proved in [13] for arbitrary independent random variables has the form

$$(1+|x|^3)|F_n(x)-\Phi(x)| \leqslant \begin{cases} 17.37\,\frac{\beta_3(p)}{\sqrt{n}}, & \text{if } \beta_3(p) < 6.07, \\ 15.7\,\frac{\beta_3(p)+0.646}{\sqrt{n}}, & \text{if } \beta_3(p) \geqslant 6.07. \end{cases} \tag{45}$$

Note that $\beta_3(p)$ is a decreasing function on $(0, 0.5]$; therefore, $\beta_3(p) \leqslant \beta_3(0.02) < 6.87$, when $p \in [0.02, 0.5]$. We compare (25) and (45) in the case $\beta_3(p) < 6.07$. Note that the root of the equation $\beta_3(p) = 6.07$ is equal to $0.02517 \ \ldots$. One can show that under the condition $0.0252 \leqslant p \leqslant 0.5$ the bound (25) is more accurate than (45) if $\sqrt{6} < x < 2.36\, n^{1/6}$ and $n \geqslant 200$. In view of limitations on the size of the paper, we skip these arguments.

4. Now we mention one result of W. Feller [14] and C. Lenart [15], which we formulate in our notations and for the case of Bernoulli summands.

Theorem C. *If* $0 < x < \frac{1}{12}\sqrt{\frac{np}{q}}$ *then*

$$1 - F_n(x) = e^{-x^2\Lambda(x/\sqrt{n})/2}\Big(1 - \Phi(x) + \tau_n\sqrt{\frac{q}{np}}\, e^{-x^2/2}\Big),$$

where $|\tau_n| < 7.465$, $\Lambda(z) = \sum_{\nu=1}^{\infty} q_\nu z^\nu$, $q_1 = \alpha_3(p) = \frac{1-2p}{\sqrt{pq}}$, $|q_\nu| \leqslant \frac{1}{8}\left(12\sqrt{\frac{q}{p}}\right)^\nu$, $\nu = 2, 3, \ \ldots$.

It follows from Theorem C that

$$F_n(x) - \Phi(x) = \Big(1-\Phi(x)\Big)\Big(1 - e^{-x^2\Lambda(x/\sqrt{n})/2}\Big) - \tau_n\sqrt{\frac{q}{np}}\exp\{-x^2/2 - x^2\Lambda(x/\sqrt{n})/2\}. \tag{46}$$

The absolute value of the right-hand side of (46) for $x > 0$ and $0 < p \leqslant 0.5$ is majorized by the expression

$$A \equiv \frac{\varphi(x)}{\sqrt{npq}}\left(\frac{(1-2p)}{6}\, x^3 M(x) + \tau_n q\sqrt{2\pi}\right),$$

where $M(x) := \frac{1-\Phi(x)}{\varphi(x)}$ is the Mills function. Compare this expression with the main term from (19) which in turn is majorized by

$$B \equiv \frac{\varphi(x)}{\sqrt{npq}}\Big(\frac{|x^2-1|(1-2p)}{6} + \frac{1}{2}\Big).$$

Put $x = 3$. If $p \to 0.5$, then $A/B \to \tau_n\sqrt{2\pi} < 18.712$, and if $p \to 0$, then

$$A/B \to \Big(6\tau_n\sqrt{2\pi} + 8.224\Big)/11 < 10.96.$$

Computations show that $10.9 < A/B < 18.712$; for example, for $p = 0.25$ we have $A/B \approx 12.6$.

5. Note that in the work of V. V. Senatov [16, Theorem 5] the bound

$$G_n(x) - \Phi(x) = Q_n(x) + R$$

is obtained for the case of lattice distributions, where x belongs to a special lattice, $R = O(1/n)$, and for R the bound is found in an explicit but rather complicated form.

2. Proof of Lemma 1

Let x_0 be a discontinuity point of the function G. Since $G(x)$ is constant for $x_0 < x < x_0 + h$ and $x_0 - h < x < x_0$, for $0 < s < h$ we have

$$G(x_0 \pm s) = G(x_0 \pm). \tag{47}$$

Consider the case "+." It follows from (47) that $\delta(x_0 + s) = \delta(x_0+) - \Big[G_0(x_0 + s) - G_0(x_0)\Big]$. Then

$$(P * \delta)(x_0 + h/2) = \frac{1}{h} \int_{-h/2}^{h/2} \delta(x_0 + h/2 - y)\, dy =$$

$$= \delta(x_0+) - \frac{1}{h} \int_{-h/2}^{h/2} \big[G_0(x_0 + h/2 - y) - G_0(x_0)\big]\, dy = \delta(x_0+) - \frac{1}{h} \int_{0}^{h} \big(G_0(x_0 + s) - G_0(x_0)\big)\, ds.$$

Similarly we obtain (7) in the case "−."

3. Proof of Lemma 2

Lemma 3. *Let $U(x)$ be a distribution function, $G(x)$ and $G_0(x)$ be some functions with the bounded variations. Denote*

$$\delta(x) = G(x) - G_0(x),\ \widehat{U}(t) = \int_{-\infty}^{\infty} e^{itx} dU(x),$$

$$\widehat{G}(t) = \int_{-\infty}^{\infty} e^{itx} dG(x),\ \widehat{G}_0(t) = \int_{-\infty}^{\infty} e^{itx} dG_0(x),\ (U * \delta)(x) = \int_{-\infty}^{\infty} \delta(x - y)\, dU(y).$$

If

$$\int_{-\infty}^{\infty} |(U * \delta)(x)|\, dx < \infty, \tag{48}$$

$$\int_{-\infty}^{\infty} \frac{1}{|t|} \left|\big(\widehat{G}(t) - \widehat{G}_0(t)\big)\, \widehat{U}(t)\right| dt < \infty, \tag{49}$$

then for every $x \in \mathbb{R}$ the following equality holds:

$$(U * \delta)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\widehat{G}(t) - \widehat{G}_0(t)}{-it}\, e^{-itx}\, \widehat{U}(t)\, dt.$$

Proof. One can prove this lemma using, for example, [8, pp. 211, 212] or [17, p. 127]. We shall follow [8]. Condition (48) ensures the possibility of integration by parts of $\int_{-\infty}^{\infty} e^{itx} d(U*\delta)(x)$. As a result we obtain

$$\frac{\widehat{G}(t)-\widehat{G}_0(t)}{-it}\,\widehat{U}(t)=\int_{-\infty}^{\infty} e^{itx}(U*\delta)(x)\,dx. \tag{50}$$

It follows from condition (49) that the right-hand side of Eq. (50), being the Fourier transform of the function $(U*\delta)(x)$, is absolutely integrable. Therefore, the inversion formula holds:

$$(U*\delta)(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\int_{-\infty}^{\infty} e^{ity}(U*\delta)(y)\,dy\,dt,$$

which coincides with the equation to be proved.

Denote by $f(t)$ the characteristic function of the random variable $X=\frac{Z-p}{\sqrt{pq}}$ (see (1)).

As in [1], we will use the algebraic equality

$$a^n-b^n=n(a-b)b^{n-1}+(a-b)^2\sum_{j=0}^{n-2}(j+1)a^{n-2-j}b^j, \tag{51}$$

putting $a=f(t/\sqrt{n})$, $b=e^{-t^2/(2n)}$.

We shall also use the notation

$$a_j=\frac{i^j\,\mathbf{E}(X^j-Y^j)}{j!},\quad \widetilde{a}_j=i^j\int_0^1\frac{(1-\theta)^{j-1}}{(j-1)!}\,\mathbf{E}\Big[e^{it\theta X}X^j-e^{it\theta Y}Y^j\Big]\,d\theta,$$

where Y is the standard normal random variable.

Lemma 4. *The following equality holds*:

$$f^n(t/\sqrt{n})-e^{-t^2/2}-\frac{(it)^3\alpha_3(p)}{3!\sqrt{n}}\,e^{-(n-1)t^2/(2n)}=\bigg(\sum_{j=4}^{5}\frac{t^j\,a_j}{n^{-1+j/2}}+\frac{t^6}{n^2}\,\widetilde{a}_6\bigg)e^{-(n-1)t^2/(2n)}+$$
$$+\Big(f(t/\sqrt{n})-e^{-t^2/(2n)}\Big)^2\sum_{j=0}^{n-2}(j+1)f^{n-2-j}(t/\sqrt{n})\,e^{-jt^2/(2n)}. \tag{52}$$

Proof. By virtue of the Taylor formula for every $k\geqslant 4$

$$f(t)-e^{-t^2/2}=\sum_{j=3}^{k-1}t^j\,a_j+t^k\,\widetilde{a}_k. \tag{53}$$

Putting $k=6$, we get from (53) that

$$f(t/\sqrt{n})-e^{-t^2/(2n)}=\Big(\frac{it}{\sqrt{n}}\Big)^3\frac{\alpha_3(p)}{3!}+\sum_{j=4}^{5}\frac{t^j\,a_j}{n^{j/2}}+\frac{t^6}{n^3}\,\widetilde{a}_6. \tag{54}$$

Formula (52) is a consequence of equalities (51) and (54).

Introduce the notation

$$\widehat{P}_n(t) = \int_{-\infty}^{\infty} e^{itx}\, dP_n(x) = \frac{\sin(h_n t/2)}{(h_n t/2)}, \tag{55}$$

$$v(t) = \frac{f^n(t/\sqrt{n}) - e^{-t^2/2} - \frac{\alpha_3(p)}{3!\sqrt{n}}\,(it)^3\, e^{-(n-1)t^2/(2n)}}{-it}\, \widehat{P}_n(t)\, e^{-itx}, \tag{56}$$

$$I(x) = \int_{-\infty}^{\infty} v(t)\, dt \tag{57}$$

(cf. [1, p. 217]). Note that

$$\frac{\alpha_3(p)}{3!\sqrt{n}}\,(it)^3\, e^{-(n-1)t^2/(2n)} = c_n^3 \int_{-\infty}^{\infty} e^{itx}\, dQ_n(p, xc_n).$$

Putting in Lemma 3

$$G(x) = F_n(x), \quad G_0(x) = \Phi_n(p,x), \quad \delta(x) = \delta_n(p,x), \quad U(x) = P_n(x),$$

we can verify that the conditions of the lemma are fulfilled, and consequently,

$$(P_n * \delta_n)(x) = \frac{1}{2\pi}\, I(x), \quad x \in \mathbb{R}.$$

Using (52) we can write

$$v(t) = v_1(t) + v_2(t),$$

where

$$v_1(t) = \frac{1}{-it}\left(\sum_{j=4}^{5} \frac{t^j\, a_j}{n^{-1+j/2}} + \frac{t^6}{n^2}\,\widetilde{a}_6\right) e^{-(n-1)t^2/(2n)}\, \widehat{P}_n(t)\, e^{-itx}, \tag{58}$$

$$v_2(t) = \frac{(f(t/\sqrt{n}) - e^{-t^2/(2n)})^2}{-it}\, \widehat{P}_n(t)\, e^{-itx} \sum_{j=0}^{n-2} (j+1) f^{n-2-j}(t/\sqrt{n})\, e^{-jt^2/(2n)}\,. \tag{59}$$

Denote $\tau_n = \left(\frac{6\sqrt{n}}{\beta_3(p)}\right)^{1/3}$,

$$I_1(x) = \int_{|t|\leqslant\tau_n} v_1(t)\, dt, \quad I_2(x) = \int_{|t|\leqslant\tau_n} v_2(t)\, dt, \quad I_3(x) = \int_{|t|>\tau_n} v(t)\, dt. \tag{60}$$

It is not hard to verify that

$$I(x) = I_1(x) + I_2(x) + I_3(x). \tag{61}$$

Hence,

$$(P_n * \delta_n)(x) = \frac{1}{2\pi}\Big(I_1(x) + I_2(x) + I_3(x)\Big), \quad x \in \mathbb{R}. \tag{62}$$

Denote (see [1])

$$\begin{aligned} &\sigma_n = \sigma_n(p) = \sqrt{npq}, \quad \omega_3(p) = q - p, \\ &\omega_4(p) = |q^3 + p^3 - 3pq|, \quad \omega_5(p) = q^4 - p^4, \quad \omega_6(p) = q^5 + p^5 + 15(pq)^2, \end{aligned} \tag{63}$$

$$T_1(p,n) = \frac{\omega_4(p)}{\sigma_n^2\,12\pi}\,c_n^4 + \frac{\omega_5(p)}{40\sqrt{2\pi}\sigma_n^3}\,c_n^5 + \frac{\omega_6(p)}{90\pi\,\sigma_n^4}\,c_n^6. \tag{64}$$

Note that the function $T_1(p,n)$ is less than $K_1(p,n)$ from [1, p. 218] by the quantity $\frac{\omega_3(p)}{4\sigma_n\sqrt{2\pi}(n-1)}\bigl(1+\frac{1}{4(n-1)}\bigr)$.

Lemma 5. *The following inequality holds*:

$$\frac{1}{2\pi}\,|I_1(x)| \leqslant T_1(p,n). \tag{65}$$

Proof. It follows from (58) and (60) that

$$|I_1(x)| \leqslant \sum_{j=4}^{5}\frac{|a_j|}{n^{-1+j/2}}\int_{-\infty}^{\infty}|t|^{j-1}e^{-(n-1)t^2/(2n)}\,dt + \frac{|\widetilde{a}_6|}{n^2}\int_{-\infty}^{\infty}|t|^5 e^{-(n-1)t^2/(2n)}\,dt.$$

We have

$$\int_{-\infty}^{\infty}|t|^{j-1}e^{-(n-1)t^2/(2n)}\,dt = \Bigl(\frac{n}{n-1}\Bigr)^{j/2}\sqrt{2\pi}\,\mathbf{E}|Y|^{j-1}, \quad \mathbf{E}|Y|^3 = \frac{4}{\sqrt{2\pi}}, \quad \mathbf{E}Y^4 = 3, \quad \mathbf{E}|Y|^5 = \frac{16}{\sqrt{2\pi}},$$

$$\mathbf{E}Y^6 = 15, \quad |a_4| = \frac{1}{4!}\,|\alpha_4(p)-3|, \quad |a_5| = \frac{|\alpha_5(p)|}{5!} = \frac{\alpha_5(p)}{5!}, \quad |\widetilde{a}_6| \leqslant \frac{\mathbf{E}X^6+\mathbf{E}Y^6}{6!} = \frac{\alpha_6(p)}{6!} + \frac{1}{48}.$$

Hence

$$\frac{1}{2\pi}\,|I_1(x)| \leqslant \frac{|\alpha_4(p)-3|}{12\pi n}\,c_n^4 + \frac{\alpha_5(p)}{40\sqrt{2\pi}\,n^{3/2}}\,c_n^5 + \frac{\alpha_6(p)+15}{90\pi\,n^2}\,c_n^6. \tag{66}$$

Taking into account the notation (63), it is easy to verify that the right-hand side of inequality (66) coincides with $T_1(p,n)$.

The following statement is proved in [1, pp. 227–230].

Lemma 6. *The bound*

$$\frac{1}{2\pi}\,|I_2(x)| \leqslant K_2(p,n)$$

holds, where $K_2(p,n)$ is the function defined in [1, p. 253].

Lemma 7. *Under condition* (9) *the following bound holds*:

$$\frac{1}{2\pi}\,|I_3(x)| \leqslant K_3(p,n) + T_4(p,n),$$

where $K_3(p,n)$ is the function from [1, p. 219],

$$T_4(p,n) = \frac{\omega_3(p)\sqrt{2}}{6\pi\sigma_n}\,c_n^3\Gamma\biggl(\frac{3}{2}, \Bigl(\frac{6\sigma_n}{\omega(p)}\Bigr)^{2/3}\frac{1}{2c_n^2}\biggr), \tag{67}$$

and $\Gamma(\alpha,x) = \int_x^{\infty} t^{\alpha-1}e^{-t}\,dt$ is an incomplete gamma-function.

Proof. It was proved in [1, see p. 219 and pp. 230–237] that under condition (9)

$$\frac{1}{2\pi}\int_{|t|\geqslant\tau_n}\frac{|f^n(t/\sqrt{n})| + e^{-t^2/2}}{|t|}\,|\widehat{P}_n(t)|\,dt \leqslant K_3(p,n).$$

Therefore, to obtain the statement of the lemma, it suffices to show that

$$\frac{\alpha_3(p)}{3!\,2\pi\sqrt{n}} \int\limits_{|t|\geqslant\tau_n} t^2 e^{-(n-1)t^2/(2n)}\,dt = T_4(p,n). \tag{68}$$

We have

$$\int\limits_{|t|\geqslant\tau_n} t^2 e^{-t^2(n-1)/(2n)}\,dt = \Big(\frac{2n}{n-1}\Big)^{3/2} \int\limits_{\frac{\tau_n^2(n-1)}{2n}}^{\infty} x^{1/2}e^{-x}\,dx = \Big(\frac{2n}{n-1}\Big)^{3/2}\Gamma\Big(\frac{3}{2}, \frac{\tau_n^2(n-1)}{2n}\Big).$$

Taking into account the equalities

$$\tau_n = \Big(\frac{6\sigma_n}{\omega(p)}\Big)^{1/3}, \quad \alpha_3(p) = \frac{\omega_3(p)}{\sqrt{pq}},$$

it is easy to prove the validity of (68).

Proof of Lemma 2. Inequality (10) follows from (62) and Lemmas 5–7.

Analyzing every summand on the right-hand side of (10), it is easy to verify that $T(p,n) = O(1/n)$, decreasing in n, and moreover, the bounds (11) and (12) hold (see also Remark 2). These arguments lead to the conclusion that $nT_j(p,n)$, $j = 1, 2, 3$, decrease in n as well (but do not converge to 0). It remains to show the decrease of $nT_4(p,n)$ in n. To this end it suffices to prove that $\sqrt{n}\,\Gamma\Big(\frac{3}{2}, \big(\frac{6\sigma_n}{\omega(p)}\big)^{2/3}\frac{1}{2c_n^2}\Big)$ decreases. After elementary calculations we conclude that the function $x^{3/2}\Gamma\big(\frac{3}{2}, x\big)$ decreases for $x \geqslant 2.2$. It remains to note that $x = \big(\frac{6\sigma_n}{\omega(p)}\big)^{2/3}\frac{n-1}{2n} > 2.6$ under condition (9).

4. Proof of Theorem 1

The following Lemmas 8 and 9 are of a general nature. Therefore we will introduce special notations.

By $Q_n(x)$ we denote the first term of the Edgeworth expansion for the distribution function of $\frac{1}{\sqrt{n}}\sum_{j=1}^n \xi_j$, where $\xi_1, \xi_2, \ldots$ are i.i.d. random variables with $\mathbf{E}\xi_1 = 0$, $\mathbf{E}\xi_1^2 = 1$, $\mathbf{E}\xi_1^3 = \alpha_3$. It is known that $Q_n(x) = -\frac{\alpha_3}{3!\sqrt{n}}\varphi''(x)$. Denote $\Phi_n(x) = \Phi(x) + Q_n(x)$.

Lemma 8. *For all $x \in \mathbb{R}$ and $h > 0$*

$$\frac{1}{h}\int\limits_0^h \Big(\Phi_n(x+s) - \Phi_n(x)\Big)\,ds = \frac{h}{2}\,\varphi(x) + r_1^+(h,x), \tag{69}$$

where

$$|r_1^+(h,x)| \leqslant \frac{h^2}{6}\sup_{y\in J^+(h,x)}|\varphi'(y)| + \frac{h\,|\alpha_3|}{12\sqrt{n}}\sup_{y\in J^+(h,x)}|\varphi^{(3)}(y)|. \tag{70}$$

Proof. The lemma is easily derived from the representations

$$\Phi(x+s) - \Phi(x) = s\varphi(x) + \frac{s^2}{2}\,\varphi'(x+\lambda_1 s), \quad 0 \leqslant \lambda_1 \leqslant 1, \tag{71}$$

$$\varphi''(x+s) - \varphi''(x) = s\varphi^{(3)}(x+\lambda_2 s), \quad 0 \leqslant \lambda_2 \leqslant 1. \tag{72}$$

Let ξ_1 be a discrete random variable. Denote

$$G_n(x) = \mathbf{P}\Big(\frac{1}{\sqrt{n}}\sum_{j=1}^n \xi_j < x\Big), \quad \delta_n(x) = G_n(x) - \Phi_n(x).$$

Let x_0 be a discontinuity point of the function G_n, and assume that the interval (x_0-h, x_0+h) contains no other such points. As above, let $P(x)$ be the uniform distribution on $[-h/2, h/2]$.

Lemma 9. *For every $x \in J^+(h, x_0)$ the following equality holds*:

$$\delta_n(x) = (P * \delta_n)(x_0 + h/2) + \Big(\frac{h}{2} - x + x_0\Big)\varphi(x) + r^+(h, x), \tag{73}$$

where

$$|r^+(h, x)| \leqslant \frac{7}{6}\, h^2 \sup_{y \in J^+(h,x_0)} |\varphi'(y)| + \frac{h\,|\alpha_3|}{4\sqrt{n}} \sup_{y \in J^+(h,x_0)} |\varphi^{(3)}(y)|. \tag{74}$$

Proof. Since the function G_n is constant in the interval $J^+(h, x_0)$, $G_n(x) = G_n(x_0+)$. Hence,

$$\delta_n(x) = \delta_n(x_0+) - \Big(\Phi_n(x) - \Phi_n(x_0)\Big). \tag{75}$$

In view of (71) and (72),

$$\Phi_n(x) - \Phi_n(x_0) = (x - x_0)\,\varphi(x_0) + r_2(h, x), \tag{76}$$

where

$$r_2(h, x) = \frac{(x - x_0)^2}{2}\,\varphi'\big(x_0 + \lambda(x - x_0)\big) - \frac{\alpha_3}{3!\sqrt{n}}\,(x - x_0)\,\varphi^{(3)}\big(x_0 + \lambda(x - x_0)\big).$$

Obviously,

$$|r_2(h, x)| \leqslant \frac{h^2}{2} \sup_{y \in J^+(h,x_0)} |\varphi'(y)| + \frac{h\,|\alpha_3|}{6\sqrt{n}} \sup_{y \in J^+(h,x_0)} |\varphi^{(3)}(y)|. \tag{77}$$

Using Lemmas 1 and 8 we get

$$\delta_n(x_0+) = (P * \delta_n)(x_0 + h/2) + \frac{h}{2}\,\varphi(x_0) + r_1^+(h, x_0). \tag{78}$$

It follows from (75), (76), (78) that

$$\delta_n(x) = (P * \delta_n)(x_0 + h/2) + \frac{h}{2}\,\varphi(x_0) + r_1^+(h, x_0) - (x - x_0)\,\varphi(x_0) - r_2^+(h, x). \tag{79}$$

The lemma follows from (70), (77), (79) and the simple bound: for $x \in J^+(h, x_0)$

$$\Big|\Big(\frac{h}{2} - x + x_0\Big)\Big(\varphi(x_0) - \varphi(x)\Big)\Big| \leqslant \frac{h^2}{2} \sup_{y \in J^+(h,x_0)} |\varphi'(y)|.$$

Lemma 10. *Let $n \geqslant 1$. For all $x \in J^+(h_n, x_k)$ the following equality holds,*

$$\delta_n(p, x) = (P_n * \delta_n)(x_k + h_n/2) + h_n S(np + x/h_n)\,\varphi(x) + R^+(h_n, x), \tag{80}$$

where

$$|R^+(h_n, x)| \leqslant \frac{7}{6}\, h_n^2 \sup_{y \in J^+(h_n,x_k)} |\varphi'(y)| + \frac{h_n\,\alpha_3(p)}{4\sqrt{n}} \sup_{y \in J^+(h_n,x_k)} |\varphi^{(3)}(y)|. \tag{81}$$

Proof. According to Lemma 9,

$$\delta_n(p, x) = (P_n * \delta_n)(x_k + h_n/2) + \Big(\frac{h_n}{2} - x + x_k\Big)\varphi(x) + R^+(h_n, x) \tag{82}$$

for $x \in J^+(h_n, x_k)$, where $R^+(h_n, x)$ satisfies the inequality (81).

Note that $k < np + x/h_n < k + 1$ for $x \in J^+(h_n, x_k)$. By the definition of the function $S(x)$ for $x \in J^+(h_n, x_k)$, we have

$$h_n S(np + x/h_n) = x_k - x + \frac{h_n}{2}.$$

Proof of Theorem 1. The statement of the theorem follows from Lemmas 2 and 10.

5. Proof of Corollary 1

Lemma 11. *The following bound holds*:

$$\big|\widetilde{Q}_n(p,x) - Q_n(p,x)\big| \leqslant \frac{\alpha_3(p)}{12n^{3/2}}\Big(|x|c_n^2 \max_{|x|\leqslant|y|\leqslant|x|c_n} |\varphi^{(3)}(y)| + 3c_n^3|\varphi''(xc_n)|\Big). \tag{83}$$

Proof. Using the Taylor formula, it is easy to verify that

$$\varphi''(xc_n) - \varphi''(x) = x(c_n-1)\varphi^{(3)}\Big(x\big(1+\theta(c_n-1)\big)\Big), \quad 0<\theta<1, \quad c_n - 1 = \varepsilon_1, \quad 0<\varepsilon_1<\frac{1}{2n}\,c_n^2,$$

and $c_n^3 = 1+\varepsilon_2$, $0<\varepsilon_2<\frac{3}{2n}\,c_n^3$ as well. Hence,

$$c_n^3 Q_n(p,xc_n) - Q_n(p,x) = Q_n(p,xc_n) - Q_n(p,x) + \varepsilon_2 Q_n(p,xc_n) =$$
$$= -\frac{\alpha_3(p)}{6\sqrt{n}}\,x\varepsilon_1\varphi^{(3)}\Big(x\big(1+\theta(c_n-1)\big)\Big) - \frac{\alpha_3(p)}{6\sqrt{n}}\,\varepsilon_2\varphi''(xc_n),$$

which implies the lemma.

Proof of Corollary 1. The statement of the corollary follows from Theorem 1 and Lemma 11.

REFERENCES

1. S. V. Nagaev and V. I. Chebotarev, "On the bound of proximity of the binomial distribution to the normal one," *Theory Probab. Appl.*, **56**, No. 2, 213–239 (2012).
2. C.-G. Esseen, "A moment inequality with an application to the central limit theorem," *Scand. Actuar. J.*, **3–4**, 160–170 (1956).
3. V. M. Zolotarev, "An absolute estimate of the remainder in the central limit theorem," *Teor. Veroyatn. Primen.*, **11**, No. 1, 108–119 (1966).
4. V. M. Zolotarev, "A sharpening of the inequality of Berry–Esseen," *Z. Wahrscheinlichkeitstheor. verw. Geb.*, **8**, 332–342 (1967).
5. I. G. Shevtsova, "On the smoothing inequality," *Dokl. Akad. Nauk*, **430**, No. 5, 600–602 (2010).
6. H. Prawitz, "Limits for a Distributions, if the Characteristic Function is given in a Finite Domain," *Skand. Aktuar. J.*, **55**, 138–154 (1972).
7. C.-G. Esseen, "Fourier analysis of distribution function. A muthematical study of the Laplace–Gaussian law," *Acta Math.*, **77**, 1–125 (1945).
8. B. V. Gnedenko and A. N. Kolmogorov, *Limit Distributions for Sums of Independent Random Variables*, Gos. Izd. Tekhn.-Teor. Lit., Moscow, Leningrad (1949).
9. H. Bergström, "On asymptotic expansions of probability functions," *Skand. Aktuar. J.*, **1–2**, 1–34 (1951).
10. A. M. Zubkov and A. A. Serov, "A complete proof of universal inequalities for distribution function of binomial law," *Theory Probab. Appl.*, **57**, No. 3, 539–544 (2013).
11. D. Alfers and H. Dinges, "A normal approximation for Beta and Gamma tail probabilities," *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, **65**, 399–420 (1984).
12. J. V. Uspensky, *Introduction to Mathematical Probability*, McGraw Hill, New York (1937).
13. I. G. Shevtsova, *Optimization of the structure of the moment bounds for accuracy of normal approximation for the distributions of sums of independent random variables*, Dissertation for the degree of Doctor of Physico–Mathematical Sciences, Moscow State University, Moscow (2013).

14. W. Feller. “Generalization of a probability limit theorem of Cramér,” *Trans. Am. Math. Soc.*, **54**, No. 3, 361–372 (1943).

15. C. Lénart, “On certain theorems of Berry and a limit theorem of Feller,” *Mat. Časopis*, **18**, No. 1, 59–75 (1968).

16. V.V. Senatov, “About non-uniform bounds of approximation accuracy,” *Theory Probab. Appl.*, **59**, No. 2, 276–312 (2013).

17. A. C. Berry, “The accuracy of the Gaussian approximation to the sum of independent variates”, *Trans. Am. Math. Soc.*, **49**, 122–136 (1941).