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SOME LIMIT THEOREMS FOR STATIONARY MARKOV CHAINS

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Introduction

Let X be a space of points ξ and F_X a σ-algebra of its subsets. Let $p(\xi, A)$, $\xi \in X$, $A \in F_X$ be a stochastic transition function. For fixed ξ the function $p(\xi, A)$ is a probability measure, and for fixed A is measurable with respect to F_X. The transition probabilities in n steps, $p^{(n)}(\xi, A)$, are calculated by the formula

$$p^n(\xi, A) = \int_A p^{(n-1)}(\eta, A)p(\xi, d\eta). \tag{0.1}$$

If the initial probability distribution $\pi(\cdot)$ is given, then $p(\cdot, \cdot)$ determines a sequence of random variables

$$x_1, x_2, \cdots, x_n, \cdots,$$

which are connected in a stationary Markov chain and where, moreover,

$$\Pr(x_1 \in A) = \pi(A),$$

$$\Pr(x_n \in A) = \int_X p^{(n-1)}(\xi, A)\pi(d\xi). \tag{0.2}$$

Let $f(\xi)$ be a real function defined on X and measurable with respect to F_X while $F_n(x)$ is the distribution function of the sum

$$S_n = \frac{1}{B_n} \sum_1^n f(x_i) - A_n, \tag{0.3}$$

where A_n and $B_n > 0$ are constants.

Just as in the study of sums of identically distributed independent random variables, there arises the question as to what laws and under what conditions can a sequence $F_n(x)$ converge. But even if this problem is completely solved for independent random variables, it is far from a final solution for random variables which are connected in a stationary Markov chain. The investigation is made more difficult by the fact that the behavior of $F_n(x)$ depends in an important way on the ergodic properties of the chain. In order to simplify the problem it becomes necessary to consider chains with sufficiently strong ergodic properties.

The most completely investigated conditions are the sufficient conditions for the validity of the central limit theorem. Already Markov had proved the central limit theorem for three states subject to the condition that all transition probabilities be positive. Different variants of the proof of this theorem for a finite number of states have been given by Romanovskii, Mihoc and Schulz.

In 1937, with certain restrictions on the stochastic transition function, Doeblin [3] proved the central limit theorem for an arbitrary set of states assuming that $f(\xi)$ is bounded. Doob in [12], and Dynkin in [11], assume, instead of boundedness of $f(\xi)$, that, for some $\delta > 0$,

$$\int_X |f(\xi)|^{2+\delta}\, p(d\xi) < \infty, \tag{0.4}$$

where $p(\cdot)$ is a stationary probability distribution corresponding to $p(\cdot, \cdot)$. As regards convergence to limit laws other than the normal, this question has been studied very little. In 1938 Doeblin [4] proved that in the case of a denumerable chain one can reduce the study of sums of the form (0.3) to the study of sums of identically distributed independent random variables. Doeblin's method makes it possible to assert that for a definite class of denumerable chains the set of possible limit laws coincides with the set of stable laws. Stable laws for the number of occurrences in a fixed state of a denumerable chain were obtained in 1949 by Feller [13].

Another important trend is the study of the conditions under which the local limit theorem is valid. The local limit theorem for finite chains was proved in 1949 by Kolmogorov [9] who used the above-mentioned Doeblin method.

Finally, the refinement of limit theorems deserves considerable attention. In 1952 S. Kh. Sirazhdinov [10] succeeded in obtaining refinements of the local and integral limit theorems for a finite number of states.

With this we conclude the short survey of the fundamental results in that area of Markov chain theory to which this article is devoted. Its only purpose was to make the latter's contents more comprehensible.

The article consists of three chapters. The asymptotic properties of the characteristic functions of sums of random variables which are connected in a stationary Markov chain are studied in the first chapter. Convergence to the normal law and to stable laws other than the normal is investigated in the second chapter. The local limit theorem is proved and asymptotic expansions are obtained in the third chapter. The basic method of investigation is the method of characteristic functions.

CHAPTER I

SOME PROPERTIES OF CHARACTERISTIC FUNCTIONS

1. Structure of a Linear Operator Determined by a Stochastic Transition Function

Let $p(\cdot, \cdot)$ satisfy the following condition:

There exists a positive integer k such that

$$\sup_{\xi, \eta, A} |p^{(k)}(\xi, A) - p^{(k)}(\eta, A)| = \delta < 1, \qquad \xi, \eta \in X, A \in F_X. \tag{1.1}$$

It is easy to show that when condition (1.1) is fulfilled there exists a probability distribution $p(\cdot)$ such that

$$|p(A) - p^{(n)}(\xi, A)| \leqq \delta^{(n/k)-1}, \qquad \xi \in X, \ A \in F_X. \tag{1.2}$$

Obviously,

$$p(A) = \int_X p(\xi, A) p(d\xi), \tag{1.3}$$

that is, $p(\cdot)$ is a stationary distribution.

If we set $\delta^{1/k} = \rho$ and $\delta^{-1} = \gamma$, we get

$$|p(A) - p^{(n)}(\xi, A)| \leqq \gamma \rho^n \tag{1.4}$$

unifromly with respect to $\xi \in X$ and $A \in F_X$. Conversely it follows from (1.4) that condition (1.1) is fulfilled.

Let $\mathfrak{M}$ be the space of all bounded complex-valued functions $g(\xi)$, $\xi \in X$, which are measurable with respect to F_X and with norm $\|g(\xi)\| = \sup_{\xi \in X} |g(\xi)|$ and let $\mathfrak{M}^*$ be the space of all complex-valued completely additive set functions $\mu(A)$, $A \in F_X$, with norm $\|\mu\| = V\,|\mu|$ ($V\,|\mu|$ is the total variation of $\mu(\cdot)$ on X).

Define the operators P and P^* in the following manner:

$$Pg(\cdot) = \int_X g(\xi)p(\cdot, d\xi), \tag{1.5}$$

$$P^*\mu(\cdot) = \int_X p(\xi, \cdot)\mu(d\xi).$$

Obviously,

$$\|P\| = \|P^*\|.$$

In the future we often will call the completely additive set function $\Phi(\xi, A)$, $\xi \in X$, $A \in F_X$, with bounded total variation for fixed ξ, which determines in $\mathfrak{M}$ an operator of the type P, the kernel of this operator.

Let P_1 be an operator in $\mathfrak{M}$ determined by a stationary probability distribution $p(\cdot)$. Obviously,

$$PP_1 = P_1P = P_1 = P_1^2, \tag{1.6}$$

and P_1 projects $\mathfrak{M}$ onto the one-dimensional subspace $\mathfrak{M}_1$ which is generated by the function $\psi(\cdot) \equiv 1$.

It is easy to see that the spectrum of the operator P is equal to the sum of the spectra of the operators P_1 and $P - P_1$ which are considered in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, respectively, where $\mathfrak{M}_2$ consists of those elements g for which $P_1g = 0$.

It follows from (1.6) that

$$(P - P_1)^n = P^n - P_1. \tag{1.7}$$

Denote by $V_n(\xi, A)$ the total variation of the measure

$$p^{(n)}(\xi, B) - p(B)$$

on the set A.

As a consequence of (1.4),

$$V_n(\xi, A) \leqq 2\gamma\rho^n. \tag{1.8}$$

Therefore

$$\|(P - P_1)^n\| \leqq 2\gamma\rho^n. \tag{1.9}$$

Since all points z for which

$$|z| > \lim \|(P - P_1)^n\|^{1/n}$$

belong to the resolvent set of the operator $P - P_1$, it follows from (1.9) (see [16], p. 454)[TN-1] that the spectrum of the operator $P - P_1$ lies in the circle of radius ρ with center at 0. The spectrum of the operator P_1, obviously, consists of the single point 1. Consequently, the region, which is exterior to the circle of radius ρ with center at the point 0, from which the point 1 has been deleted, lies entirely within the resolvent set of the operator P.

It is easy to see that the resolvent of the operator P is

$$R(z) = \frac{1}{z-1} P_1 + \sum_0^\infty (P - P_1)z^{-n-1} \tag{1.10}$$

where $|z| > \rho$.

[TN-1] The relevant reference to editions in all languages and issues is Section 148.

2. The Operator $P(\theta)$ and its Spectral Decomposition

Now consider the operator $P(\theta)$ in $\mathfrak{M}$ which is generated by the kernel of

$$p(\theta, \xi, A) = \int_A e^{i\theta f(\eta)} p(\xi, d\eta), \qquad A \in F_X,$$

where $f(\eta)$ is a real function which is measurable with respect to F_X. If $\|P(\theta) - P\| < 1/\|R(z)\|$, then the series

$$\sum_0^\infty R(z)[(P(\theta) - P)R(z)]^k \tag{1.11}$$

converges and determines the resolvent operator $R(z, \theta)$ for $P(\theta)$.

Let I_1 and I_2 be circles with centers at 1 and 0, respectively, and radii $\rho_1 = (1 - \rho)/3$ and $\rho_2 = (1 + 2\rho)/3$.

In the future we will always denote by (g, μ) the functional

$$\int_X g(\xi)\mu(d\xi), \qquad g \in \mathfrak{M},\ \mu \in \mathfrak{M}^*.$$

Lemma 1.1. *There exists an $\varepsilon > 0$ such that for $\|P - P(\theta)\| < \varepsilon$*

$$P^n(\theta) = \lambda^n(\theta)P_1(\theta) + O(\rho_2{}^n), \tag{1.12}$$

where

$$P_1(\theta) = \frac{1}{2\pi i}\int_{I_1} R(z, \theta)dz, \qquad \lambda(\theta) = \frac{(P(\theta)P_1(\theta)\psi, p)}{(P_1(\theta)\psi, p)}, \tag{1.13}$$

and by $O(\rho_2{}^n)$ we denote the operator T_n whose norm $\|T_n\| = O(\rho_2{}^n)$.

Proof. Let M_δ be the sup $\|R(z)\|$ in the region which lies outside the circles with centers at 0 and 1 and radii $\delta + \rho$ and δ, respectively, and also where $\delta < \rho_1$ and $\delta + \rho < \rho_2$.

It is easy to see that if

$$\|P(\theta) - P\| < \frac{1}{M_\delta}, \tag{1.14}$$

then I_1 and I_2 lie within the resolvent set of the operator $P(\theta)$.

Consider the projections

$$P_1(\theta) = \frac{1}{2\pi i}\int_{I_1} R(z, \theta)dz, \qquad P_2(\theta) = \frac{1}{2\pi i}\int_{I_2} R(z, \theta)dz. \tag{1.15}$$

Let $\mathfrak{M}_1(\theta)$ be the subspace onto which $P_1(\theta)$ projects $\mathfrak{M}$. If

$$\|P_1(\theta) - P_1\| < 1, \tag{1.16}$$

then $\mathfrak{M}_1(\theta)$ is one-dimensional. Indeed let us assume the contrary. Then there exist two linearly independents element h_1 and h_2 in $\mathfrak{M}_1(\theta)$. Obviously $P_1 h_1 = \gamma P_1 h_2$, where γ is some complex number.

If $h = h_1 - \gamma h_2$, then $P_1 h = 0$. Consequently,

$$(P_1(\theta) - P_1)h = h,$$

but this contradicts (1.16).

So let us assume that (1.16) is fulfilled. Denote by $\psi(\theta)$ the element which determines $\mathfrak{M}_1(\theta)$. Obviously,

$$P(\theta)P_1(\theta)\psi(\theta) = P_1(\theta)P(\theta)\psi(\theta) = \lambda(\theta)\psi(\theta). \tag{1.17}$$

On the other hand, we can choose $\psi(\theta)$ so that $\psi(\theta) = P_1(\theta)\psi$.

Then it follows from (1.17) that

$$(P(\theta)P_1(\theta)\psi, p) = \lambda(\theta)(P_1(\theta)\psi, p). \tag{1.18}$$

Further,

$$P^n(\theta) = P^n(\theta)P_1(\theta) + P^n(\theta)P_2(\theta), \tag{1.19}$$

$$P^n(\theta)P_2(\theta) = \frac{1}{2\pi i}\int_{I_2} z^n R(z, \theta)dz. \tag{1.20}$$

It follows from (1.17) that

$$P^n(\theta)P_1(\theta) = \lambda^n(\theta)P_1(\theta). \tag{1.21}$$

By virtue of (1.20)

$$\|P^n(\theta)P_2(\theta)\| \leqq \sup_{z\in I_2} \|R(z, \theta)\| \rho_2{}^n. \tag{1.22}$$

If ε is such that (1.14) and (1.16) are fulfilled, then the assertion of the lemma follows from (1.18), (1.19), (1.21) and (1.22).

3. Construction of the Eigenvalue of the Operator $P(\theta)$ with Maximum Modulus

Denote the operator $(P(\theta) - P)R(z)$ by $A(z, \theta)$. Then, by virtue of (1.11) and (1.15),

$$\begin{aligned} P_1(\theta) &= P_1 + \frac{1}{2\pi i}\sum_1^\infty \int_{I_1} R(z)A^k(z, \theta)dz, \\ P(\theta)P_1(\theta) &= P_1 + \frac{1}{2\pi i}\sum_1^\infty \int_{I_1} zR(z)A^k(z, \theta)dz. \end{aligned} \tag{1.23}$$

It follows from (1.13) and (1.23) that

$$\lambda(\theta) = \frac{1 + \sum_1^\infty B_k(\theta)}{1 + \sum_1^\infty C_k(\theta)}, \tag{1.24}$$

where

$$(1.25)\qquad \begin{aligned} B_k(\theta) &= \frac{1}{2\pi i}\left(\int_{I_1} zR(z)A^k(z,\theta)dz\psi, p\right),\\ C_k(\theta) &= \frac{1}{2\pi i}\left(\int_{I_1} R(z)A^k(z,\theta)dz\psi, p\right). \end{aligned}$$

Lemma 1.2. *If* $|\sum_1^\infty C_m(\theta)| < 1$, *then*

$$(1.26)\qquad \lambda(\theta) = \int_X e^{i\theta f(\xi)}p(d\xi) + \sum_1^\infty\left\{\int_X e^{i\theta f(\xi)}p(d\xi)\int_X e^{i\theta f(\eta)}p^k(\xi, d\eta) - \left(\int_X e^{i\theta f(\xi)}p(d\xi)\right)\right\} + W(\theta),$$

where

$$(1.27)\qquad \begin{aligned} W(\theta) = &-\sum_3^\infty C_m(\theta) + \sum_3^\infty B_m(\theta) + \sum_2^\infty(-1)^k\left(\sum_2^\infty C_m(\theta)\right)^k\\ &+\sum_1^\infty B_m(\theta)\left[\sum_1^\infty(-1)^k\left(\sum_2^\infty C_m(\theta)\right)^k\right]. \end{aligned}$$

Proof. Denote $P(\theta) - P$ by $G(\theta)$. The operator $G(\theta)$ is determined by the kernel of

$$(1.28)\qquad \int_A \tilde{f}(\eta, \theta)p(\xi, d\eta),$$

where

$$(1.29)\qquad \tilde{f}(\eta, \theta) = if(\eta)\int_0^\theta e^{itf(\eta)}dt.$$

Let us transform the expressions for $C_1(\theta)$ and $B_1(\theta)$ in (1.25). To do this consider the integrals

$$K_1 = \frac{1}{2\pi i}\int_{I_1} zR(z)A(z,\theta)dz$$

and

$$K_2 = \frac{1}{2\pi i}\int_{I_1} R(z)A(z,\theta)dz.$$

Denote $P_1/(z-1)$ and $\sum_0^\infty (P^k - P_1)z^{-k-1}$, respectively, by $R_1(z)$ and $R_2(z)$.

Through simple computations we get

$$(1.30)\qquad \begin{aligned} K_1 &= P_1G(\theta)P_1 + P_1G(\theta)R_2(1) + R_2(1)G(\theta)P_1,\\ K_2 &= P_1G(\theta)R_2(1) + R_2(1)G(\theta)P_1. \end{aligned}$$

Since $R_2(1)\psi = 0$, it follows from (1.30) that

$$B_1(\theta) = \int_X e^{i\theta f(\eta)} p(d\eta) - 1, \tag{1.31}$$
$$C_1(\theta) = 0.$$

Similarly,

$$B_2(\theta) - C_2(\theta) = \sum_1^\infty \left\{ \int_X e^{i\theta f(\xi)} p(d\xi) \int_X e^{i\theta f(\eta)} p^k(\xi, d\eta) - \left(\int_X e^{i\theta f(\xi)} p(d\xi) \right)^2 \right\}. \tag{1.32}$$

(1.26) follows from (1.24), (1.31) and (1.32).

Lemma 1.3. *There exists a constant Δ, depending on ν and $f(\cdot)$, such that for $|\theta| < \Delta$, $0 < \alpha \leqq 1$, $0 < \nu \leqq 1$,*

$$\left| \lambda(\theta) - \int_X e^{i\theta f(\xi)} p(d\xi) \right| < \sup_\xi \int_X |f(\eta)|^\nu p(\xi, d\eta) \cdot \int_X |f(\eta)|^\alpha p(d\eta) |\theta|^{\alpha+\nu}. \tag{1.33}$$

Proof. By virtue of (1.24),

$$\lambda(\theta) = \int_X e^{i\theta f(\eta)} p(d\eta) + \sum_1^\infty (-1)^k \left(\sum_2^\infty C_m(\theta)\right)^k \left[\sum_1^\infty B_k(\theta) + 1\right] + \sum_2^\infty B_k(\theta) \tag{1.34}$$

if $|\sum_1^\infty C_m(\theta)| < 1$.

It is easy to see that

$$C_k(\theta) = \frac{1}{2\pi i} \int_{I_1} P_1 R(z) G(\theta) A^{k-1}(z, \theta) \psi dz. \tag{1.35}$$

By virtue of (1.28),

$$\|A(z, \theta)\| \leqq 2 \, \|R(z)\| \sup_\xi \int_X |f(\eta)|^\nu p(\xi, d\eta) \, |\theta|^\nu. \tag{1.36}$$

Further,

$$P_1 R(z)(P(\theta) - P) = \frac{1}{z-1} (P_1(\theta) - P_1), \tag{1.37}$$
$$\|P_1(\theta) - P_1\| \leqq 2 \int_X |f(\eta)|^\alpha p(d\eta) \, |\theta|^\alpha.$$

From (1.35)—(1.37) it follows that

$$\|C_k(\theta)\| \leqq Q_1{}^{k-1} \, |\theta|^{\alpha+(k-1)\nu} [\sup_\xi \int_X |f(\eta)|^\nu p(\xi, d\eta)]^{k-1} \int_X |f(\eta)|^\alpha p(d\eta), \tag{1.38}$$

where Q_1 is a constant independent of k, α, ν and $f(\cdot)$. Similarly,

$$\|B_k(\theta)\| \leqq Q_2{}^{k-1} \, |\theta|^{\alpha+(k-1)\nu} [\sup_\xi \int_X |f(\eta)|^\nu (\xi, d\eta)]^{k-1} \int_X |f(\eta)|^\alpha p(d\eta), \tag{1.39}$$

where Q_2 is a constant independent of k, α, ν and $f(\cdot)$. The assertion of the lemma follows from (1.34), (1.38) and (1.39).

4. Estimate for $\lambda''(\theta)$

Lemma 1.4. *There exist a function $\Phi(t) > 0$ and a constant $\Delta > 0$, both independent of $f(\cdot)$, such that*

$$\lim_{t\to 0} \Phi(t) = 0$$

and

$$|\lambda''(\theta) - \lambda''(0)| < \Phi(M\theta) \int_X f^2(\xi)p(d\xi) \tag{1.40}$$

if $|\theta| < \Delta/M$, where $M = \sup_{\xi\in X} |f(\xi)|$.

PROOF. Let us first estimate $B_k''(\theta)$ and $C_k''(\theta)$ for $k \geqq 2$. Let the operators $G^{(1)}(\theta)$ and $G^{(2)}(\theta)$ be determined by the corresponding kernels of

$$\int_A \frac{d}{d\theta}\tilde{f}(\eta, \theta)p(\xi, d\eta)$$

and

$$\int_A \frac{d^2}{d\theta^2}\tilde{f}(\eta, \theta)p(\xi, d\eta).$$

By virtue of (1.25), $B_k''(\theta)$ consists of terms $b^{j_1j_2\cdots j_k}_{i_1i_2\cdots i_{k+1}}(\theta)$ of the form

$$\frac{1}{2\pi i}\left(\int_{I_1} zR_{i_1}G^{(j_1)}(\theta)R_{i_2}(z)G^{(j_2)}(\theta)\cdots G^{(j_k)}(\theta)R_{i_{k+1}}(z)dz\psi, p\right), \tag{1.41}$$

where i_m and j_m take on the values 1, 2 and 0, 1, 2, respectively, $\sum_1^k j_m = 2$, and $G^{(0)}(\theta) = G(\theta)$.

If $i_{k+1} = 2$, then $b^{j_1j_2\cdots j_k}_{i_1i_2\cdots i_{k+1}}(\theta) = 0$ since $R_2(z)\psi \equiv 0$. Therefore we can consider $i_{k+1} = 1$.

By the definition of $R_2(z)$,

$$\begin{aligned} &G^{(i_1)}(\theta)R_2(z)\cdots G^{(i_k)}(\theta)R_2(z)G^{(i_{k+1})}(\theta) \\ &= \sum_{n_1, n_2, \cdots, n_k} G^{(i_k)}(\theta)(P^{n_1} - P_1)\cdots(P^{n_k} - P_1)G^{(i_{k+1})}(\theta)z^{-n_1-n_2\cdots-n_k-k}. \end{aligned} \tag{1.42}$$

The operator

$$M^{n_1n_2\cdots n_k}_{i_1i_2\cdots i_{k+1}}(\theta) = P_1G^{(i_1)}(\theta)(P^{n_1} - P_1)G^{(i_2)}(\theta)(P^{n_2} - P_1)\cdots(P^{n_k} - P_1)G^{(i_{k+1})}(\theta),$$

$$0 \leqq i_m \leqq 2,\ m = 1, 2, \cdots, k+1,$$

is determined by the kernel of

$$\begin{aligned} &\int_A g^{(i_1)}(\xi_1, \theta)p(d\xi_1)\int_X g^{(i_2)}(\xi_2, \theta)(p^{(n_1+1)}(\xi_1, d\xi_2) - p(d\xi_2)) \\ &\cdots \int_X g^{(i_{k+1})}(\xi_{k+1}, \theta)(p^{(n_{k+1})}(\xi_k, d\xi_{k+1}) - p(d\xi_{k+1})). \end{aligned} \tag{1.43}$$

Let us first consider the case when some $i_l = 2,\ l \neq i_1,\ i_m = 0,\ m \neq l$. By Hölder's inequality

$$\|M^{n_1 n_2 \cdots n_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq \int_X |g^{(i_1)}(\xi_1, \theta)|\, p(d\xi_1) \int_X |g^{(i_2)}(\xi_2, \theta)|\, V_{n_1+1}(\xi_1, d\xi_2) \cdots \int_X |g^{(i_{k+1})}(\xi_{k+1}, \theta)|\, V_{n_k+1}(\xi_k, d\xi_{k+1})$$

$$\leqq \left[\int_X |g^{(i_1)}(\xi_1, \theta)|^r\, p(d\xi_1)\right]^{1/r} \left[\int_X p(d\xi_1)\left(\int_X |g^{(i_2)}(\xi_2, \theta)|\, V_{n_1+1}(\xi_1, d\xi_2) \cdots \int_X |g^{(i_{k+1})}(\xi_{k+1}, \theta)|\, V_{n_k+1}(\xi_k, d\xi_{k+1})\right)^s\right]^{1/s}, \tag{1.44}$$

$$\int_X |g^{(i_2)}(\xi_2, \theta)|\, V_{n_1+1}(\xi_1, d\xi_2) \cdots \int_X |g^{i_{k+1}}(\xi_{k+1}, \theta)|\, V_{n_k+1}(\xi_k, d\xi_{k+1}) \tag{1.45}$$

$$\leqq V^{1/r}_{n_1+1}(\xi_1, X) \left[\int_X V_{n_1+1}(\xi_1, d\xi_2)\, |g^{(i_2)}(\xi_2, \theta)|^s \left(\int_X |g^{(i_3)}(\xi_3, \theta)|\, V_{n_2+1}(\xi_2, d\xi_3) \cdots \int_X |g^{(i_{k+1})}(\xi_{k+1}, \theta)|\, V_{n_k+1}(\xi_k, d\xi_{k+1})\right)^s\right]^{1/s},$$

where $1/s + 1/r = 1$.

By applying inequalities (1.45) and (1.8) in succession, and taking into account the fact that $V_n(\xi, A) \leqq p(A) + p^{(n)}(\xi, A)$, we get

$$\|M^{n_1 n_2 \cdots n_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq 2^k \gamma^{k/r} \rho^{\left(\sum_1^k n_i + k\right)} \left(\sup_{\xi, q \neq l} |g^{(i_q)}(\xi, \theta)|\right)^{k-1} \mathbf{E}^{1/r}\, |g^{(i_1)}(x_1, \theta)|^r\, \mathbf{E}^{1/s}\, |g^{(i_l)}(x_1, \theta)|^s, \tag{1.46}$$

where x_1 has a stationary distribution and $\mathbf{E}$ denotes the expectation. As a consequence of (1.28)

$$\sup_{\xi, q \neq l} |g^{(i_q)}(\xi, \theta)| \leqq M\, |\theta|, \qquad |g^2(\xi, \theta)| \leqq f^2(\xi), \tag{1.47}$$

whence it follows that $|g^{(0)}(\xi, \theta)| \leqq |f(\xi)|\, |\theta|$,

$$\mathbf{E}^{1/r}\, |g^{(0)}(x_1, \theta)|^r\, \mathbf{E}^{1/s} f^{2s}(x_1) \leqq M\, |\theta|\, \mathbf{E}^{1/r}\, |f(x_1)|^{r\delta} \mathbf{E}^{1/s}\, |f(x_1)|^{2s - \delta s}. \tag{1.48}$$

If we set $s = 2/(2 - \delta)$, then it follows from (1.46) and (1.48) that

$$\|M^{n_1 n_2 \cdots n_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq 2^k \gamma^{k\delta/2} M^k\, |\theta|^k\, \rho^{\delta/2 \left(\sum_1^k n_i + k\right)}\, \mathbf{E} f^2(x_1)$$

or, since δ can be made arbitrarily close to 2,

$$\|M^{n_1 n_2 \cdots n_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq 2^k \gamma^k M^k\, |\theta|^k \rho^{\left(\sum_1^k n_i + k\right)}\, \mathbf{E} f^2(x_1). \tag{1.49}$$

It is easy to see that estimate (1.49) is true also in the case when $i_1 = 2$, $i_m = 0$ and $m \neq i_1$.

If for some s and p, $i_s = 1$, $i_p = 1$, $i_m = 0$, and $m \neq p, s$, then similar reasoning yields the estimate

$$(1.50) \qquad \|M^{n_1 n_2 \cdots n_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq 2^k \gamma^k M^{k-1} |\theta|^{k-1} \rho^{\left(\sum_1^k n_i + k\right)} \mathbf{E} f^2(x_1).$$

Further, by starting with (1.46), it is easy to show that

$$(1.51) \qquad \|M^{n_1 n_2 \cdots n_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq \begin{cases} 2^k \gamma^k M^k |\theta|^k \rho^{\sum_1^k n_i + k} \mathbf{E}\, |f(x_1)| & \text{if} \quad \sum_1^{k+1} i_m = 1, \\ 2^k \gamma^k M^{k+1} |\theta|^{k+1} \rho^{\sum_1^k n_i + k} & \text{if} \quad \sum_1^{k+1} i_m = 0. \end{cases}$$

Denote the operator

$$R_1(z) G^{(i_1)}(\theta) R_2(z) \cdots R_2(z) G^{(i_{k-1})}(\theta) R_2(z) G^{(i_k)}(\theta)$$

by

$$L(i_1, i_2, \cdots, i_k, z, \theta).$$

From (1.43) and (1.50)—(1.52) we conclude that

$$(1.52) \qquad \|L(i_1, i_2, \cdots, i_{k+1}, z, \theta)\| \leqq \frac{2^k}{|z-1|} \gamma^k |M\theta|^{k+1-\sum_1^{k+1} i_m} \times \frac{1}{\left(1 - \frac{\rho}{|z|}\right)^k} \mathbf{E}\, |f(x_1)|^{\sum_1^{k+1} i_m}.$$

On the other hand,

$$(1.53) \qquad \left(\int_{I_1} z R_1(z) G^{(j_1)}(\theta) R_{i_2}(z) \cdots G^{(j_k)}(\theta) R_1(z) dz \psi, p\right) = \left(\int_{I_1} z L(j_1, j_2, \cdots, j_{s_1}, z, \theta) L(j_{s_1+1}, \cdots, j_{s_2}, z, \theta) \cdots L(j_{s_m+1}, \cdots, j_k, z, \theta) R_1(z) dz \psi, p\right)$$

if

$$i_{s_1} = i_{s_2} = \cdots = i_{s_m} = 1.$$

It is easy to see that

$$(1.54) \qquad \left(\int_{I_1} z R_2(z) G^{(j_1)}(\theta) R_{i_2}(z) \cdots G^{(j_k)}(\theta) R_1(z) dz \psi, p\right) = \int_{I_1} z P_1 R_2(z) G^{(j_1)}(\theta) R_{i_2}(z) \cdots G^{(j_k)}(\theta) R_1(z) \psi dz = 0.$$

It follows from (1.41) and (1.52)—(1.54) that

$$(1.55) \qquad \|b^{j_1 \cdots j_k}_{i_1 i_2 \cdots i_{k+1}}(\theta)\| \leqq B^k |M\theta|^{k-2} \mathbf{E} f^2(x_1),$$

where B is a constant independent of k and $f(\cdot)$. The number of different combinations of indices j_m is equal to k^2, while the number of different combinations of indices i_m when $i_1 = i_{k+1} = 1$ is equal to 2^{k-1}.

Therefore

$$|B_k''(\theta)| \leqq 2^{k-1}B^k k^2 \, |M\theta|^{k-2} \, \mathbf{E}f^2(x_1). \tag{1.56}$$

Similarly

$$|C_k''(\theta)| \leqq 2^{k-1}C^k k^2 \, |M\theta|^{k-2} \, \mathbf{E}f^2(x_1), \tag{1.57}$$

where C is a constant which is independent of k and $f(\cdot)$.

Note that $B_k'(0) = 0$ and $C_k'(0) = 0$ for $k \geqq 2$. Consequently,

$$\begin{aligned} |B_k'(\theta)| &\leqq \sup_{|t| \leqq |\theta|} |B_k''(t)| \, |\theta| \leqq 2^{k-1}B^k k^2 M^{k-2} \, |\theta|^{k-1} \, \mathbf{E}f^2(x_1), \\ |C_k'(\theta)| &\leqq \sup_{|t| \leqq |\theta|} |C_k''(t)| \, |\theta| \leqq 2^{k-1}C^k k^2 M^{k-2} \, |\theta|^{k-1} \, \mathbf{E}f^2(x_1). \end{aligned} \tag{1.58}$$

As regards $B_k(\theta)$ and $C_k(\theta)$, it is easy to obtain the following estimates:

$$\begin{aligned} |B_k(\theta)| &\leqq \sup_{z \in I_1} |z| \, (\sup_{z \in I_1} \|R(z)\|)^{k+1} \, |M\theta|^k, \\ |C_k(\theta)| &\leqq (\sup_{z \in I_1} \|R(z)\|)^{k+1} \, |M\theta|^k. \end{aligned} \tag{1.59}$$

If

$$|\theta| < \frac{1}{MN}, \tag{1.60}$$

where

$$N = z \max (B, C, \sup_{z \in I_1} \|R(z)\|),$$

then the series $\sum_1^\infty B_k(\theta)$, $\sum_1^\infty B_k'(\theta)$, $\sum_1^\infty B_k''(\theta)$, $\sum_1^\infty C_k(\theta)$, $\sum_1^\infty C_k'(\theta)$ and $\sum_1^\infty C_k''(\theta)$ converge absolutely and uniformly, and

$$\begin{aligned} &\frac{d^2}{d\theta^2} [\sum_2^\infty C_m(\theta)]^k \\ &= k(k-1)[\sum_1^\infty C_m(\theta)]^{k-2} \sum_2^\infty \frac{d}{d\theta} C_m(\theta) + k[\sum_2^\infty C_m(\theta)]^{k-1} \sum_2^\infty \frac{d^2}{d\theta^2} C_m(\theta). \end{aligned} \tag{1.61}$$

If at the same time

$$\begin{aligned} &|MN\theta| < \tfrac{1}{2}, \\ &|MN\theta|^2 \, N < \tfrac{1}{2}, \end{aligned} \tag{1.62}$$

then

$$|\sum_2^\infty C_k(\theta)| < 1,$$

and from (1.27), (1.56)—(1.59) and (1.61) it follows that

$$|W''(\theta)| < AM \, |\theta| \, \mathbf{E}f^2(x_1), \tag{1.63}$$

where A is a constant which is independent of $f(\cdot)$. Let us now take up the estimate of the difference $[B_2''(\theta) - C_2''(\theta)] - [B_2''(0) - C_2''(0)]$.

From (1.39) we find that

$$B_2''(\theta) - C_2''(\theta) = \sum_1^\infty [2\mathbf{E}e^{i\theta f(x_1)}\,\mathbf{E}f^2(x_1)e^{i\theta f(x_1)} + 2\mathbf{E}^2 f(x_1)e^{i\theta f(x_1)} - \mathbf{E}(f(x_1) + f(x_{k+1}))^2 e^{i\theta(f(x_1)+f(x_{k+1}))}],$$
$$B_2''(0) - C_2''(0) = -2\sum_1^\infty [\mathbf{E}f(x_1)f(x_{k+1}) - \mathbf{E}^2 f(x_1)]. \tag{1.64}$$

Using the same reasoning as in the derivation of (1.57), it is easy to get that

$$|B_2''(\theta) - C_2''(\theta) - 2\sum_1^\infty [\mathbf{E}^2 f(x_1)e^{i\theta f(x_1)} - \mathbf{E}f(x_1)f(x_{k+1})e^{i\theta(f(x_1)+f(x_{k+1}))}]| \leqq 4\gamma \sum_1^\infty \rho^k(|M\theta| + |M\theta|^2)\mathbf{E}f^2(x_1). \tag{1.65}$$

On the other hand it is easy to see that

$$|\mathbf{E}f(x_1)f(x_{k+1})e^{i\theta(f(x_1)+f(x_{k+1}))} - \mathbf{E}^2 f(x_1)e^{i\theta f(x_1)}| \leqq 2\gamma^{1/2}\rho^{k/2}\mathbf{E}f^2(x_1),$$
$$|\mathbf{E}^2 f(x_1)e^{i\theta f(x_1)} - \mathbf{E}^2 f(x_1)| \leqq 2M\,|\theta|\,\mathbf{E}^2\,|f(x_1)|,$$
$$|\mathbf{E}f(x_1)f(x_{k+1})e^{i\theta(f(x_1)+f(x_{k+1}))} - \mathbf{E}f(x_1)f(x_{k+1})| \leqq 2M\,|\theta|\,\mathbf{E}f^2(x_1). \tag{1.66}$$

It follows from (1.64)—(1.66) that

$$|B_2''(\theta) - C_2''(\theta) - B_2''(0) + C_2''(0)| \leqq 4\gamma\frac{\rho}{1-\rho}(M\,|\theta| + |M\theta|^2)\mathbf{E}f^2(x_1) + 8\gamma^{1/2}\sum_{[|1/M\theta|^{1/2}]}^\infty \rho^{k/2}\,\mathbf{E}f^2(x_1) + 8M\,|\theta|\,\frac{1}{|M\theta|^{1/2}}\,\mathbf{E}f^2(x_1). \tag{1.67}$$

(Square brackets denote the largest integer not exceeding the given number).
The assertion of the lemma follows from (1.63), (1.64) and (1.67).

CHAPTER II

LIMIT THEOREMS

1. General Form of Limit Distributions

Throughout this entire chapter we will assume that condition (1.1) is fulfilled.

Theorem 2.1. *A sequence of distribution functions $F_n(x)$ of sums of the form* (0.3) *can converge only to a stable law. If the stable law to which $F_n(x)$ converges has characteristic exponent α, then*

$$B_n = n^{1/\alpha}h(n), \tag{2.1}$$

where $h(n)$ is a slowly varying function, that is,

$$\lim_{n\to\infty} \frac{h(k_n)}{h(n)} = 1$$

for an arbitrary integer $k > 0$.

PROOF. Let us assume that x_1 has a stationary distribution. The general case is easily reduced to this special case. Let the sequence $F_n(x)$ converge to some proper distribution function $F(x)$. Then

$$B_n \to \infty,$$

$$\frac{B_{n+1}}{B_n} \to 1.$$

Let a_1 and a_2 be two arbitrary positive numbers. Choose two sequences $m = m(n)$ and $l = l(n)$ so that

$$\lim_{n\to\infty} \frac{B_m}{B_n} = \frac{a_1}{a_2}, \qquad \lim_{n\to\infty} l(n) = \infty, \tag{2.2}$$

and

$$\operatorname{p}\lim_{n\to\infty} \frac{1}{B_n} \sum_1^l f(x_l) = 0$$

(the symbol $\operatorname{p}\lim\limits_{n\to\infty}$ denotes limit in probability).

Consider the sum

$$\begin{aligned} &\frac{1}{a_1}\left(\frac{1}{B_n}\sum_1^n f(x_k) - A_n - b_1\right) + \frac{1}{a_1 B_n}\sum_{n+1}^{n+l} f(x_k) \\ &+ \frac{B_m}{a_1 B_n}\left(\frac{1}{B_m}\sum_{n+l+1}^{n+l+m} f(x_k) - A_m - b_2\right) = \frac{1}{a_1 B_n}\sum_1^{n+l+m} f(x_k) - C_n, \end{aligned} \tag{2.3}$$

where

$$C_n = \frac{1}{a_1 B_n}(B_n A_n + B_m A_m + b_1 B_n + b_2 B_m).$$

The distribution functions of the first and third summands of the left side of (2.3) converge to $F(a_1 x + b_1)$ and $F(a_2 x + b_2)$, respectively. On the other hand, it is easy to see that

$$\begin{aligned} &\left| F_{n+l+m}\left(\frac{a_1 B_n}{B_{n+l+m}} x - A_{n+m+l} + C_n \frac{B_n a_1}{B_{n+l+m}}\right)\right. \\ &\qquad \left. - F(a_1 x + b_1) * F_m\left(\frac{a_1 B_n}{B_m} x + b_2\right)\right| \leqq 2\gamma e^l. \end{aligned} \tag{2.4}$$

By virtue of (2.2) and (2.4) the distribution function of the left side of (2.3) converges to the composition of $F(ax_1 + b_1)$ and $F(ax_2 + b_2)$.

The distribution function of the right side can converge only to a distribution function of the form $F(a_3x + b_3)$, where $a_3 > 0$ and b_3 are some constants.

Consequently,

$$F(a_1x + b_1) * F(a_2x + b_2) = F(a_3x + b_3), \tag{2.5}$$

that is, $F(x)$ is a stable law.

For the proof of the second part of the theorem we will use a method suggested by V. M. Zolotarev. Let $F_n(x)$ converge to a stable law with exponent α. If $v(\theta)$ is the characteristic function of this law, then, as is well-known ([6], p. 101),

$$|v(\theta)| = e^{-c|\theta|^\alpha}, \qquad c > 0.$$

Consequently,

$$\lim_{n\to\infty} \left| v_n\left(\frac{\theta}{B_n}\right)\right| = e^{-c|\theta|^\alpha}, \tag{2.6}$$

where $v_n(\theta)$ is the characteristic function of $\sum_1^n f(x_k)$. On the other hand it is easy to show that

$$\lim_{n\to\infty} \left| v_{nk}\left(\frac{\theta}{B_{nk}}\right)\right| = \lim_{n\to\infty} \left| v_n\left(\frac{\theta}{B_{nk}}\right)\right|^k, \tag{2.7}$$

from which in consequence of (2.6) it follows that

$$\lim_{n\to\infty} \left| v_n\left(\frac{\theta}{B_{nk}}\right)\right| = e^{-c|\theta|^\alpha/k}. \tag{2.8}$$

From (2.6) and (2.8) it follows that

$$\lim_{n\to\infty} \frac{B_{nk}}{B_n} = k^{1/\alpha}. \tag{2.9}$$

Obviously relation (2.9) is equivalent to (2.1).

2. Central Limit Theorem

Theorem 2.2. *If*

$$\int_X |f(\eta)|^2 p(d\eta) < \infty \tag{2.10}$$

and

$$\lim_{n\to\infty} \mathbf{E}\left[\frac{1}{\sqrt{n}} \sum_1^n \left(f(x_k) - \int_X f(\eta)p(d\eta)\right)\right]^2 = \sigma^2 > 0, \tag{2.11}$$

then for an arbitrary initial distribution

$$\lim_{n\to\infty} \Pr\left\{\frac{1}{\sqrt{n}} \sum_1^n \left(f(x_k) - \int_X f(\eta)p(d\eta)\right) < x\right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2\sigma^2}\, du. \tag{2.12}$$

This theorem is an analogue of a famous theorem due to P. LÉVY concerning sums of identically distributed independent random variables.

PROOF. As before, we will restrict ourselves to the case when the x_i are identically distributed. Without loss of generality we can assume that $\mathbf{E}f(x_i) = 0$. Let us consider, along with the function $f(\cdot)$, the function $f_n(\cdot)$ which is defined in the following manner:

$$f_n(\xi) = f(\xi) \text{ if } |f(\xi)| < M(n),$$
$$f_n(\xi) = 0 \quad \text{if } |f(\xi)| \geqq M(n),$$

where $M(n)$ is a constant which is independent on n. Denote $(1/\sqrt{n}) \sum_1^n f(x_k)$ and $(1/\sqrt{n}) \sum_1^n f_n(x_k)$ by S_n and S_n', respectively. It is easy to see that

$$|\Pr(S_n < x) - \Pr(S_n' < x)| \leqq 2n \int_{|x| \geqq M(n)} dF(x), \tag{2.13}$$

where

$$F(x) = \Pr(f(x_i) < x).$$

Since

$$n \int_{|x| \geqq \varepsilon\sqrt{n}} dF(x) \leqq \frac{1}{\varepsilon^2} \int_{|x| \geqq \varepsilon\sqrt{n}} x^2 dF(x),$$

it follows that

$$n \int_{|x| \geqq \varepsilon\sqrt{n}} dF(x) = 0 \tag{2.14}$$

for arbitrary $\varepsilon > 0$.

Consequently it is possible to choose a sequence $\psi(n) > 0$ such that

$$\lim_{n\to\infty} \psi(n) = 0$$

and

$$\lim_{n\to\infty} n \int_{|x| \geqq \psi(n)\sqrt{n}} dF(x) = 0. \tag{2.15}$$

Suppose

$$M(n) = \psi(n)\sqrt{n}. \tag{2.16}$$

It is easy to see that

$$\mathbf{E}e^{i\theta S_n'} = \left(P_n{}^{n-1}\left(\frac{\theta}{\sqrt{n}}\right)\psi, \quad p_n\left(\frac{\theta}{\sqrt{n}}\right)\right). \tag{2.17}$$

The operator $P_n(\theta)$ is defined by the kernel of

$$p_n(\theta, \xi, A) = \int_A e^{i\theta f_n(\eta)} p(\xi, d\eta),$$

and $p_n(\theta, A)$ is the completely additive set function $\int_A \exp(i\theta f_n(\eta)) p(d\eta)$.

By virtue of (1.28)

$$\|P_n(\theta) - P\| \leqq M(n)\, |\theta|. \tag{2.18}$$

Consequently, by Lemma 1.1, from some $n = n(\theta)$ on,

$$P_n^n\left(\frac{\theta}{\sqrt{n}}\right) = \lambda_n^n\left(\frac{\theta}{\sqrt{n}}\right) P_{1n}\left(\frac{\theta}{\sqrt{n}}\right) + O(\rho_2^n). \tag{2.19}$$

Expanding $\log \lambda_n(\theta/\sqrt{n})$ into a series, we obtain

$$\log \lambda_n^n\left(\frac{\theta}{\sqrt{n}}\right) = n\left[\lambda_n\left(\frac{\theta}{\sqrt{n}}\right) - 1 + O\left(\left(\lambda_n\left(\frac{\theta}{\sqrt{n}}\right) - 1\right)^2\right)\right]. \tag{2.20}$$

On the other hand,

$$\lambda_n\left(\frac{\theta}{\sqrt{n}}\right) = 1 + \frac{\theta}{\sqrt{n}}\lambda_n'(0) + \frac{\theta^2}{2n}\lambda_n''(0) + \frac{\theta^2}{2n} \times \left[\lambda_n''\left(\frac{\bar\theta}{\sqrt{n}}\right) - \lambda_n''(0)\right], \qquad 0 < |\bar\theta| < |\theta|.$$

It follows from (1.26), (1.64) and (1.66) that

$$\begin{aligned} -\lim_{n\to\infty} \lambda_n''(0) &= \mathbf{E} f^2(x_1) + 2\sum_1^\infty \mathbf{E} f(x_1) f(x_{k+1}) \\ &= \lim_{n\to\infty} \mathbf{E}\left[\frac{1}{\sqrt{n}}\sum_1^n (f(x_k) - \mathbf{E} f(x_k))\right]^2 = \sigma^2. \end{aligned} \tag{2.21}$$

It follows from (1.40) and (2.16) that

$$\lim_{n\to\infty}\left[\lambda_n''\left(\frac{\bar\theta}{\sqrt{n}}\right) - \lambda_n''(0)\right] = 0. \tag{2.22}$$

Further, as is easy to see,

$$\begin{aligned} \lambda_n'(0) &= i\int_{|x|<M(n)} x\,dF(x) = -i\int_{|x|\geqq M(n)} x\,dF(x), \\ |\lambda_n'(0)| &\leqq \left(\int_{|x|\geqq M(n)} x^2\,dF(x)\right)^{1/2}\left(\int_{|x|\geqq M(n)} dF(x)\right)^{1/2}. \end{aligned} \tag{2.23}$$

In consequence of (2.15), (2.16) and (2.25),

$$\lim_{n\to\infty} \sqrt{n}\lambda_n'(0) = 0. \tag{2.24}$$

It is easy to obtain from (2.20)—(2.24)

$$\lim_{n\to\infty} \log \lambda_n^n\left(\frac{\theta}{\sqrt{n}}\right) = -\sigma^2\frac{\theta^2}{2}. \tag{2.25}$$

By virtue of (2.16) and (2.18),

$$\lim_{n\to\infty}\left\| R_n\left(z, \frac{\theta}{\sqrt{n}}\right) - R(z)\right\| = 0,$$

and consequently

$$\lim_{n\to\infty} \left\| P_{n1}\left(\frac{\theta}{\sqrt{n}}\right) - P_1 \right\| = 0. \tag{2.26}$$

Similarly

$$\lim_{n\to\infty} \left\| p_n\left(\frac{\theta}{\sqrt{n}}\right) - p \right\| = 0. \tag{2.27}$$

From (2.17), (2.19), (2.22) and (2.25)—(2.27) we finally conclude that

$$\lim_{n\to\infty} \mathbf{E}e^{i\theta S_n'} = e^{-\theta^2\sigma^2/2}, \tag{2.28}$$

but by virtue of (2.13) this is equivalent to the assertion of the theorem.

3. Convergence to Laws other than the Normal

Theorem 2.3. *Let $u_1, u_2, \cdots, u_n, \cdots$ be a sequence of independent random variables with the distribution function*

$$F(x) = \Pr(f(x_1) < x),$$

where x_1 has a stationary distribution.

If for some sequence of constants A_n and $B_n > 0$

$$\lim_{n\to\infty} \Pr\left(\frac{\sum_1^n u_k}{B_n} - A_n < x\right) = V_\alpha(x), \tag{2.29}$$

where $V_\alpha(x)$ is a stable law with exponent α, and for some $0 < \nu \leqq 1$ and some $\alpha > \varepsilon > 0$,

$$\lim_{n\to\infty} nB_n^{-\nu-\min(1,\alpha-\nu)} \sup_\xi \int_{|f(\eta)|<B_n\tau} |f(\eta)|^\nu p(\xi, d\eta) = 0, \tag{2.30}$$

for any $\tau > 0$, then, for arbitrary initial distribution,

$$\lim_{n\to\infty} \Pr\left(\frac{1}{B_n}\sum_1^n f(x_k) - A_n < x\right) = V_\alpha(x). \tag{2.31}$$

Proof. We will again restrict ourselves to the case when the x_i are identically distributed. Since (2.30) is fulfilled for arbitrary τ, it is possible to choose a sequence $\psi(n) > 0$ so that

$$\lim_{n\to\infty} \psi(n) = \infty,$$

$$\lim_{n\to\infty} nB_n^{-\nu-\min(1,\alpha-\varepsilon)} \sup_\xi \int_{|f(\eta)|<B_n\psi(n)} |f(\eta)|^\nu p(\xi, d\eta) = 0, \tag{2.32}$$

$$\lim_{n\to\infty} \frac{\psi(n)}{n^\delta} = 0,$$

for any $\delta > 0$.

Let $f_n(\cdot)$ be defined as follows:

$$f_n(\cdot) = f(\cdot) \text{ if } |f(\cdot)| < B_n\psi(n),$$
$$f_n(\cdot) = 0 \quad \text{ if } |f(\cdot)| \geqq B_n\psi(n).$$

Let S_n and S_n' denote

$$\frac{1}{B_n}\sum_1^n f(x_k) - A_n$$

and

$$\frac{1}{B_n}\sum_1^n f_n(x_k) - A_n,$$

respectively.

Let $P_n(\theta)$ be determined by the kernel of

$$\int_A e^{i\theta f_n(\eta)} p(\xi, d\eta).$$

By virtue of (1.28)

$$\|P_n(\theta) - P\| \leqq 2 \sup_\xi \int_{|f(\eta)| < B_n\psi(n)} |f(\eta)|^\nu \, p(\xi, d\eta) \, |\theta|^\nu. \tag{2.33}$$

By Theorem 2.1, $B_n = n^{1/\alpha}h(n)$, where $h(n)$ is a slowly varying function. Therefore it follows from (2.32) that

$$\lim_{n\to\infty} B_n^{-\nu} \sup_\xi \int_{|f(\eta)| < B_n\psi(n)} |f(\eta)|^\nu \, p(\xi, d\eta) = 0. \tag{2.34}$$

Consequently, by Lemma 1.1

$$P_n^n\left(\frac{\theta}{B_n}\right) = \lambda_n^n\left(\frac{\theta}{B_n}\right) P_{1n}\left(\frac{\theta}{B_n}\right) + O(\rho_2^n) \tag{2.35}$$

for all $n \geqq n(\theta)$.

Further, by Lemma 1.3, since

$$\int_X |f(\eta)|^{\alpha-\varepsilon} \, p(d\eta) < \infty, \qquad 0 < \varepsilon < \alpha,$$

(see [14], p. 192 TN-2), we have

$$\begin{aligned} \log \lambda_n^n\left(\frac{\theta}{B_n}\right) &= n \log \varphi_n\left(\frac{\theta}{B_n}\right) + \\ &+ O\left(n B_n^{-\nu-\min(1,\alpha-\varepsilon)} \sup_\xi \int_{|f(\eta)| < B_n\psi(n)} |f(\eta)|^\nu \, p(\xi, \eta)\right), \end{aligned} \tag{2.36}$$

where

$$\varphi_n(\theta) = \int_X e^{i\theta f_n(\eta)} p(d\eta).$$

TN-2 The page reference in the English language edition is p. 180.

It is easy to see that

$$\mathbf{E}e^{i\theta S_n'} = e^{-i\theta A_n}\left(P_n^{\,n-1}\left(\frac{\theta}{B_n}\right)\psi,\quad p_n\left(\frac{\theta}{B_n}\right)\right), \tag{2.37}$$

where $p_n(\theta)$ is the completely additive set function

$$\int\limits_A e^{i\theta f_n(\eta)}p(d\eta).$$

Let $u_k^{(n)}$ be a sequence of independent random variables with the distribution function

$$F_n(x) = \Pr(f_n(x_1) < x).$$

It is easy to verify that

$$\left|\Pr\left(\frac{1}{B_n}\sum_1^n u_k - A_n < x\right) - \Pr\left(\frac{1}{B_n}\sum_1^n u_k^{(n)} - A_n < x\right)\right| \leqq 2n \int\limits_{|x| \geqq B_n\psi(n)} dF(x). \tag{2.38}$$

It is well-known that ([14], p. 195 TN-3),

$$\lim_{n\to\infty} nF(B_n x) = \frac{c_1}{|x|^\alpha} \qquad \text{for } x < 0,$$

$$\lim_{n\to\infty} n(1 - F(B_n x)) = \frac{c_2}{x^\alpha} \quad \text{for } x > 0,$$

where $c_1 > 0$, $c_2 > 0$ are certain constants.

Therefore

$$\lim_{n\to\infty} n \int\limits_{|x| \geqq B_n\psi(n)} dF(x) = 0. \tag{2.39}$$

It follows from (2.38) and (2.39) that

$$\lim_{n\to\infty} \varphi_n^{\,n}\left(\frac{\theta}{B_n}\right)e^{-i\theta A_n} = \lim_{n\to\infty} \varphi_\infty^{\,n}\left(\frac{\theta}{B_n}\right)e^{-i\theta A_n}. \tag{2.40}$$

It follows from (2.30), (2.35)—(2.37) and (2.40) that

$$\lim_{n\to\infty} \mathbf{E}e^{i\theta S_n'} = \varphi_\alpha(\theta), \tag{2.41}$$

where $\varphi_\alpha(\theta)$ is the characteristic function of the law $V_\alpha(x)$.

On the other hand it is easy to see that

$$|\Pr(S_n < x) - \Pr(S_n' < x)| \leqq 2n \int\limits_{|x| \geqq B_n\psi(n)} dF(x),$$

TN-3 The page reference in the English language edition is p. 182.

and this in consequence of (2.39) and (2.41) means that

$$\lim_{n\to\infty} \Pr(S_n < x) = V_\alpha(x),$$

which is what was to be proven.

REMARK. It is quite likely that condition (2.30) can be weakened. However some sort of restrictions on the order of magnitude of

$$\sup_\xi \int_{|f(\eta)|<N} |f(\eta)|^\nu p(\xi, d\eta)$$

as a function of N ought to be imposed just the same. It is possible to construct examples in which

$$\sup_\xi \int_{|f(\eta)|<N} |f(\eta)| p(\xi, d\eta) = O(N)$$

and the limit distributions for the sums

$$\frac{1}{B_n}\sum_1^n u_k - A_n \quad\text{and}\quad \frac{1}{B_n}\sum_1^n f(x_k) - A_n$$

do not coincide. On the other hand, it can be shown that always

$$\lim_{n\to\infty} \mathbf{E}e^{i\theta S_n} = \lim_{n\to\infty} \lambda_n{}^n\left(\frac{\theta}{B_n}\right)e^{-i\theta A_n}.$$

We will not dwell on this.

Theorem 2.4. *If*

$$\int_X |f(\eta)|^\alpha p(d\eta) < \infty, \qquad 0<\alpha<2,$$

then, for some choice of constants A_n and arbitrary initial distribution

$$\lim_{n\to\infty} \Pr\left(\frac{1}{n^{1/\alpha}}\sum_1^n f(x_k) - A_n < x\right) = E(x), \tag{2.42}$$

where $E(x)$ is an improper law.

This theorem is proved by the same method as Theorem 2.1. The function $f_n(\cdot)$ is defined as follows:

$$f_n(\cdot) = f(\cdot) \text{ if } |f(\cdot)| \leqq n^{1/\alpha}\psi(n),$$
$$f_n(\cdot) = 0 \quad \text{ if } |f(\cdot)| > n^{1/\alpha}\psi(n),$$

where the sequence $\psi(n)$ is chosen so that

$$\lim_{n\to\infty} \psi(n) = 0 \tag{2.43}$$

and

$$\lim_{n\to\infty} \int_{|x|>n^{1/\alpha}\psi(n)} dF(x) = 0$$

(we keep the notation of § 2 and § 3).

It is easy to see that

$$\lambda_n(\theta) = \varphi_n(\theta) + O(\mathbf{E}f_n{}^2(x_1)\theta^2). \tag{2.44}$$

At the same time

$$\mathbf{E}f_n{}^2(x_1) < n^{2/\alpha-1}\psi^{2-\alpha}(n)\mathbf{E}\,|f_n(x_1)|^\alpha. \tag{2.45}$$

Consequently

$$\lim_{n\to\infty} \lambda_n{}^n\left(\frac{\theta}{n^{1/\alpha}}\right) = \lim_{n\to\infty} \varphi_n{}^n\left(\frac{\theta}{n^{1/\alpha}}\right). \tag{2.46}$$

The rest of the proof is left to the reader.

CHAPTER III

THE LOCAL LIMIT THEOREM AND ASYMPTOTIC EXPANSIONS

1. The Local Theorem

Suppose that X is a denumerable set of points ξ_i and that condition (1.1) is fulfilled for $k = 1$. Then, as is easy to see,

$$\inf_{(i,j)} \sum_{k-1}^{\infty} \min(p_{ik}, p_{jk}) > 0, \tag{3.1}$$

where $p_{ik} = p(\xi_i, \xi_k)$ (concerning condition (3.1) see, for example, [15]).

Let us suppose further that all states ξ_i are essential and constitute a positive class (see [5]).

By virtue of (3.1) this class consists of one subclass. Suppose $f(\xi_i) = a + k_i h$, where k_i is an integer, a is an arbitrary real number and $h > 0$.

Theorem 3.1. *If the greatest common divisor of the k_i is equal to* 1,

$$\sum_i f^2(\xi_i)p_i < \infty$$

and

$$\sigma > 0 \tag{3.2}$$

(*the p_i are final probabilities and σ is defined in the same way as in Theorem* 2.1), *then*

$$\lim_{n\to\infty}\left(\frac{\sigma\sqrt{n}}{h}\mathscr{P}_{\pi n}(s) - \frac{1}{\sqrt{2\pi}}e^{-z^2{}_{ns}/2}\right) = 0, \tag{3.3}$$

uniformly in s, where

$$\mathscr{P}_{\pi n}(s) = \Pr\,(\sum_1^n f(x_i) = an + sh),$$

under the condition that the initial distribution is $\pi_i = \pi(\xi_i)$, *and that*

$$\sigma\sqrt{n}\, z_{ns} = a(n+1) + sh - (n+1)\sum_1^\infty f(\xi_i)p_i.$$

The following lemma will be needed in the proof.

Lemma 3.1. *If the greatest common divisor of the* k_i *is equal to* 1, *then for* $\varepsilon \leqq |\theta| \leqq (2\pi/h) - \varepsilon$ *and* $n > n_0$

$$|\varphi_{n\pi}(\theta)| < e^{-cn}, \tag{3.4}$$

where $\varphi_{n\pi}(\theta)$ *is the characteristic function of the random variable* $\sum_1^n f(x_i)$ *subject to the condition that the initial distribution is* $\pi_i = \pi(\xi_i)$ *and* c *is a positive constant which is independent of* ε.

Proof. Denote $k_j - k_i$ by Δ_{ji}. Obviously, the greatest common divisor of the Δ_{ij} is equal to 1. Let S_k be the set of those Δ_{ji} for which $p_{kj} > 0$ and $p_{ki} > 0$. It is easy to see that the greatest common divisor d of the numbers $\Delta_{is} \in \sum_1^\infty S_k$ is equal to 1. Indeed, let us assume that $d > 1$. Then $p_{si} = 0$ if $k_i \neq a_s + m_i d$, where a_s and m are integers which depend on s and i, respectively. If $a_k \not\equiv a_i \pmod d$ for some k and i, then the inequalities $p_{kj} > 0$ and $p_{ij} > 0$ cannot be fulfilled simultaneously and this contradicts (3.1). But if $a_k \equiv a_i \pmod d$ for arbitrary i and k, then $p_{si} = 0$ for i such that $k_i \not\equiv a_1 \pmod d$ and all s, but this obviously is impossible.

Let us choose from $\sum_1^\infty S_k$ a finite set of S such that the greatest common divisor of $\Delta_{ij} \in S$ is equal to 1. Index the $\Delta_{ij} \in S$ in some manner. Let Δ_1, $\Delta_2, \cdots, \Delta_N$ be the resulting sequence. If $\Delta_s \in S_{\mu(s)}$, then for some $i(s)$ and $j(s)$ we have $p_{\mu(s)i(s)} > 0$ and $p_{\mu(s)j(s)} > 0$. In consequence of (3.1) there exists an index $\nu(s)$ such that

$$p_{i(s)\nu(s)} > 0 \text{ and } p_{j(s)\nu(s)} > 0. \tag{3.5}$$

Denote $\sum_{s=1}^\infty p_{is}p_{sj}e^{i\theta(x_s+x_j)}$ by $p_{ij}^{(2)}(\theta)$.

By virtue of (3.5), for

$$\frac{2\pi k}{h\,|\Delta_s|} + \varepsilon \leqq |\theta| \leqq \frac{2\pi(k+1)}{h\,|\Delta_s|} - \varepsilon,$$

$$p_{\mu(s)\nu(s)}^{(2)} - |p_{\mu(s)\nu(s)}^{(2)}(\theta)| > \rho_s(\varepsilon) > 0, \tag{3.6}$$

where $\rho_s(\varepsilon)$ is a constant which is dependent on s and ε.

Further,

$$\sum_j |p_{ij}^{(m+2)}(\theta)| \leqq 1 - p_{i\mu(s)}^{(m)}(p_{\mu(s)\nu(s)}^{(2)} - |p_{\mu(s)\nu(s)}^{(2)}(\theta)|), \tag{3.7}$$

where $p_{ij}^{(m+2)}(\theta)$ is an element of the matrix $P^{m+2}(\theta)$. ($P(\theta)$ is the characteristic matrix $\{p_{ik}\exp(i\theta x_k)\}$).

Without loss of generality we can assume that Δ_1 is in absolute value the smallest of the differences $\Delta_s \in S$. Suppose $|\Delta_1| \neq 1$. If $|\theta| = 2\pi k/h\,|\Delta_1|$, $k < |\Delta_1|$, then it is always possible to choose $j(k)$ and $\Delta_{i(k)} \in S$, so that

$$\frac{2\pi j(k)}{h\,|\Delta_{i(k)}|} < |\theta| < \frac{2\pi(j(k)+1)}{h\,|\Delta_{i(k)}|}. \tag{3.8}$$

Indeed, otherwise

$$\frac{j_i}{|\Delta_i|} = \frac{k}{|\Delta_1|}$$

for all $\Delta_i \in S$, that is $\Delta_i = n_i \Delta_1$, where n_i is an integer, but this contradicts the fact that the greatest common divisor of the $\Delta_i \in S$ is equal to 1.

By virtue of (3.6) and (3.8), when $\varepsilon \leqq |\theta| \leqq (2\pi/h) - \varepsilon$,

$$(3.9) \qquad \max_{s \leqq N} (p^{(2)}_{\mu(s)\nu(s)} - |p^{(2)}_{\mu(s)\nu(s)}(\theta)|) > \rho(\varepsilon) > 0,$$

where $\rho(\varepsilon)$ is a constant which is dependent on ε.

It follows from (3.7) and (3.9) that

$$(3.10) \qquad \max_i \sum_j |p^{(m+2)}_{ij}(\theta)| \leqq 1 - \min_{s \leqq N, i} p^{(m)}_{i\mu(s)}\rho(\varepsilon).$$

By virtue of (1.4), since all final probabilities $p_i > 0$, it is possible to choose m so that

$$(3.11) \qquad p^{(m)}_{i\mu(s)} > \alpha > 0$$

for all i and $s \leqq N$.

On the other hand it is easy to see that

$$(3.12) \qquad |\varphi_{n\pi}(\theta)| \leqq (\max_i \sum_j |p^{(m+2)}_{ij}(\theta)|)^{(n-1)/(m+2)-1}.$$

The lemma's assertion follows from (3.10)—(3.12).

Proof of Theorem 3.1. The following two inequalities play a fundamental role in the proof of the local limit theorem for independent identically distributed random variables using B. V. Gnedenko's method:

$$(3.13) \qquad |f_n(\theta)| < e^{-cn} \quad \text{for} \quad \varepsilon \leqq |\theta| \leqq \frac{2\pi}{h} - \varepsilon$$

and

$$(3.14) \qquad \left| f_n\left(\frac{\theta}{B_n}\right)\right| < e^{-\theta^2/4} \quad \text{for} \quad |\theta| < \varepsilon,$$

where $f_n(\theta)$ is the characteristic function of the sum of the first n random variables, $B_n{}^2$ is the variance of this sum, and ε is sufficiently small.

Lemma 3.1 shows that inequality (3.13) remains true also under the conditions of Theorem 3.1. On the other hand, as is easy to see, it follows from (3.7) that

$$(3.15) \qquad \left| \varphi_{n\pi}\left(\frac{\theta}{\sigma\sqrt{n}}\right)\right| < e^{-b^2\theta^2},$$

where $b^2 > 0$ is some constant.

Utilizing Theorem (1.1) and inequalities (3.13) and (3.15), and reasoning in the same way as in the case of independent random variables (see [14], § 49), we easily obtain (3.3).

2. Asymptotic Expansions

Suppose

$$\int_X |f(\eta)|^{k-2+\delta}\pi(d\eta) < \infty, \quad \int_X |f(\eta)|^{k+\delta}p(\xi, d\eta) < M < \infty \tag{3.16}$$

for some integer $k \geqq 3$ and $\delta > 0$.

Expanding $\exp(i\theta f(\eta))$ into a series, we get

$$P(\theta) = P + i\theta P^{(1)} + \frac{(i\theta)^2}{2!}P^{(2)} + \cdots + \frac{(i\theta)^k}{k!}P^{(k)} + O(\theta^{k+\delta}), \tag{3.17}$$

where $P^{(s)}$ is determined by the kernel of

$$i^{-s}\int_A \frac{d^s}{d\theta^s} e^{i\theta f(\eta)}|_{\theta=0}\, p(\xi, d\eta).$$

Obviously

$$\|P^s\| \leqq \sup_\xi \int_X |f(\eta)|^s\, p(\xi, d\eta).$$

It is easy to conclude from (3.17) and (1.11) that

$$R(z, \theta) = R(z) + \sum_1^k R^{(s)}(z)(i\theta)^s + O(\theta^{k+\delta}), \tag{3.18}$$

where $R^{(s)}(z)$ is a combination of $R(z)$ and $P^{(j)}$, $j \leqq s$.

It follows from (1.13) and (3.18) that, for $\|P(\theta) - P\| < \varepsilon$, we have

$$P_1(\theta) = P_1 + \sum_1^k P_1^{(s)}(i\theta)^s + O(\theta^{k+\delta}),$$

$$\lambda(\theta) = 1 + \sum_1^k \lambda^{(s)}(i\theta)^s + O(\theta^{k+\delta}), \tag{3.19}$$

where

$$P_1{}^{(s)} = \frac{1}{2\pi i}\int_{I_1} R^{(s)}(z)dz,$$

and $\lambda^{(s)}$ is a combination of $(P_1^{(j)}\psi, p)$ and $\int_{I_1} z(R^{(j)}\psi, p)dz$, $j \leqq s$.

Further,

$$\log \lambda(\theta) = \sum_1^k \alpha^{(s)}(i\theta)^s + O(\theta^{k+\delta}), \tag{3.20}$$

where $\alpha^{(s)}$ is a combination of the $\lambda^{(j)}$, $j \leqq s$.

Henceforth we will assume that

$$\alpha^{(1)} = \lambda^{(1)}(0) = 0.$$

It is easy to see that

$$\begin{aligned} \log \varphi_{n\pi}\left(\frac{\theta}{\sigma\sqrt{n}}\right) &= n \log \lambda\left(\frac{\theta}{\sigma\sqrt{n}}\right) + \log\left(P_1\left(\frac{\theta}{\sigma\sqrt{n}}\right)\psi,\ \pi\left(\frac{\theta}{\sigma\sqrt{n}}\right)\right) \\ &= -\frac{\theta^2}{2} + n\sum_3^k (\alpha^{(s)} + \beta^{(s-2)})\left(\frac{i\theta}{\sigma\sqrt{n}}\right)^{s-2} + O\left(\frac{\theta^{k-2+\delta}}{n^{(k-2+\delta)/2}}\right) + O(\theta\rho_2{}^n), \end{aligned} \tag{3.21}$$

where $\varphi_{n\pi}(\theta)$ is the characteristic function of $\sum_1^{n+1} f(x_k)$, $\pi(\theta)$ is the completely additive set function $\int_A \exp(i\theta f(\eta))\pi(d\eta)$, and $\beta^{(s)}$ is the coefficient of the term containing $(i\theta)^s$ in the expansion of $\log\,(P_1(\theta)\psi, \pi(\theta))$.

The following expansion for $\varphi_{n\pi}(\theta/\sigma\sqrt{n})$ corresponds formally to expansion (3.21):

$$e^{-\theta^2/2}\left(1 + \sum_1^\infty P_{s\pi}(i\theta)\left(\frac{1}{\sqrt{n}}\right)^s\right), \tag{3.22}$$

where $P_{s\pi}(\theta)$ is some polynomial of degree $3s$ whose coefficients depend on the initial distribution $\pi(\cdot)$.

Lemma 3.2. *If* $|\theta| < \sqrt{n}\Delta_1$ (Δ_1 *is some constant*), *then*

$$\begin{aligned} &\left|\varphi_{n\pi}\left(\frac{\theta}{\sigma\sqrt{n}}\right) - e^{-\theta^2/2}\left(1 + \sum_1^{k-2} P_{s\pi}(i\theta)\left(\frac{1}{\sqrt{n}}\right)^s\right)\right| \\ &\qquad \leqq \frac{c(k)}{(\sqrt{n})^{k-2}}\,\delta(n)(|\theta|^k + |\theta|^{3(k-2)})\,e^{-\theta^2/4} + O(\theta\rho_2{}^n), \end{aligned} \tag{3.23}$$

where $c(k)$ is a constant which depends on k, and $\lim_{n\to\infty}\delta(n) = 0$.

The proof is quite analogous to the proof of Theorem 1 in § 41 of monograph [14].

Theorem 3.2. *If the conditions of Theorem* (3.1) *and condition* (3.16) *are satisfied, then*

$$\mathscr{P}_{\pi(n+1)}(s) = \frac{h}{\sigma\sqrt{n}}\left\{\varphi(z_{ns}) + \sum_1^{k-2}\frac{1}{n^{m/2}}P_{\pi m}(-\,\varphi(z_{ns}))\right\} + o\left(\frac{1}{n^{(k-1)/2}}\right).$$

Here $\varphi(z) = (1/\sqrt{2\pi})\exp\,(-\,z^2/2)$ *and* $P_{\pi m}(-\,\varphi)$ *is computed in the same way as* $P_{\pi m}(-\,u)$ *with the substitution of* $\varphi^{(r)}$ *for* u^r.

This theorem is proved in the same way as the analogous assertion for independent random variables ([14], § 51, Theorem 1). Lemmas 3.1 and 3.2 are used for this.

Theorem 3.3. *Let X be the real line. If condition* (3.16) *is satisfied, then*

$$0 < m < p_0(\xi, \eta) < M < \infty, \qquad \eta \in C,\ \xi \in X, \tag{3.24}$$

where $p_0(\xi, \eta)$ is the density of the component of the function $p(\xi, A)$ which is absolutely continuous with respect to the Lebesgue measure, $C \in F_X$, meas $C > 0$ (*Lebesgue measure is meant*), *and*

$$\liminf_{\theta\to\infty}\int_C\int_C \sin^2(f(\xi) - f(\eta))\theta\,d\eta\,d\xi > 0, \tag{3.25}$$

then

$$F_{n\pi}(x) - \Phi(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\sum_1^{k-2}\frac{Q_{s\pi}(x)}{(\sqrt{n})^s} + o\left(\frac{1}{n^{(k-2)/2}}\right). \tag{3.26}$$

Here $F_{n\pi}(x)$ is the distribution function of $(\sum_1^n f(x_k))/\sigma\sqrt{n}$ *subject to the* *condition that the initial distribution is* $\pi(\cdot)$,

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt,$$

$$\frac{1}{\sqrt{2\pi}} Q_{s\pi}(x) e^{-x^2/2} = P_{s\pi}(-\Phi),$$

where $P_{s\pi}(-\Phi)$ *is computed in the same way as* $P_s(-u)$, *with the substitution of* $\Phi^{(r)}$ *for* u^r.

PROOF. Let $\mu(\xi, \cdot)$ be the singular component of $p(\xi, \cdot)$. It is easy to see that

$$p^{(2)}(\theta, \xi, A) = \int_A d\zeta \int_X e^{i\theta(f(\eta)+f(\zeta))} p_0(\xi, \eta) p_0(\eta, \zeta) d\eta + \mu^{(2)}(\theta, \xi, A),$$

where $p^{(2)}(\theta, \xi, A)$ is the kernel of the operator $P^2(\theta)$, and

$$\mu^{(2)}(\theta, \xi, A) = \int_A d\zeta \int_X e^{i\theta(f(\eta)+f(\zeta))} p_0(\eta, \zeta) \mu(\xi, d\eta)$$

$$+ \int_X d\eta \int_A e^{i\theta(f(\eta)+f(\zeta))} p_0(\xi, \eta) \mu(\eta, d\zeta) + \int_X e^{i\theta f(\eta)} \mu(\xi, d\eta) \int_A e^{i\theta f(\zeta)} \mu(\eta, d\zeta).$$

Consequently,

$$\|p^{(2)}(\theta, \xi, A)\| \leqq 1 - \int_X d\zeta \Big\{ \int_X p_0(\xi, \eta) p_0(\eta, \zeta) d\eta - \Big| \int_X p_0(\xi, \eta) p_0(\eta, \zeta) e^{i\theta f(\eta)} d\eta \Big| \Big\}. \tag{3.27}$$

Further,

$$\Big| \int_X e^{i\theta f(\eta)} p_0(\xi, \eta) p_0(\eta, \zeta) d\eta \Big|^2 = \int_X \int_X p_0(\xi, \eta) p_0(\eta, \zeta) \times p_0(\xi, \lambda) p_0(\lambda, \zeta) \cos \theta(f(\eta) - f(\zeta)) d\eta d\lambda.$$

Hence, in consequence of (3.24),

$$\begin{aligned} &\Big(\int_X p_0(\xi, \eta) p_0(\eta, \zeta) d\eta \Big)^2 - \Big| \int_X e^{i\theta f(\eta)} p_0(\xi, \eta) p_0(\eta, \zeta) d\eta \Big|^2 \\ &= 2 \int_X \int_X p_0(\xi, \eta) p_0(\eta, \zeta) p_0(\xi, \lambda) p_0(\lambda, \zeta) \sin^2 \frac{\theta}{2} (f(\eta) - f(\lambda)) d\eta d\lambda \\ &\geqq 2m^4 \int_C \int_C \sin^2 \frac{\theta}{2} (f(\eta) - f(\lambda)) d\eta d\lambda. \end{aligned} \tag{3.28}$$

On the other hand, for $\zeta \in C$,

$$\left| \int_X e^{i\theta f(\eta)} p_0(\xi, \eta) p_0(\eta, \zeta) d\eta \right| + \int_X p_0(\xi, \eta) p_0(\eta, \zeta) d\eta \tag{3.29}$$
$$\leqq 2 \int_X p_0(\xi, \eta) p_0(\eta, \zeta) d\eta < 2M.$$

It follows from (3.28) and (3.29) that

$$\int_X p_0(\xi, \eta) p_0(\eta, \zeta) d\eta - \left| \int_X p_0(\xi, \eta) p_0(\eta, \zeta) e^{i\theta f(\eta)} d\eta \right| \tag{3.30}$$
$$> \frac{m^4}{M} \int_C \int_C \sin^2 \frac{\theta}{2} (f(\eta) - f(\lambda)) d\eta d\lambda$$

for $\zeta \in C$.

Hence,

$$\int_X d\zeta \left\{ \int_X p_0(\xi, \eta) p_0(\eta, \zeta) d\eta - \left| \int_X e^{i\theta f(\eta)} p_0(\xi, \eta) p_0(\eta, \zeta) d\eta \right| \right\}$$
$$> \frac{m^4}{M} \operatorname{meas} C \int_C \int_C \sin^2 \frac{\theta}{2} (f(\eta) - f(\lambda)) d\eta d\lambda.$$

It follows from (3.25), (3.27) and (3.28) that

$$\|p^{(2)}(\theta, \xi, A)\| < 1 - \alpha, \qquad \xi \in X, \ |\theta| > \Delta_1,$$

where $\alpha > 0$ is some constant, that is,

$$\|P^2(\theta)\| < 1 - \alpha$$

for $|\theta| > \Delta_1$.

The remainder of the proof is carried out in exactly the same way as the proof of the theorem in § 45 in [14]. Lemma 3.2 is used here.

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SOME LIMIT THEOREMS FOR STATIONARY MARKOV CHAINS

S. V. NAGAEV *(TASHKENT)*

(*Summary*)

Let X be a space of points, F_X a σ-algebra of its subsets, and $p(\xi, A)$, $\xi \in X$, $A \in F_X$, a stochastic transition function satisfying the following condition:

an integer $k \geqq 1$ exists such that

$$\sup_{\eta, \xi \in X, A \in F_X} |p^{(k)}(\xi, A) - p^{(k)}(\eta, A)| < 1. \tag{1}$$

Let us define the sequence of random variables $x_1, x_2, \cdots, x_n, \cdots$ as follows:

$$\Pr(x_1 \in A_1, x_2 \in A_2, \cdots, x_n \in A_n) = \int_{A_1} \pi(d\xi_1) \int_{A_2} p(\xi_1, d\xi_2) \cdots \int_{A_n} p(\xi_{n-1}, d\xi_n),$$

where $\pi(\cdot)$ is the initial distribution.

Let $f(\xi)$ be a real function of $\xi \in X$ measurable with respect to F_X.

In Chapter I the asymptotic behaviour of the characteristic function of $\Sigma_1^n f(x_i)$ is studied. Chapter II is devoted to limit theorems. The central limit theorem is proved under the assumption that

$$\int_X f^2(\xi) p(d\xi) < \infty, \tag{2}$$

where $p(\cdot)$ is a stationary absolute probability distribution corresponding to $p(\cdot, \cdot)$. The sufficient conditions for convergence to stable laws are given. In chapter III the local limit theorem is proved, and asumptotic expansions are given. The characteristic function method is the basic one used.

TN-4 There is an English translation: Gnedenko, B. V., and Kolmogorov, A. N., *Limit Distributions for Sums of Independent Random Variables*, Cambridge, Mass., 1954.

TN-5 The original edition was in French, printed at Budapest, Hungary, 1952. There are now three French editions: 1952, 1953, 1955. There is also an English translation: *Functional Analysis*, translated from the second French edition. New York, 1955.