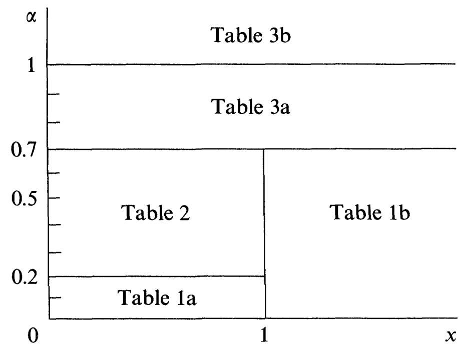

3. If $0 \leqq \alpha \leqq 0.7$ and $x \geqq 1$, then Table 1b must be used:

$$G(x, \alpha) = 10^{-4}F_1(x^{-\alpha}, \alpha).$$

4. If $0.7 \leqq \alpha < 1$ and $0 < x \leqq 20$, then the values of $G(x, \alpha)$ are derived from Table 3a:

$$G(x, \alpha) = 10^{-4}F_3(y, \alpha) = 10^{-6}G(y, 1) - 10^{-4}D_3(y, \alpha),$$

where $y = (1 - \alpha)^{-1/\alpha}x - (1 - \alpha)^{-1}$ ($G(y, 1)$ is determined from Table 3b). For $0.7 \leqq \alpha < 1$ and $x > 20$ the table added to Table 3b is used:

$$G(x, \alpha) = 10^{-4}\tilde{F}_3(z, \alpha) = 10^{-6}G(z, 1) - 10^{-4}\tilde{D}_3(z, \alpha),$$

where

$$z = [(1 - \alpha)^{-1/\alpha}x + (1 - \alpha)^{-1}]^{\alpha} = y^{\alpha}.$$

5. If $\alpha = 1$, then one must use Table 3b for the computation of $G(x, 1)$. In this table the values of the function are directly given, multiplied by 10^6.

The tables were compiled by means of the quadrature formula mentioned in Section 1. The computation was done on EBM "Strela" BECM-6.

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ON THE SPEED OF CONVERGENCE OF THE DISTRIBUTION OF MAXIMUM SUMS OF INDEPENDENT RANDOM VARIABLES

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(Translated by B. Seckler)

Let $\xi_n, n = 1, 2, 3, \cdots$, be a sequence of identically distributed independent random variables such that $\mathbf{E}\xi_1 = 0$. Set

$$\sigma^2 = \mathbf{D}\xi_1, \qquad c_3 = \mathbf{E}|\xi_1|^3, \qquad S_n = \sum_{i=1}^{n} \xi_i,$$

$$\bar{S}_n = \max_{1 \leq i \leq n} S_i, \qquad F(x) = \mathbf{P}(\xi_1 < x),$$

$$F_n(x) = \mathbf{P}(S_n < x), \qquad \bar{F}_n(x) = \mathbf{P}(\bar{S}_n < x).$$

In [1], the author proved that

$$\left|\bar{F}_n(\sigma x\sqrt{n}) - \left(\frac{2}{\pi}\right)^{1/2}\int_0^x e^{-u^2/2}\,du\right| < \frac{Lc_3^2}{\sigma^6\sqrt{n}}\min\left[\log n, \frac{1+x^2}{x^2}\right], \tag{1}$$

where L is an absolute constant. The aim of this paper is to obtain a more exact estimate.

Theorem. *There exists an absolute constant K such that*

$$\sup_{0 \leq x < \infty}\left|\bar{F}_n(\sigma x\sqrt{n}) - \left(\frac{2}{\pi}\right)^{1/2}\int_0^x e^{-u^2/2}\,du\right| < \frac{Kc_3^2}{\sigma^6\sqrt{n}}.$$

We shall retain the notation of [1] and we shall use it without making any special comments. In particular, the symbol O will be used as in [1] only where the corresponding constant is an absolute one.

In referring to the relations proved in [1], we shall add the number 1 to them, i.e., we shall write $(1.j)$ instead of (j).

PROOF OF THEOREM. Without loss of generality, we may assume that $\sigma^2 = 1$. The estimate $O(n^{-1/2}c_3^2\min[\log n, (1+x^2)/x^2])$ which was obtained in [1] instead of $O(c_3^2/n^{1/2})$ is due to the fact that the estimates (1.65) and (1.66) for $\Omega_{1n}(x)$ and (1.70) for $\Omega_{2n}(x)$ are not sufficiently exact.

If we succeed in obtaining the estimate $O(c_3^2/n^{1/2})$ for $\Omega_{1n}(x)$ and $\Omega_{2n}(x)$, the theorem will have been proved.

We first consider the estimation of $\Omega_{2n}(x)$. In [1] the principal role in estimating $\Omega_{2n}(x)$ was played by Lemma 5.

We shall supplement the estimate in this lemma with another one which turns out to be more useful when $x = o(\sqrt{n})$, namely, the following holds.

Lemma.

$$G_n'(x) = O\left(\frac{C_3}{\sqrt{n}}\min\left[\frac{1}{|x|}, \frac{|x|}{n}\right] + \frac{c_3^2}{n^{3/2}}\right) + r_n, \tag{2}$$

where $\sum_{n=1}^{\infty}|r_n| = O(c_3)$.

PROOF. First of all,

$$G_n'(x) = \frac{1}{2\pi}\int_{|t| \leq c} e^{-itx}(f^n(t) - e^{-nt^2/2})h(5c_3 t)\,dt; \tag{3}$$

here and later on $c = (5c_3)^{-1}$. Using (1.14), we obtain

$$\int_{|t| \leq c} e^{-itx}(f^n(t) - e^{-nt^2/2})h(5c_3 t)\,dt = \int_{|t| \leq c} e^{-itx}(f^n(t) - e^{-nt^2/2})\,dt + O\left(nc_3^3\int_0^c t^5 e^{-nt^2/4}\,dt\right), \tag{4}$$

since

$$h(t) = 1 + O(t^2).$$

Let us now estimate the integral

$$I_n \equiv \int_{|t| \leq c} e^{-itx}(f^n(t) - e^{-nt^2/2})\,dt.$$

It is not hard to see that

$$f(t) - e^{-t^2/2} = t^4 R_1(t) - it^3 I(t) + 1 - \frac{t^2}{2} - e^{-t^2/2}, \tag{5}$$

where

$$R_1(t) = \frac{1}{t^4}\int_{-\infty}^{\infty}\left(\cos tx - 1 + \frac{t^2x^2}{2}\right)dF(x),$$

$$I(t) = \frac{1}{t^3}\int_{-\infty}^{\infty} (\sin tx - tx)\, dF(x).$$

Let us estimate the integral

$$I_{1n} \equiv \int_{|t|\leq c} t^3 I(t)\, e^{-nt^2/2} \sin tx\, dt.$$

It is not hard to see that

$$I_{1n} = \int_{-\infty}^{\infty} dF(y) \int_{|t|\leq c} (\sin ty - ty)\, e^{-nt^2/2} \sin tx\, dt.$$

Set

$$\varphi(u) = \frac{\sin u - u}{u^3}.$$

Clearly,

(6) $$I_{1n} = \int_{-\infty}^{\infty} y^3\, dF(y) \int_{|t|\leq c} t^3 \varphi(ty)\, e^{-nt^2/2} \sin tx\, dt.$$

Further,

(7) $$\int_{|t|\leq c} t^3\varphi(ty)\, e^{-nt^2/2} \sin tx\, dt = -t^3\varphi(ty)\, e^{-nt^2/2} \frac{\cos tx}{x}\Bigg|_{t=-c}^{t=c} + \frac{1}{x}\int_{|t|\leq c} \frac{d}{dt}(t^3\varphi(ty)\, e^{-nt^2/2}) \cos tx\, dt.$$

Clearly,

(8) $$\frac{d}{dt} t^3\varphi(ty)\, e^{-nt^2/2} = -nt^4\, e^{-nt^2/2}\varphi(ty) + 3t^2\, e^{-nt^2/2}\varphi(ty) + yt^3\, e^{-nt^2/2}\varphi'(ty).$$

Observe that

$$\sup_u |\varphi(u)| < \infty \quad \text{and} \quad \sup_u |u\varphi'(u)| < \infty.$$

Therefore,

(9) $$\begin{aligned} \int_{|t|\leq c} |yt^3\, e^{-nt^2/2}\varphi'(ty)|\, dt &= O\left(\int_0^{\infty} t^2\, e^{-nt^2/2}\, dt\right) = O(n^{-3/2}), \\ n\int_{|t|\leq c} t^4\, e^{-nt^2/2}|\varphi(ty)|\, dt &= O(n^{-3/2}), \\ \int_{|t|\leq c} t^2\, e^{-nt^2/2}|\varphi(ty)|\, dt &= O(n^{-3/2}), \\ t^3\varphi(ty)\, e^{-nt^2/2} &= O(n^{-3/2}). \end{aligned}$$

From (6)–(9) it follows that

(10) $$I_{1n} = O\left(\frac{c_3}{|x|n^{3/2}}\right).$$

On the other hand,

(11) $$I_{1n} = O\left(|x|\int_{|t|\leq c} t^4 |I(t)|\, e^{-nt^2/2}\, dt\right) = O(c_3|x|n^{-5/2}),$$

since $I(t) = O(c_3)$. By (5),

(12) $$\begin{aligned} (f(t) - e^{-t^2/2})\, e^{-kt^2/2} f^{n-k}(t) = {} & (t^4 R_1(t) - it^3 I(t))\, e^{-nt^2/2} \\ & + (f(t) - e^{-t^2/2})(f^{n-k}(t) - e^{-(n-k)t^2/2})\, e^{-kt^2/2} \\ & + \left(1 - \frac{t^2}{2} - e^{-t^2/2}\right) e^{-nt^2/2}. \end{aligned}$$

It is not hard to show that

$$(13)\qquad \int_{|t|\le c}(t^4R_1(t) - it^3I(t))\,e^{-itx-nt^2/2}\,dt = \int_{|t|\le c} t^4R_1(t)\,e^{-nt^2/2}\cos tx\,dt + I_{1n}.$$

The relations (12), (13), (10), (1.19), (1.21), (1.22) and (1.24) lead to the estimate

$$(14)\qquad I_n = O\left(c_3\min\left[\frac{|x|}{n^{3/2}}, \frac{1}{|x|\sqrt{n}}\right] + \frac{c_3^2}{n^{3/2}}\right) + r_n,$$

where r_n satisfies the condition

$$\sum_{n=1}^{\infty}|r_n| = O(c_3).$$

Taking into account that

$$\int_0^c t^5\,e^{-nt^2/4}\,dt = O\left(\frac{1}{c_3}\int_0^\infty t^4\,e^{-nt^2/4}\,dt\right) = O\left(\frac{1}{c_3n^{5/2}}\right),$$

we obtain from (3), (4) and (14) the assertion of the lemma.

We now continue with the proof of the theorem. We first estimate $\Omega_{2n}(x)$. From (1.67) we can deduce without difficulty

$$(15)\qquad \Omega_{2n}(x) = \sum_{k=1}^{n-1}\int_{-\infty}^0 G_k'(x-u)\bar{F}_{n-k}(u)\,du.$$

By Lemma 5 of [1],

$$(16)\qquad G_k'(x) = O\left(\frac{c_3}{n}\right)$$

for $k \geqq n/2$. Applying (16) and (1.69), we obtain

$$(17)\qquad \sum_{k=n/2}^{n}\int_{-\infty}^0 G_k'(x-u)\bar{F}_{n-k}(u)\,du = O\left(\frac{c_3}{n}\sum_{k=1}^{n/2}\bar{a}_k\right) = O\left(\frac{c_3}{\sqrt{n}}\right).$$

It is not hard to show that

$$\min\left[\frac{1}{|x|}, \frac{|x|}{k}\right] = O\left(\min\left[\frac{1}{|x|+1}, \frac{|x|}{k}\right]\right).$$

Therefore, the estimate (2) can be rewritten as follows:

$$G_k'(x) = O\left(\frac{c_3}{\sqrt{k}}\min\left[\frac{1}{|x|+1}, \frac{|x|}{k}\right] + \frac{c_3^2}{k^{3/2}}\right) + r_k.$$

Consequently,

$$(18)\qquad \begin{aligned}\sum_{k=1}^{n/2-1}\int_{-\infty}^0 G_k'(x-u)\bar{F}_{n-k}(u)\,du &= O\left(\sum_{k=1}^{(x^2+1)c_3^2}\frac{c_3}{\sqrt{k}}\int_{-\infty}^0\frac{1}{x-u+1}\bar{F}_{n-k}(u)\,du\right)\\ &+ O\left(c_3\sum_{k=(x^2+1)c_3^2}^{n/2}k^{-3/2}\int_{-\infty}^0(x-u)\bar{F}_{n-k}(u)\,du\right)\\ &+ O\left(\sum_{k=1}^{n/2}\left(|r_k| + \frac{c_3^2}{k^{3/2}}\right)(n-k)^{-1/2}\right),\end{aligned}$$

since there is no loss of generality in assuming that $(x^2+1)c_3^2 > n/2$. By virtue of (1.69),

$$\int_{-\infty}^0(x-u)\bar{F}_n(u)\,du = -\int_{-\infty}^0 u\bar{F}_n(u)\,du + O(x/\sqrt{n}).$$

By Lemma 4 of [1],

$$\int_{-\infty}^0 u\bar{F}_n(u)\,du = O\left(\frac{c_3^2}{\sqrt{n}}\right).$$

Therefore,

$$\int_{-\infty}^{0} (x-u)\bar{F}_n(u)\,du = O\left(\frac{|x| + c_3^2}{\sqrt{n}}\right).$$

Hence,

$$\sum_{k=(x^2+1)c_3^2}^{n/2-1} \frac{c_3}{k^{3/2}} \int_{-\infty}^{0} (x-u)\bar{F}_{n-k}(u)\,du = O\left(c_3(c_3^2 + |x|) \sum_{k=(x^2+1)c_3^2}^{n/2} \frac{1}{(n-k)^{1/2}k^{3/2}}\right).$$

Clearly,

$$\sum_{k=(x^2+1)c_3^2}^{n/2} \frac{1}{(n-k)^{1/2}k^{3/2}} = O\left(\frac{1}{\sqrt{n}} \sum_{k=(x^2+1)c_3^2}^{\infty} \frac{1}{k^{3/2}}\right) = O\left(\frac{1}{\sqrt{n}c_3(|x|+1)}\right).$$

Thus,

$$c^3 \sum_{k=(x^2+1)c_3^2}^{n/2-1} \frac{1}{k^{3/2}} \int_{-\infty}^{0} (x-u)\bar{F}_{n-k}(u)\,du = O\left(\frac{c_3^2}{\sqrt{n}}\right). \tag{19}$$

Clearly,

$$\int_{-\infty}^{0} \frac{1}{x-u+1}\bar{F}_n(u)\,du \leqq -\frac{\bar{a}_n}{x+1}, \qquad x \geqq 0.$$

Therefore,

$$\sum_{k=1}^{(x^2+1)c_3^2} \frac{c_3}{\sqrt{n}} \int_{-\infty}^{0} \frac{1}{x-u+1}\bar{F}_{n-k}(u)\,du = O\left(\frac{c_3}{(x+1)\sqrt{n}} \sum_{k=1}^{(x^2+1)c_3^2} \frac{1}{\sqrt{k}}\right) = O\left(\frac{c_3^2}{\sqrt{n}}\right), \qquad x \geqq 0. \tag{20}$$

Further,

$$\sum_{k=1}^{n/2-1} \left(|r_k| + \frac{c_3^2}{k^{3/2}}\right)(n-k)^{-1/2} = O\left(\frac{c_3^2}{\sqrt{n}}\right). \tag{21}$$

From (18)–(21) results

$$\sum_{k=1}^{n/2-1} \int_{-\infty}^{0} G_k'(x-u)\bar{F}_{n-k}(u)\,du = O\left(\frac{c_3^2}{\sqrt{n}}\right). \tag{22}$$

In turn, (17) and (22) lead by virtue of (15), to the estimate

$$\Omega_{2n}(x) = O\left(\frac{c_3^2}{\sqrt{n}}\right). \tag{23}$$

Let us now estimate $\Omega_{1n}(x)$. Observe first of all that

$$\tilde{\varphi}_n(t) = -\int_{-\infty}^{0} e^{itx} p_n^{(2)}(x)\,dx,$$

where

$$p_n^{(2)}(x) = \int_{-\infty}^{x} \bar{F}_n(y)\,dy$$

and

$$t^2 e^{-kt^2/2} = S\left(-\frac{x}{\sqrt{2\pi}k^{3/2}} e^{-x^2/2k}\right).$$

Hence,

$$\Omega_{1n}(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n-1} k^{-3/2} \int_{-\infty}^{0} (x-y)\,e^{-(x-y)^2/2k} p_{n-k}^{(2)}(y)\,dy. \tag{24}$$

It is not hard to see that, for $x \geqq 0$,

$$(25)\quad \int_{-\infty}^{0} (x-y)\, e^{-(x-y)^2/2k} p_{n-k}^{(2)}(y)\, dy \leqq x\, e^{-x^2/2k} \int_{-\infty}^{0} p_{n-k}^{(2)}(y)\, dy + \int_{-\infty}^{0} |y|\, e^{-y^2/2k} p_{n-k}^{(2)}(y)\, dy.$$

Applying Lemma 4 of [1], we obtain

$$(26)\qquad \int_{-\infty}^{0} p_n^{(2)}(y)\, dy = \frac{\bar{b}_n}{2} = O\left(\frac{c_3^2}{\sqrt{n}}\right).$$

From (26) and (1.84) results

$$(27)\qquad \begin{aligned} x \sum_{k=1}^{n-1} k^{-3/2} e^{-x^2/2k} \int_{-\infty}^{0} p_{n-k}^{(2)}(y)\, dy &= O\left(c_3^2 x \sum_{k=1}^{n-1} k^{-3/2}(n-k)^{-1/2} e^{-x^2/2k}\right) \\ &= O\left(\frac{c_3^2}{\sqrt{n}}\right), \qquad x \geqq 0. \end{aligned}$$

Clearly,

$$\bar{F}_n(x) \geqq \bar{F}_{n+1}(x).$$

Therefore,

$$p_n^{(2)}(x) \geqq p_{n+1}^{(2)}(x).$$

Hence,

$$\sum_{k=1}^{n/2} \frac{1}{k^{3/2}} \int_{-\infty}^{0} |y|\, e^{-y^2/2k} p_{n-k}^{(2)}(y)\, dy \leqq \sum_{k=1}^{n/2} \frac{1}{k^{3/2}} \int_{-\infty}^{0} |y|\, e^{-y^2/2k} p_{n/2}^{(2)}(y)\, dy.$$

In consequence of (1.84),

$$|y| \sum_{k=1}^{n/2} \frac{1}{k^{3/2}} e^{-y^2/2k} \leqq \sqrt{n}|y| \sum_{k=1}^{n} \frac{e^{-y^2/2k}}{k^{3/2}(n-k)^{1/2}} = O(1).$$

Taking (26) into account, we find

$$(28)\qquad \sum_{k=1}^{n/2} \frac{1}{k^{3/2}} \int_{-\infty}^{0} |y| e^{-y^2/2k} p_{n-k}^{(2)}(y)\, dy = O(\bar{b}_{n/2}) = O\left(\frac{c_3^2}{\sqrt{n}}\right).$$

Further

$$\int_{-\infty}^{0} |y|\, e^{-y^2/2k} p_{n-k}^{(2)}(y)\, dy = O\left(\sqrt{k} \int_{-\infty}^{0} p_{n-k}^{(2)}(y)\, dy\right) = O\left(\frac{c_3^2\sqrt{k}}{\sqrt{n-k}}\right).$$

Therefore,

$$(29)\qquad \sum_{k=n/2}^{n} \frac{1}{k^{3/2}} \int_{-\infty}^{0} |y|\, e^{-y^2/2k} p_{n-k}^{(2)}(y)\, dy = O\left(\sum_{k=n/2}^{n} \frac{c_3^2}{k\sqrt{n-k}}\right) = O\left(\frac{c_3^2}{\sqrt{n}}\right).$$

The relations (24), (25) and (27)–(29) imply

$$\Omega_{1n}(x) = O\left(\frac{c_3^2}{\sqrt{n}}\right).$$

The proof is complete.

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