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THE BERRY–ESSEEN BOUND FOR SELF-NORMALIZED SUMS

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Abstract

In the article we first estimate a constant in the Berry–Esseen bound for self-normalized sums of independent identically distributed (i.i.d.) random variables. We obtain different versions of the Berry–Esseen bound, namely, under the assumption of the third or sixth moments and for bounded random variables. Our approach to proving the Berry–Esseen bound differs principally from that accepted by Bentkus and Götze. In the case when the sixth moment is finite, we compare our result with that which can be derived from bounds on the reminder in the two-dimensional central limit theorem (CLT).

Key words and phrases: Berry–Esseen bound, self-normalized sum, Student's statistic, central limit theorem.

1. Introduction and results

Let $X, X_1, X_2, \dots, X_n$ be independent identically distributed (i.i.d.) random variables with $\mathbb{E}X = 0$, $\sigma^2 = \mathbb{E}X^2$, $\beta_j = \mathbb{E}|X|^j$. Put $S_n = \sum_1^n X_k$ and $V_n^2 = \sum_1^n X_k^2$ and consider the statistic

$$T_n = \frac{S_n}{V_n}.$$

This statistic is close to the statistic $\sqrt{n}\,t_n$, where

$$t_n = \frac{\overline{X}}{\widehat{\sigma}}, \quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \widehat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 = \frac{1}{n}V_n^2 - \overline{X}^2.$$

Denote by Φ the standard normal law.

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Bentkus and Götze proved in [1] that an absolute constant c exists such that

$$\sup_r \left|\mathbb{P}(\sqrt{n}\, t_n < r) - \Phi(r)\right| < \frac{c\beta_3}{\sigma^3\sqrt{n}}. \tag{1.1}$$

The article [1] also contains a survey of the previous results.

As the authors of [1] observed, the (1.1) is valid for the statistic T_n as well. Put

$$\Delta_n = \sup_r \left|\mathbb{P}(T_n < r) - \Phi(r)\right|.$$

The purpose of the present article is to estimate a constant c in the Berry–Esseen bounds for Δ_n. Observe that this problem is not considered in [1].

Our main result is stated as follows:

Theorem 1. *There exist absolute constants c_1 and c_2 such that*

$$\Delta_n < \left(c_1\frac{\beta_3}{\sigma^3} + c_2\right)n^{-1/2}, \tag{1.2}$$

where

$$c_1 < 36, \quad c_2 < 9. \tag{1.3}$$

Of course, the value of c_1 in Theorem 1 looks very large, especially in comparison with that in the classical Berry–Esseen bound or with the Edgeworth expansion for Student's statistic (see, e.g., [2, 4]). Unfortunately, we are not able to say anything definite on accuracy of the bounds (1.3). However, we can decrease the constant in the Berry–Esseen bound if the moments of order higher than 3 are recruited. We will obtain one such bound on assuming $\beta_6 < \infty$. On the other hand, the additional restriction gives us an opportunity to some extent to describe our approach to the proof of Theorem 1 in Section 3. Observe that our approach differs radically from that in [1]. Denote

$$X_k(n,r) = X_k - \frac{r}{2n}(X_k^2 - \sigma^2),$$
$$X(n,r) = X - \frac{r}{2n\sigma}(X^2 - \sigma^2),$$
$$S_n' = \sum_1^n X_k(n,r),$$
$$\sigma_n^2(r) = \mathbb{E}X^2(n,r),$$
$$\eta_n = \frac{1}{n}\sum_1^n (X_k^2 - \sigma^2).$$

It is easily seen that

$$\left\{S_n\sqrt{n} < rV_n\right\} = \left\{S_n < r\sqrt{\sigma^2 + \eta_n}\right\}$$
$$= \left\{S_n - \frac{\eta_n r}{2\sigma} < \sigma r - \frac{\eta_n^2 r}{\sigma\left(\sqrt{\sigma^2+\eta_n} + \sigma\right)^2}\right\}.$$

Hence, for every $\varepsilon > 0$,

$$\mathbb{P}\left(S_n - \frac{\eta_n r}{2\sigma} < \sigma r(1-\varepsilon)\right) - \mathbb{P}\left(\frac{\eta_n^2}{\sigma^4} > \varepsilon\right)$$
$$< \mathbb{P}(S_n < rV_n) < \mathbb{P}\left(S_n - \frac{\eta_n r}{2\sigma} < \sigma r\right). \tag{1.4}$$

Assume that $\varepsilon = \frac{\beta_3}{\sigma^3\sqrt{n}}$. Then

$$\mathbb{P}\left(\frac{\eta_n^2}{\sigma^4} > \varepsilon\right) < \frac{\beta_4}{\sigma\beta_3\sqrt{n}}, \tag{1.5}$$

$$\Delta_n' := \sup_{r'}\left|\mathbb{P}\left(\frac{S_n'}{\sigma} < r'\right) - \Phi\left(\frac{\sigma r'}{\sigma_n(r)\sqrt{n}}\right)\right| < c_0\frac{\beta_3(n,r)}{\sigma_n^3(r)\sqrt{n}}, \tag{1.6}$$

where $\beta_3(n,r) = \mathbb{E}\left|X(n,r)\right|^3$.

It is known that $c_0 \le 0.7655$ (see [6, 7]).

Clearly,

$$\mathbb{E}\left|X(n,r)\right|^3 < \left(\beta_3 + \frac{r\bar\beta_6}{2n\sigma^3}\right)\left(1 + \frac{r}{2n}\right)^2. \tag{1.7}$$

where $\bar\beta_6 = \mathbb{E}\left|X^2 - \sigma^2\right|^3$. Furthermore,

$$\sigma_n^2(r) > \sigma^2 - \frac{r\beta_3}{n\sigma}.$$

Assume that

$$r < n^{3/4}\sqrt{\frac{\sigma^3}{\beta_3} \vee \frac{\sigma\beta_3}{\beta_4}} \tag{1.8}$$

and

$$\left(\frac{\beta_4}{\sigma\beta_3} \vee \frac{\beta_3}{\sigma^3}\right)\frac{1}{\sqrt{n}} < 0.23. \tag{1.9}$$

Then

$$\sigma_n^2(r) > 0.52\sigma^2. \tag{1.10}$$

It follows from (1.6)–(1.10) that

$$\Delta_n' < 4.1\left(\frac{\beta_3}{\sigma^3} + 0.24\frac{\bar\beta_6}{\sigma^6}\right)n^{-1/2}. \tag{1.11}$$

Next,

$$\left|\Phi\left(\frac{r\sigma}{\sqrt{n}\,\sigma_n(r)}\right) - \Phi\left(\frac{r}{\sqrt{n}}\right)\right| < \frac{r}{\sqrt{2\pi n}}\left|\frac{\sigma-\sigma_n(r)}{\sigma_n(r)}\right|\exp\left\{-\frac{r^2}{2n}\left(\frac{\sigma}{\sigma_n(r)}\wedge 1\right)\right\}. \tag{1.12}$$

Clearly,

$$\left|\frac{\sigma}{\sigma_n(r)} - 1\right| = \frac{|\sigma^2-\sigma_n^2(r)|}{\sigma_n(r)(\sigma+\sigma_n(r))}. \tag{1.13}$$

It is easy to verify that

$$\sigma_n^2(r) - \sigma^2(r) = \frac{r^2}{4n^2}\frac{\mathbb{E}(X^2-\sigma^2)^2}{\sigma^2} - \frac{r\mathbb{E}X^3}{n\sigma}. \tag{1.14}$$

Hence, under the condition (1.9), we obtain

$$|\sigma_n^2(r) - \sigma^2(r)| < \frac{r\beta_3}{\sigma n}\left(1 + \frac{r}{4n}\frac{\beta_4}{\beta_3\sigma}\right) < 1.12\frac{r\beta_3}{\sigma n}. \tag{1.15}$$

It follows from (1.10), (1.12), and (1.15) that

$$\sup_r\left|\Phi\left(\frac{r\sigma}{\sqrt{n}\,\sigma_n(r)}\right) - \Phi\left(\frac{r}{\sqrt{n}}\right)\right| < 1.12\frac{\beta_3}{\sqrt{2\pi e}\,\sqrt{n}\,\sigma\wedge\sigma_n(r)(\sigma+\sigma_n(r))} < 0.21\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{1.16}$$

Under the condition (1.9), we have

$$\Phi\left(\frac{r}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{r(1-\varepsilon)}{\sigma\sqrt{n}}\right) < \frac{r\varepsilon}{\sigma\sqrt{2\pi n}}\exp\left\{-\frac{(r-\varepsilon)^2}{2\sigma^2 n}\right\} < \frac{\varepsilon}{(1-\varepsilon)\sqrt{2\pi e}} < 0.07\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{1.17}$$

Collecting the bounds (1.4), (1.5), (1.11), (1.16), and (1.17), we conclude that, under the conditions (1.8) and (1.9),

$$|\mathbb{P}(T_n < r) - \Phi(r)| < \left(4.4\frac{\beta_3}{\sigma^3} + \frac{\beta_4}{\beta_3\sigma} + \frac{\bar\beta_6}{\sigma^6}\right)n^{-1/2}. \tag{1.18}$$

If $\left(\frac{\beta_3}{\sigma^3}\vee\frac{\beta_4}{\sigma\beta_3}\right)n^{-1/2} \ge 0.23$ then

$$|\mathbb{P}(T_n < r) - \Phi(r)| < 1 < 4.35\left(\frac{\beta_3}{\sigma^3}\vee\frac{\beta_4}{\sigma\beta_3}\right)n^{-1/2}. \tag{1.19}$$

Finally, consider the case

$$r > n^{3/4}\sqrt{\frac{\sigma^3}{\beta_3}\vee\frac{\sigma\beta_3}{\beta_4}}, \quad \left(\frac{\beta_3}{\sigma^3}\vee\frac{\beta_4}{\sigma\beta_3}\right)n^{-1/2} < 0.23.$$

It is easy to see that

$$\left\{\frac{\eta_n^2}{\sigma^4} < \frac{\beta_3}{\sigma^3\sqrt{n}}\right\} \subset \left\{\frac{V_n^2}{n} > \sigma^2\left(1 - \left(\frac{\beta_3}{\sigma^3\sqrt{n}}\right)^{1/2}\right)\right\} \subset \left\{\frac{V_n^2}{n} > 0.52\sigma^2\right\}.$$

Hence, by (1.5),

$$\mathbb{P}\left(\frac{V_n^2}{n} > 0.52\sigma^2\right) > 1 - \frac{\beta_4}{\sigma\beta_3\sqrt{n}}. \tag{1.20}$$

On the other hand,

$$\mathbb{P}\left(\frac{S_n}{V_n} > n^{3/4}\sqrt{\frac{\sigma^3}{\beta_3}},\ V_n^2 > 0.46\sigma^2\right) < \mathbb{P}\left(\frac{S_n}{\sqrt{0.52}\,\sigma} > n^{3/4}\sqrt{\frac{\sigma^3}{\beta_3}}\right) < 1.93\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{1.21}$$

It follows from (1.20) and (1.21) that

$$\mathbb{P}\left(\frac{S_n}{V_n} > n^{3/4}\sqrt{\frac{\sigma^3}{\beta_3}}\right) < 1.93\frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{\beta_4}{\sigma\beta_3\sqrt{n}} \tag{1.22}$$

under the condition $\left(\frac{\beta_3}{\sigma^3}\vee\frac{\beta_4}{\sigma\beta_3}\right)n^{-1/2} < 0.23$. Comparing (1.18), (1.19), and (1.22), we conclude that the estimate (1.18) is valid without restrictions on r and the moments β_3 and β_4.

Similar arguments were used recently by S. Y. Novak (see [5]). Combining them with the preliminary truncation of summands he obtained the estimate

$$\sup_r|\mathbb{P}(t_n < r) - \Phi(r)| \le c_1\frac{\beta_3}{\sqrt{n}} + c_2\frac{\sqrt{\mathrm{Var}\,(X^2)}}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),$$

where

$$c_1 = \frac{8}{e\sqrt{2\pi}} + C_*, \quad C_* \le 0.7655,$$

$$c_2 = \left(\frac{2}{\pi e}\right)^{1/4}.$$

This estimate is intermediate between the Berry–Esseen bound and asymptotic expansion. It is difficult to compare this estimate with the bound (1.18) until the constant in the remainder is computed. Assuming only that $\beta_3 < \infty$, S. Y. Novak obtain the bound $\Delta_n = O(n^{-2/7})$ without evaluating the constants.

Observe that, in the case $\beta_4 < \infty$, we can obtain the bound for Δ_n with explicit constants using S. Y. Novak's results (see [5]).

If random variables X_i are bounded then we can take a smaller value of the absolute constant c.

Theorem 2. *Let* $|X| \leq L < \infty$. *Then*

$$\Delta_n < 3.5\frac{L}{\sigma\sqrt{n}}. \tag{1.23}$$

It is tempting to use the two-dimensional CLT for estimating Δ_n.

We try to realize this idea in the case $\beta_6 < \infty$, $\mathbb{E}X^3 = 0$.

It is easily seen that

$$\mathbb{P}(|T_n| < r) = \mathbb{P}\left(\left(\frac{S_n}{\sigma\sqrt{n}}, \frac{Z_n}{\widetilde{\sigma}\sqrt{n}}\right) \in G_{n,r}\right) := P_{n,r}, \tag{1.24}$$

where

$$Z_n = \sum_1^n (X_j^2 - \sigma_j^2),$$

$$G_{n,r} = \left\{x, y : x^2 < \sqrt{\frac{\widetilde{\sigma}}{\sigma^2}\frac{r^2 y}{\sqrt{n}} + r^2}\right\}, \quad \widetilde{\sigma}^2 = \beta_4 - \sigma^4.$$

Let Φ_2 be a standard normal law in $\mathbb{R}_2$. Assume for simplicity that $\frac{\widetilde{\sigma}}{\sigma^2} = 1$. According to (13.44) in [3], we have

$$\limsup_{n\to\infty} \sqrt{n}\,|P_{n,r} - \Phi_2(G_{n,r})| < \left[a_1(2) + \frac{128}{3}\pi^{-1/3}2^{4/3}\frac{\Gamma(3/2)}{\Gamma(1)}\right]\rho_3, \tag{1.25}$$

where

$$\rho_3 = \mathbb{E}\left(\frac{X^2}{\sigma^2} + \frac{(X^2-\sigma^2)^4}{\widetilde{\sigma}^2}\right)^{3/2},$$

$$\begin{aligned} a_1(2) &= \frac{2}{9}(2\pi)^{-1/2}(4e^{-3/2}+1) \\ &\quad + \frac{2}{3}(\pi e^{1/2})^{-1}2^{1/2}\left(1-\frac{1}{2}\right) \\ &\quad + \frac{2}{3}\left(\frac{2}{\pi}\right)^{3/2}\left(2^{3/2} - 3\sqrt{2} + 2\right) \\ &\approx 0.4571. \end{aligned}$$

Clearly,

$$\rho_3 \geq \frac{\beta_6}{\sigma^6}.$$

Denoting by $c(2)$ the coefficient of ρ_3 in (1.25), we obtain

$$c(2) \leq 65.5136. \tag{1.26}$$

Put

$$G_{n,r}^+ = \left\{(x,y) : (x,y) \in G_{n,r},\ 0 \leq y < \sqrt{\ln n}\right\},$$

$$G_{n,r}^- = \left\{(x,y) : (x,y) \in G_{n,r},\ -\sqrt{\ln n} < y < 0\right\}.$$

Obviously,

$$\begin{aligned} \Phi_2(G_{n,r}^-) &= \frac{1}{2\pi}\int_{G_{n,r}^-} e^{-(x^2+y^2)/2}dxdy \\ &= \frac{1}{2\pi}\int_{-\sqrt{\ln n}<y<o} e^{-y^2/2}dy \int_{|x|<r_n(y)} e^{-x^2/2}dx, \end{aligned} \tag{1.27}$$

where $r_n(y) = r\sqrt{1 + y/\sqrt{n}}$. It is easy to see that, for $0 > y > -\sqrt{n}$,

$$0 < \int_{|x|<r} e^{-x^2/2}dx - \int_{|x|<r_n(y)} e^{-x^2/2}dx < 2\big(r - r_n(y)\big)e^{-r_n^2(y)/2}. \tag{1.28}$$

Using the identity

$$r_n(y) - r = \frac{ry}{\sqrt{n}}\frac{1}{1+\sqrt{1+y/\sqrt{n}}}$$

and the inequality

$$\sqrt{\frac{\ln n}{n}} < \frac{1}{e}, \tag{1.29}$$

we conclude that, for $-\sqrt{\ln n} < y < 0$,

$$r - r_n(y) < \frac{r|y|}{(\sqrt{1-e^{-1}}+1)\sqrt{n}}. \tag{1.30}$$

On the other hand, by (1.29),

$$r_n(y) > r\sqrt{1-e^{-1}}. \tag{1.31}$$

Combining (1.30), (1.31), and the inequality

$$re^{-r^2/2} < \frac{1}{\sqrt{e}}$$

we obtain, for $-\sqrt{\ln n} < y < 0$,

$$\big(r - r_n(y)\big)e^{-r_n^2(y)/2} < \frac{1}{\big(1+\sqrt{1-e^{-1}}\big)\sqrt{e-1}}\,\frac{|y|}{\sqrt{n}}. \tag{1.32}$$

It follows from (1.27), (1.28), and (1.32) that, uniformly with respect to r,

$$0 < \Phi_2(D_{n,r}^-) - \Phi_2(G_{n,r}^-) < \frac{0.271}{\sqrt{n}}, \tag{1.33}$$

where

$$D_n^- = \Big\{x, y : |x| < r,\ -\sqrt{\ln n} < y < 0\Big\}.$$

Similarly,

$$0 < \Phi_2(G_{n,r}^+) - \Phi_2(D_{n,r}^+) < \frac{0.215}{\sqrt{n}}, \tag{1.34}$$

where

$$D_{n,r}^+ = \{x, y : |x| < r,\ -\ln n < y < 0\}.$$

Next, for $n \ge 3$ and every $r > 0$, we have

$$\Phi_2(\overline{G}_{n,r}) < \sqrt{\frac{2}{\pi n}}, \quad \overline{G}_{n,r} = \mathbb{R}_2 - G_{n,r}\,.$$

Collecting the bounds (1.25)–(1.27) and (1.33)–(1.35) and taking account of (1.24), we conclude that, for $n \ge 3$,

$$\sup_r \big|\mathbb{P}\big(|T_n| < r\big) - \Phi(r)\big| < \left(c_1\frac{\beta_6}{\sigma^6} + c_2\right)n^{-1/2}, \tag{1.35}$$

where $c_1 \ge c(2)$ and $c_2 < 1.3$.

Comparing (1.2) and (1.18) with (1.35), we see that both bounds are sharper than (1.35).

The case $\beta_4 = \infty$ is most difficult since we do not have an appropriate estimate for $\mathbb{P}\Big(\frac{1}{n}\sum_1^n (X_i^2 - \sigma^2) > \varepsilon\Big)$. We will briefly describe how this difficulty can be overcome. The main idea consists in reducing the general case to that of bounded summands. We may represent the distribution of every summand as a mixture of two distributions, the first of which is concentrated on the interval $(-L, L)$. As a result, the distribution of T_n is written as a mixture of distributions of self-normalized sums consisting of summands of two types described above. Given the values of the second type summands, we obtain the self-normalized sum of bounded i.d.d. random variables. Using the same approach as in proving (1.18), we approximate the distribution of every such statistics with a Gaussian law whose shift and variance depend on the above condition.

It means in turn that $\mathbb{P}(T_n < r)$ is approximated for every r with the expectation of the smooth functional of the sum of a random number of i.i.d. random variables (r.v.'s) of the second type (see (3.84) and (3.108)). In the sequel, we apply the Berry–Esseen bounds to the summands of the second type and for the number of successes in the Bernoulli trials.

2. Auxiliary results. Proof of Theorem 2

Throughout this section, we suppose that $|X| \le L < \infty$. In contrast to Section 1, we do not require that $\mathbb{E}X = 0$. Denote $a = \mathbb{E}X$ and $b^2 = \mathbb{E}X^2$. We begin with

Lemma 2.1. *For every x, y, and z such that $y + z > 0$ and $x + z > 0$ we have*

$$\sqrt{x+z} - \sqrt{y+z} = \frac{x-y}{2\sqrt{y+z}} - \frac{(x-y)^2}{\sqrt{y+z}\big(\sqrt{x+z}+\sqrt{y+z}\big)^2}. \tag{2.1}$$

Proof. Obviously,

$$\sqrt{x+z} - \sqrt{y+z} = \frac{x-y}{\sqrt{x+z}+\sqrt{y+z}}. \tag{2.2}$$

On the other hand,

$$\frac{1}{\sqrt{x+z}+\sqrt{y+z}} - \frac{1}{2\sqrt{y+z}} = \frac{\sqrt{y+z}-\sqrt{x+z}}{\sqrt{y+z}\big(\sqrt{x+z}+\sqrt{y+z}\big)} = \frac{y-x}{\sqrt{y+z}\big(\sqrt{x+z}+\sqrt{y+z}\big)^2}.$$

Combining these two identities, we obtain the desired result. □

Denote

$$\eta_n = \frac{1}{n}\sum_{k=1}^n (X_k^2 - b^2), \quad X_n(r) = X - \frac{r(X^2-b^2)}{2n\sqrt{\alpha^2 b^2 + y^2}},$$

$$\sigma_n^2(r) = \operatorname{Var} X_n(r), \quad \sigma^2 = \operatorname{Var} X, \quad \alpha^2 = \frac{k}{n}.$$

Lemma 2.2. *Let*

$$r < \alpha\gamma n^{3/4}\left(\frac{b}{L}\right)^{1/2}. \tag{2.3}$$

Then

$$\sigma_n^2(r) > \sigma^2 - 2\gamma b^2\left(\frac{L}{b}\right)^{1/2} n^{-1/4}. \tag{2.4}$$

Proof. It is easily seen that

$$\sigma^2-\sigma_n^2(r)=\frac{r}{n\sqrt{\alpha^2b^2+y^2}}\mathbb{E}(X-a)(X^2-b^2)-\frac{r^2}{4n^2}\frac{\mathbb{E}(X^2-b^2)^2}{\alpha^2b^2+y^2}. \quad (2.5)$$

Notice that

$$\left|\mathbb{E}(X-a)(X^2-b^2)\right|=\left|\mathbb{E}X^3-ab^2\right|<2\beta_3<2Lb^2. \quad (2.6)$$

Combining (2.5) and (2.6), we come to the desired result. □

Lemma 2.3. *Under the condition* (2.3), *the following inequality is valid:*

$$\left|\sigma_n^2(r)-\sigma^2\right|<\frac{2r\beta_3}{\alpha bn}\left(1+\frac{\gamma}{8}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right). \quad (2.7)$$

Proof. Observe that

$$\mathbb{E}(X^2-b^2)^2<\mathbb{E}X^4<L\beta_3. \quad (2.8)$$

Combining (2.3), (2.5), (2.6), and (2.8), we obtain the desired result. □

Lemma 2.4. *Let the condition* (2.3) *and the inequality*

$$\gamma\left(\frac{L}{b\sqrt{n}}\right)^{1/2}<2\nu,\quad 0<\nu\le 1, \quad (2.9)$$

hold. Then

$$\sigma_n^{-3}(r)\mathbb{E}\left|X_n(r)-a\right|^3<\left(1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)^{-1/2}\frac{(1+\nu)L+|a|}{\sigma}. \quad (2.10)$$

Proof. It is easy to verify that

$$X-\frac{rX^2}{2n\sqrt{\alpha^2b^2+y^2}}<X_n(r)<X+\frac{rb^2}{2n\sqrt{\alpha^2b^2+y^2}}. \quad (2.11)$$

By (2.3) and (2.9), we have

$$\frac{r|X|}{2n\sqrt{\alpha^2b^2+y^2}}<1. \quad (2.12)$$

It follows from (2.11) and (2.12) that

$$|X_n(r)|<(1+\nu)|X|.$$

Furthermore,

$$\frac{\mathbb{E}\left|X_n(r)-a\right|^3}{\sigma_n^3(r)}\le\frac{\sup|X_n(r)|+|a|}{\sigma_n(r)}.$$

The last bounds and (2.4) yield the inequality (2.10). □

Lemma 2.5. *Under the conditions of Lemma 2.4,*

$$\left|\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sigma_n(r)\sqrt{k}}\right)-\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)\right|$$
$$<\sqrt{\frac{2}{\pi}}\frac{\beta_3}{n\sigma b^2\alpha^2}\left(1+\frac{\gamma}{8}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)\left(1-2\gamma\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\frac{b^2}{\sigma^2}\right)^{-1/2}$$
$$\times\left(\frac{2}{e}\sqrt{k}+e^{-1/2}\frac{|z|}{\sigma}\left(2-2\gamma\left(\frac{L}{b\sqrt{2}}\right)^{1/2}\frac{b^2}{\sigma^2}\right)^{-1/2}\right). \quad (2.13)$$

Proof. It is easy to see that

$$\triangle:=\left|\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sigma_n(r)\sqrt{k}}\right)-\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)\right|$$
$$<\frac{\left|\left(\sigma_n^2(r)-\sigma^2\right)\left(r\sqrt{\alpha^2b^2+y^2}+z\right)\right|}{\sqrt{2\pi k}(\sigma_n(r)+\sigma)\sigma_n(r)\sigma}\exp\left\{-\frac{\left(r\sqrt{\alpha^2b^2+y^2}+z\right)^2}{2(\sigma^2\vee\sigma_n^2(r))k}\right\}. \quad (2.14)$$

Clearly,

$$r\left|r\sqrt{\alpha^2b^2+y^2}+z\right|<\frac{\left(r\sqrt{\alpha^2b^2+y^2}+z\right)^2}{\sqrt{\alpha^2b^2+y^2}}+\frac{\left|z\left(r\sqrt{\alpha^2b^2+y^2}+z\right)\right|}{\sqrt{\alpha^2b^2+y^2}}. \quad (2.15)$$

Furthermore,

$$\frac{\left(r\sqrt{\alpha^2b^2+y^2}+z\right)^2}{\sqrt{k}}\exp\left\{-\frac{\left(r\sqrt{\alpha^2b^2+y^2}+z\right)^2}{2(\sigma^2\vee\sigma_n^2(r))k}\right\}<\frac{2}{e}\sqrt{k}(\sigma^2\vee\sigma_n^2(r)), \quad (2.16)$$

$$\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sqrt{k}}\exp\left\{-\frac{\left(r\sqrt{\alpha^2b^2+y^2}+z\right)^2}{2(\sigma^2\vee\sigma_n^2(r))k}\right\}<e^{-1/2}\sqrt{\sigma^2\vee\sigma_n^2(r)}. \quad (2.17)$$

It follows from (2.14)–(2.17) that

$$\triangle<\frac{\left|\sigma_n^2(r)-\sigma^2\right|}{\alpha br\sqrt{2\pi}(\sigma_n(r)+\sigma)\sigma\sigma_n(r)}$$
$$\times\left(\frac{2}{e}\sqrt{k}(\sigma^2\vee\sigma_n^2(r))+e^{-1/2}|z|(\sigma\vee\sigma_n(r))\right). \quad (2.18)$$

Taking account of (2.4), we infer that

$$\frac{\sigma^2\vee\sigma_n^2(r)}{\sigma\sigma_n(r)\big(\sigma_n(r)+\sigma\big)} < \frac{1}{\sigma\wedge\sigma_n(r)} < \frac{1}{\sigma\sqrt{1-2\gamma\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\frac{b^2}{\sigma^2}}},$$

$$\frac{\sigma\vee\sigma_n(r)}{\sigma\sigma_n(r)} < \frac{1}{\sigma\wedge\sigma_n(r)} < \frac{1}{\sigma\sqrt{1-2\gamma\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\frac{b^2}{\sigma^2}}}. \tag{2.19}$$

By (2.7),

$$\frac{|\sigma_n^2(r)-\sigma^2|}{r} < \frac{2\beta_3}{nb\alpha}\left(1+\frac{\gamma}{8}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right). \tag{2.20}$$

The claim of the lemma is immediate from (2.14)–(2.16). □

Lemma 2.6. *For every* $0<\varepsilon<1$,

$$\left|\Phi\left(\frac{r(1-\varepsilon)\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)-\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)\right| < \frac{\varepsilon}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{e}(1-\varepsilon)}+\frac{|z|}{\sigma\sqrt{k}}\right). \tag{2.21}$$

Proof. It is easy to see that

$$\triangle := \Phi\left|\left(\frac{r(1-\varepsilon)\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)-\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)\right| < \frac{\varepsilon r\sqrt{b^2+y^2}}{\sigma\sqrt{2\pi k}}\exp\left\{-\left(\frac{(1-\varepsilon)r\sqrt{\alpha^2b^2+y^2}+z}{\sigma\sqrt{k}}\right)^2\right\}.$$

Using the identity

$$r\sqrt{\alpha^2b^2+y^2}=\left(r\sqrt{\alpha^2b^2+y^2}+z\right)-z,$$

we obtain

$$\triangle < \frac{\varepsilon}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{e}(1-\varepsilon)}+\frac{|z|}{\sigma\sqrt{k}}\right).$$

This completes the proof. □

Lemma 2.7. *Let* (2.3) *and* (2.9) *hold and* $0<\frac{L}{b\sqrt{n}}\delta<1$. *Then, for every* x *and* $y\ge 0$, *we have*

$$\left|\mathbb{P}\left(S_k+x<r\sqrt{\frac{1}{n}\sum_1^k X_i^2+y^2}\right)-\Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}-x-ka}{\sigma\sqrt{k}}\right)\right|$$

$$< c_0\frac{(1+\nu)L+|a|}{\sigma\sqrt{k\left(1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)}}+\frac{2}{e}\sqrt{\frac{2}{\pi}}\frac{\beta_3}{\sqrt{k}\,\sigma b^2}\frac{1+\frac{\gamma}{8}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}}{\sqrt{1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}}}$$

$$+\frac{16}{9}\frac{L\beta_3}{b^4k}+\frac{4}{9}\frac{\sqrt{n}\,\beta_3}{\delta kb^3}+\frac{1}{\sqrt{2\pi e}}\frac{\delta L}{\left(1-\frac{\delta L}{b\sqrt{n}}\right)b\sqrt{n}}$$

$$+\frac{|z|}{\sigma}\left(\frac{\delta L}{\sqrt{2\pi}\,b\sqrt{nk}}+\frac{1}{\sqrt{2\pi e}}\times\frac{\beta_3}{\sigma b^2\sqrt{k}}\frac{1+\frac{\gamma}{8}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}}{\sqrt{\left(1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)\left(1-\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)}}\right), \tag{2.22}$$

where $c_0<0.7655$ *and* $z=x+ka$.

Proof. Without loss of generality we may confine exposition to the case $r>0$. Denote

$$\eta=\frac{1}{n}\sum_1^k(X_i^2-b^2),\quad S=S_k-ka,\quad z=x+ka,\quad \alpha^2=\frac{k}{n}.$$

Using Lemma 2.1, we obtain

$$\left\{S_k+x<r\sqrt{\frac{1}{n}\sum_1^k X_i^2+y^2}\right\}$$

$$=\left\{S<r\sqrt{\alpha^2b^2+y^2+\eta}-z\right\}$$

$$=\left\{S-\frac{\eta r}{2\sqrt{\alpha^2b^2+y^2}}<r\left(\sqrt{\alpha^2b^2+y^2}-\frac{\eta^2}{\sqrt{\alpha^2b^2+y^2}\left(\sqrt{\alpha^2b^2+y^2}+\sqrt{\alpha^2b^2+y^2+\eta}\right)^2}\right)-z\right\}. \tag{2.23}$$

Employing the Berry–Esseen bound and Lemma 2.4, we conclude that, under the conditions (2.3) and (2.9),

$$\left|\mathbb{P}\left(S-\frac{\eta r}{2\sqrt{\alpha^2b^2+y^2}}<u\right)-\Phi\left(\frac{u}{\sigma_n(r)\sqrt{k}}\right)\right| < c_0\frac{\mathbb{E}\left|X_n(r)-a\right|^3}{\sigma_n^3(r)\sqrt{k}} < c_0\frac{(1+\nu)L+|a|}{\sigma\sqrt{\left(1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)k}}. \tag{2.24}$$

Therefore,

$$\mathbb{P}\left(S_k+x<r\sqrt{\frac{1}{n}\sum_1^k X_i^2+y^2}\right) < \mathbb{P}\left(S-\frac{\eta r}{2\sqrt{\alpha^2b^2+y^2}}<r\sqrt{\alpha^2b^2+y^2}-z\right) < \Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}-z}{\sigma_n(r)\sqrt{k}}\right)+c_0\frac{(1+\nu)L+|a|}{\sigma\sqrt{\left(1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)k}}. \tag{2.25}$$

Similarly,

$$\mathbb{P}\left(S_k+x<r\sqrt{\frac{1}{n}\sum_1^k X_i^2+y^2}\right) > \mathbb{P}\left(S-\frac{\eta r}{2\sqrt{\alpha^2b^2+y^2}}<r(1-\varepsilon)\sqrt{\alpha^2b^2+y^2}-z\right) - \mathbb{P}\left(\frac{\eta^2}{\sqrt{b^2+y^2}\left(\sqrt{\alpha^2b^2+y^2+\eta}+\sqrt{\alpha^2b^2+y^2}\right)^2}>\varepsilon\sqrt{\alpha^2b^2+y^2}\right). \tag{2.26}$$

If $\eta > -\frac{3}{4}\alpha^2b^2$ then

$$\alpha^2b^2+y^2+\eta>\frac{\alpha^2}{4}b^2.$$

Taking account of (2.8), we infer that

$$\mathbb{E}\eta^2<\frac{k}{n^2}\mathbb{E}X^4<\frac{kL\beta_3}{n^2}.$$

Therefore,

$$\mathbb{P}\left(\frac{\eta^2}{(\alpha^2b^2+y^2)\left(\sqrt{\alpha^2b^2+y^2+\eta}+\sqrt{\alpha^2b^2+y^2}\right)^2}>\varepsilon\right) < \mathbb{P}\left(\eta^2>\frac{9\alpha^4}{4}b^4\varepsilon\right)+\mathbb{P}\left(\eta<-\frac{3\alpha^2}{4}b^2\right) < \left(\frac{4}{9\varepsilon}+\frac{16}{9}\right)\frac{\mathbb{E}\eta^2}{\alpha^4b^4} < \frac{4}{9}\left(4+\frac{1}{\varepsilon}\right)\frac{L\beta_3}{kb^4}. \tag{2.27}$$

It follows from (2.24), (2.25), and (2.27) that

$$\mathbb{P}\left(S_k+x<r\sqrt{\frac{1}{n}\sum_1^k X_i^2+y^2}\right) > \Phi\left(\frac{r(1-\varepsilon)\sqrt{\alpha^2b^2+y^2}-z}{\sigma_n(r)\sqrt{k}}\right) - c_0\frac{(1+\nu)L+|a|}{\sigma\sqrt{\left(1-2\gamma\frac{b^2}{\sigma^2}\left(\frac{L}{b\sqrt{n}}\right)^{1/2}\right)k}}-\frac{4}{9}\left(4+\frac{1}{\varepsilon}\right)\frac{L\beta_3}{kb^4}. \tag{2.28}$$

Combining (2.13), (2.23) and (2.28), we obtain

$$\Phi\left(\frac{r(1-\varepsilon)\sqrt{\alpha^2b^2+y^2}-z}{\sigma\sqrt{k}}\right)-\delta_1 < \mathbb{P}\left(S_k+x<r\sqrt{\frac{1}{n}\sum_1^k X_i^2+y^2}\right) < \Phi\left(\frac{r\sqrt{\alpha^2b^2+y^2}-z}{\sigma\sqrt{k}}\right)+\delta_2,$$

where

$$\delta_2 = c_0 \frac{(1+\nu)L + |a|}{\sigma\sqrt{k\Big(1 - 2\gamma\frac{b^2}{\sigma^2}\big(\frac{L}{b\sqrt{n}}\big)^{1/2}\Big)}}$$

$$+ \sqrt{\frac{2}{\pi}}\,\frac{\beta_3}{n\sigma\alpha b^2}\,\frac{1+\frac{\gamma}{8}\big(\frac{L}{b\sqrt{n}}\big)^{1/2}}{\sqrt{1-2\gamma\frac{b^2}{\sigma^2}\big(\frac{L}{\sigma\sqrt{n}}\big)^{1/2}}}\left(\frac{2}{e}\sqrt{k}+\frac{|z|}{\sigma\sqrt{2e\Big(1-\gamma\frac{b^2}{\sigma^2}\big(\frac{L}{b\sqrt{n}}\big)^{1/2}\Big)}}\right),$$

$$\delta_1 = \delta_2 + \frac{4}{9}\left(4 + \frac{1}{\varepsilon}\right)\frac{L\beta_3}{kb^4}.$$

Hence, by (2.21),

$$\left|\mathbb{P}\left(S_k + x < r\sqrt{\frac{1}{n}\sum_1^k X_i^2 + y^2}\right) - \Phi\left(\frac{r\sqrt{b^2+y^2} - z}{\sigma\sqrt{k}}\right)\right|$$

$$< \delta_1 + \frac{\varepsilon}{\sqrt{2\pi}}\left(\frac{1}{(1-\varepsilon)\sqrt{e}} + \frac{|z|}{\sigma\sqrt{k}}\right).$$

Letting $\varepsilon = \frac{L\delta}{b\sqrt{n}}$, we arrive at the claim of the lemma. □

Lemma 2.8. *For every $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(\sum_1^n X_i^2 < nb^2\varepsilon\right) \le \exp\left\{-(1-\varepsilon)^2\frac{nb^4}{2\beta_4}\right\}. \tag{2.29}$$

Proof. For every $h > 0$ and $x > 0$, we have

$$\mathbb{P}\left(\sum_1^n X_i^2 < x\right) < e^{hx}\left(\mathbb{E}e^{-hX^2}\right)^n,$$

$$\mathbb{E}e^{-hX^2} = 1 + \mathbb{E}(e^{-hX^2} - 1) \le 1 - hb^2 + \frac{h^2}{2}\beta_4 < \exp\left(\frac{h^2}{2}\beta_4 - hb^2\right).$$

These inequalities imply that

$$\mathbb{P}\left(\sum_1^n X_i^2 < x\right) < \exp\left(h(x - nb^2) + n\frac{h^2}{2}\beta_4\right). \tag{2.30}$$

It is clear that, for $x < nb^2$,

$$\min_h\left(h(x - nb^2) + \frac{nh^2}{2}\beta_4\right) = -\frac{(x-nb^2)^2}{2\beta_4 n}. \tag{2.31}$$

Combining (2.30) and (2.31), we conclude that

$$\mathbb{P}\left(\sum_1^n X_i^2 < x\right) \le \exp\left(-\frac{(x-nb^2)^2}{2\beta_4 n}\right).$$

Putting $x = nb^2\varepsilon$, we obtain the bound (2.29). □

Notice that Lemma 2.8 is valid without the restriction $|X_i| \le L$.

Corollary. *For every $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(\sum_1^n X_i^2 < nb^2\varepsilon\right) \le \exp\left\{-(1-\varepsilon)^2\frac{nb^2}{2L^2}\right\}. \tag{2.32}$$

Remark. From Lemma 12 in [5], we easily derive the inequalities

$$\mathbb{P}\left(\sum_{i=1}^n X_i^2 < \frac{nb^2}{3}\right) < \exp\left(-\frac{nb^4}{36\beta_4}\right) < \exp\left(-\frac{nb^2}{36L^2}\right).$$

Putting $\varepsilon = \frac{1}{3}$ in (2.29), we obtain the sharper bound

$$\mathbb{P}\left(\sum_{i=1}^n X_i^2 < \frac{nb^2}{3}\right) < \exp\left(-\frac{2nb^4}{9\beta_4}\right).$$

Lemma 2.9. *For every $0 < \varepsilon < 1$,*

$$\mathbb{P}\left(\sum_1^n X_i^2 < n\varepsilon b^2\right) < \exp\left(-\frac{8(1-\varepsilon)^3 nb^6}{27\beta_3^2}\right).$$

Proof. It is easy to see that

$$\mathbb{E}e^{-hX^2} = 1 - hb^2 + \mathbb{E}\left(e^{-hX^2} - 1 + hX^2\right).$$

It is obvious that

$$\mathbb{E}\left(e^{-hX^2} - 1 + hX^2\right) = \mathbb{E}\left(e^{-hX^2} - 1 + hX^2;\ hX^2 \le \sqrt{2}\right)$$
$$+ \mathbb{E}\left(e^{-hX^2} - 1 + hX^2;\ hX^2 > \sqrt{2}\right)$$
$$= E_1 + E_2.$$

The bounds

$$E_1 < \frac{h^2}{2}\mathbb{E}\left(X^4;\ hX^2 \le \sqrt{2}\right) \le \frac{h^{3/2}}{\sqrt{2}}\mathbb{E}\left(|X|^3;\ hX^2 \le \sqrt{2}\right),$$

$$E_2 < h\mathbb{E}\left(X^2;\ hX^2 > \sqrt{2}\right) \le \frac{h^{3/2}}{\sqrt{2}}\mathbb{E}\left(|X|^3; hX^2 > \sqrt{2}\right)$$

hold. Thus,

$$\mathbb{E}e^{-hX^2} \le 1 - b^2h + \frac{\beta_3}{\sqrt{2}}h^{3/2}.$$

Hence,

$$\mathbb{P}\Big(\sum_1^n X_i^2 < x\Big) < e^{hx}(\mathbb{E}e^{-hX^2})^n < \exp\Big((x - nb^2)h + \frac{n\beta_3}{\sqrt{2}}h^{3/2}\Big).$$

Notice that, for $x < nb^2$,

$$\min_h\Big((x - nb^2)h + \frac{n\beta_3}{\sqrt{2}}h^{3/2}\Big)$$

is achieved at the point h_0 satisfying the equation

$$(x - nb^2) + \frac{3}{2\sqrt{2}}n\beta_3h^{1/2} = 0.$$

Clearly,

$$h_0^{1/2} = \frac{2\sqrt{2}}{3}\frac{nb^2 - x}{n\beta_3}.$$

As a result, we obtain the bound

$$\mathbb{P}\Big(\sum_1^n X_i^2 < x\Big) < \exp\Big(-\frac{8(x - nb^2)^3}{27n\beta_3^2}\Big).$$

We are left with putting $x = n\varepsilon b^2$. □

We are able now to prove Theorem 2. To this end, put $k = n$, $a = 0$, $b = \sigma$, $y = 0$, and $x = 0$ in Lemma 2.7. Let the condition (2.3) hold for $\gamma = 0.51$. Without loss of generality we may assume that

$$\frac{L}{\sigma\sqrt{n}} < \frac{1}{3.5} < 0.286. \tag{2.33}$$

Therefore,

$$\gamma\Big(\frac{L}{\sigma\sqrt{n}}\Big)^{1/2} < 0.273,$$

i.e., the condition (2.9) holds with $\nu = 0.137$. Hence,

$$\sqrt{1 - 2\gamma\Big(\frac{L}{\sigma\sqrt{n}}\Big)^{1/2}} > 0.674.$$

As a result, we obtain the bounds

$$c_0\frac{1.275L}{\sigma\sqrt{1 - 2\gamma\Big(\frac{L}{\sigma\sqrt{n}}\Big)^{1/2}}} < 1.3\frac{L}{\sigma},$$

$$\frac{2}{e}\sqrt{\frac{2}{\pi}}\,\frac{1 + \frac{\gamma}{8}\Big(\frac{L}{\sigma\sqrt{n}}\Big)^{1/2}}{\sqrt{1 - 2\gamma\Big(\frac{L}{\sigma\sqrt{n}}\Big)^{1/2}}} < 0.91. \tag{2.34}$$

Furthermore, by (2.33),

$$\frac{16}{9}\frac{L\beta_3}{\sigma^4 n} < \frac{16}{9}\frac{L^2}{\sigma^2 n} < 0.51\frac{L}{\sigma\sqrt{n}}. \tag{2.35}$$

Putting $\delta = 1$, we have

$$\frac{4}{9}\frac{\beta_3}{\delta\sigma^3} + \frac{1}{\sqrt{2\pi e}}\,\frac{\delta L}{\Big(1 - \frac{\delta L}{\sigma\sqrt{n}}\Big)\sigma} < 0.78\frac{L}{\sigma}. \tag{2.36}$$

It follows from (2.22) and (2.34)–(2.36) that, for $|u| \le 0.51n^{1/4}\sqrt{\frac{\sigma}{L}}$,

$$\big|\mathbb{P}(T_n < u) - \Phi(u)\big| < 3.5\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{2.37}$$

In the case $|r| > 0.51n^{3/4}\sqrt{\frac{\sigma}{L}}$, we apply Lemma 2.8 and the Chebyshev inequality.

According to (2.32),

$$\mathbb{P}\Big(\sum_1^n X_i^2 < \frac{n\sigma^2}{2}\Big) < \exp\Big\{-\frac{n\sigma^2}{8L^2}\Big\}.$$

Thus,

$$\mathbb{P}\Big(\sum_1^n X_i^2 < \frac{n\sigma^2}{2}\Big) < \exp\Big\{-\frac{n\sigma^2}{8L^2}\Big\} < \frac{8}{e}\frac{L^2}{n\sigma^2} < 0.842\frac{L}{\sigma\sqrt{n}}. \tag{2.38}$$

If $|r| > r_n = 0.51n^{3/4}\sqrt{\frac{\sigma}{L}}$ and $\sum_1^n X_i^2 > \frac{n}{2}\sigma^2$ simultaneously then, by the Berry–Esseen bound,

$$\mathbb{P}\Big(S_n > \frac{r}{n}\sqrt{\sum X_i^2}\Big) < \mathbb{P}\Big(S_n > \frac{r_n}{\sigma\sqrt{2}}\Big) \le 1 - \Phi\Big(\frac{r_n}{\sigma\sqrt{2n}}\Big) + c_0\frac{L}{\sigma\sqrt{n}}. \tag{2.39}$$

Using the inequality

$$1 - \Phi(u) < \frac{1}{\sqrt{2\pi}\, u} e^{-u^2/2} < \frac{1}{\sqrt{2\pi e}\, u^2}, \tag{2.40}$$

we obtain

$$1 - \Phi\left(\frac{r_n}{\sqrt{2n}}\right) < \sqrt{\frac{2}{\pi e}} \frac{n}{r_n^2}. \tag{2.41}$$

Hence,

$$1 - \Phi\left(\frac{r_n}{\sqrt{2n}}\right) < \frac{0.49}{\gamma^2} \frac{L}{\sigma\sqrt{n}} < 1.89 \frac{L}{\sigma\sqrt{n}}. \tag{2.42}$$

It follows from (2.38), (2.39), and (2.42) that

$$\mathbb{P}\left(S_n > \frac{r_n}{n}\sqrt{\sum X_i^2}\right) < 3.5 \frac{L}{\sigma\sqrt{n}}. \tag{2.43}$$

On the other hand, by (2.40), for $u > 0.51 n^{1/4}\sqrt{\frac{\sigma}{L}}$, we have

$$1 - \Phi(u) < \frac{1}{\sqrt{2\pi e}\, u^2} < 0.94 \frac{L}{\sigma\sqrt{n}}. \tag{2.44}$$

Combining (2.43) and (2.44), we obtain

$$\left|\mathbb{P}(T_n < u) - \Phi(u)\right| < 3.5 \frac{L}{\sigma\sqrt{n}}. \tag{2.45}$$

The bounds (2.37) and (2.45) imply the claim of the lemma. □

3. Proof of Theorem 1

Let a random variable (r.v.) $\overline{X}$ be defined as

$$\overline{X} = \begin{cases} X, & |X| \le \sigma\sqrt{n}, \\ \sigma\sqrt{n}, & |X| > \sigma\sqrt{n}. \end{cases}$$

Let α_0 satisfy the condition

$$\alpha_0 \mathbb{P}(|\overline{X}| \le L) + \mathbb{P}(|\overline{X}| > L) = \frac{1}{2}. \tag{3.1}$$

Here and in the sequel $L = 2\frac{\beta_3}{\sigma^2}$. Without loss of generality we have

$$L < \sigma\sqrt{n}.$$

Therefore, we may replace $\overline{X}$ with X in (3.1). Notice that

$$\mathbb{P}(|\overline{X}| > L) = \mathbb{P}(|X| > L) < \frac{\beta_3}{L^3} < \frac{1}{8}. \tag{3.2}$$

Consequently,

$$\frac{1}{2} \ge \alpha_0 = \frac{\frac{1}{2} - \mathbb{P}(|\overline{X}| > L)}{\mathbb{P}(|\overline{X}| \le L)} = 1 - \frac{1}{2\mathbb{P}(|\overline{X}| \le L)} > \frac{3}{7}. \tag{3.3}$$

Put

$$F_1(A) = 2(1 - \alpha_0)\mathbb{P}(\overline{X} \in A;\, |\overline{X}| \le L),$$

$$F_2(A) = 2\Big(\alpha_0 \mathbb{P}(\overline{X} \in A;\, |\overline{X}| \le L) + \mathbb{P}(\overline{X} \in A;\, |X| > L)\Big).$$

Let $X(1), X_1(1), \ldots$ be i.i.d. r.v.'s with distribution $F_1(\cdot)$. Accordingly, $X(2), X_1(2), X_2(2), \ldots$ are defined as i.i.d. r.v.'s with distribution $F_2(\cdot)$.

Assume the sequences $\{X_j(1)\}_1^\infty$ and $\{X_j(2)\}_1^\infty$ independent. Let $\overline{X}_j$, $j = \overline{1, n}$, have the same distribution as $\overline{X}$ and be independent.

Put

$$S_k(1) = \sum_1^k X_j(1), \qquad S_{n-k}(2) = \sum_{k+1}^n X_j(2),$$

$$U_k(1) = \frac{1}{n}\sum_1^k X_j^2(1), \quad U_{n-k}(2) = \frac{1}{n}\sum_{k+1}^n X_j^2(2), \quad \bar{S}_n = \sum_1^n \overline{X}_j.$$

Consider the statistic

$$\overline{Z}_n = \frac{\bar{S}_n}{\sqrt{\frac{1}{n}\sum_1^n \overline{X}_j^2}}.$$

Obviously,

$$\left|\mathbb{P}(Z_n) - \mathbb{P}(\overline{Z}_n)\right| \le n\mathbb{P}(X \ne \overline{X}) \le \frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{3.4}$$

It is easy to see that

$$\mathbb{P}(\overline{Z}_n < x) = \frac{1}{2^n}\sum_{k=1}^n \binom{n}{k} \mathbb{P}\left(\frac{S_k(1) + S_{n-k}(2)}{\sqrt{U_k(1) + U_{n-k}(2)}} < x\right). \tag{3.5}$$

In what follows, we use the notation

$$\mathbb{E}_k f(k) = \frac{1}{2^n}\sum_1^n \binom{n}{k} f(k).$$

In these terms, we may rewrite (3.5) as

$$\mathbb{P}(\overline{Z}_n < r) = \mathbb{E}_k G_{n,k}(r) + 2^{-n} G_{n,0}(r), \tag{3.6}$$

where

$$G_{n,k}(r) = \mathbb{P}\left(\frac{S_k(1) + S_{n-k}(2)}{\sqrt{U_k(1) + U_{n-k}(2)}} < r\right).$$

Denote

$$a(j) = \mathbb{E}X(j), \quad b^2(j) = \mathbb{E}X^2(j), \quad \bar{b}^2 = \frac{b^2(1) + b^2(2)}{2}, \quad \sigma^2(j) = \operatorname{Var} X(j),$$

$$\beta_3(j) = \mathbb{E}|X(j)|^3, \quad \bar{\beta}_3(j) = \mathbb{E}|X(j) - \mathbb{E}X(j)|^3,$$

$$b_n^2(k) = k\sigma^2(1) + (n-k)\sigma^2(2),$$

$$\bar{\sigma}^2 = \frac{\sigma^2(1) + \sigma^2(2)}{2}, \quad A_n(k) = ka(1) + (n-k)a(2),$$

$$D_{n,k} = \sqrt{\frac{k}{n} b^2(1) + \frac{n-k}{n} b^2(2)}, \quad D'_{n,k} = \sqrt{\frac{k}{n} b^2(1) + U_{n-k}(2)},$$

$$\xi(j) = X^2(j) - b^2(j), \quad \xi_i(j) = X_i^2(j) - b^2(j), \quad \eta_{n,k} = \frac{1}{n}\sum_1^k \xi_i(2),$$

$$\overline{X}(2) = X(2) - a(2), \quad \overline{X}_j(2) = X_j(2) - a(2), \quad \overline{S}_k(2) = S_k(2) - ka(2).$$

Lemma 3.1. *The following inequality holds:*

$$|a(1)| \le \frac{2}{7}\sigma. \tag{3.7}$$

Proof. It is easy to see that

$$a(1) = 2(1-\alpha_0)\mathbb{E}\{X; |X| \le L\}. \tag{3.8}$$

Obviously,

$$\mathbb{E}\{X; |X| \le L\} = -\mathbb{E}\{X; |X| > L\}. \tag{3.9}$$

Furthermore,

$$|\mathbb{E}\{X; |X| > L\}| \le \frac{\beta_3}{L^2} < \frac{\sigma}{4}. \tag{3.10}$$

Infer from (3.8)–(3.10) that

$$|a(1)| \le (1-\alpha_0)\frac{\sigma}{2}. \tag{3.11}$$

To complete the proof, it remains to use the bound (3.3). □

Lemma 3.2. *The inequality*

$$|a(2)| < \frac{2}{7}\sigma + \frac{2\beta_3}{\sigma^2 n} \tag{3.12}$$

is valid.

Proof. It is clear that

$$\frac{a(2)}{2} = \mathbb{E}\{\overline{X}; |\overline{X}| > L\} + \alpha_0 \mathbb{E}\{\overline{X}; |\overline{X}| \le L\}.$$

On the other hand, we have

$$|\mathbb{E}\{\overline{X}; \overline{X} > L\} - \mathbb{E}\{X; |X| > L\}| < \frac{\beta_3}{\sigma^2 n}.$$

Thus,

$$\frac{|a(2)|}{2} < (1-\alpha_0)|\mathbb{E}\{X; |X| < L\}| + \frac{\beta_3}{\sigma^2 n}.$$

Hence, combining (3.9), (3.10), and (3.3), we obtain the desired result. □

Lemma 3.3. *The following inequalities hold:*

$$\frac{\sigma^2}{2} < b^2(1) < \frac{8}{7}\sigma^2. \tag{3.13}$$

Proof. It is easy to verify that

$$\mathbb{E}\{X^2; |X| \le L\} = \sigma^2 - \mathbb{E}\{X^2; |X| > L\} \ge \frac{\sigma^2}{2}.$$

Hence, by (3.3),

$$b^2(1) = 2(1-\alpha_0)\,\mathbb{E}\{X^2; |X| \le L\} \ge \frac{\sigma^2}{2}. \tag{3.14}$$

According to (3.3), we have $1 - \alpha_0 < \frac{4}{7}$. Therefore,

$$b^2(1) < 2(1-\alpha_0)\sigma^2 < \frac{8}{7}\sigma^2.$$

The lemma is proven. □

Lemma 3.4. *The inequalities*

$$\frac{6}{7}\left(\sigma^2 - \frac{\beta_3}{\sigma\sqrt{n}}\right) < b^2(2) < \frac{3}{2}\sigma^2 \tag{3.15}$$

are valid.

Proof. First, observe that

$$\begin{aligned}\frac{b^2(2)}{2} &= \mathbb{E}\{X^2;|X|\le L\}\alpha_0 + \mathbb{E}\{\overline{X}^2;|\overline{X}|>L\}\\ &\le \alpha_0\sigma^2 + (1-\alpha_0)\mathbb{E}\{X^2;|X|>L\} < \alpha_0\sigma^2 + (1-\alpha_0)\frac{\sigma^2}{2}\\ &= \frac{1+\alpha_0}{2}\sigma^2 < \frac{3}{4}\sigma^2.\end{aligned}$$

Hence,

$$b^2(2) < \frac{3}{2}\sigma^2.$$

We see that

$$\mathbb{E}\{\overline{X}^2;|\overline{X}|>L\} > \mathbb{E}\{X^2;|X|>L\} - \frac{\beta_3}{\sigma\sqrt{n}}.$$

Consequently,

$$\frac{b^2(2)}{2} > \alpha_0\left(\sigma^2 - \frac{\beta_3}{\sigma\sqrt{n}}\right) > \frac{3}{7}\left(\sigma^2 - \frac{\beta_3}{\sigma\sqrt{n}}\right).$$

We have used here (3.3) as well. Lemma 3.4 is proven. □

Lemma 3.5. *The following inequality holds:*

$$\sigma^2(1) > \frac{7}{16}\sigma^2. \tag{3.16}$$

Proof. Using (3.11) and (3.14), we obtain

$$\begin{aligned}\sigma^2(1) &= b^2(1) - a^2(1) > 2(1-\alpha_0)\frac{\sigma^2}{2} - (1-\alpha_0)^2\frac{\sigma^2}{4}\\ &= (1-\alpha_0)(3+\alpha_0)\frac{\sigma^2}{4} > \frac{7}{16}\sigma^2.\end{aligned}$$ □

Lemma 3.6. *The inequality*

$$\mathbb{E}|X(2)| \le \frac{5}{4}\sigma \tag{3.17}$$

is valid.

Proof. We easily deduce that

$$\begin{aligned}\frac{1}{2}\mathbb{E}|X(2)| &= \mathbb{E}(|X|;|X|\le L)\alpha_0 + \mathbb{E}(|\overline{X}|;|\overline{X}|>L)\\ &\le \mathbb{E}(|X|;|X|\le L)\alpha_0 + \mathbb{E}(|X|;|X|>L)\\ &= \alpha_0\mathbb{E}|X| + (1-\alpha_0)\mathbb{E}(|X|;|X|>L)\\ &\le \alpha_0\sigma + (1-\alpha_0)\frac{\sigma}{4}\\ &= \frac{3\alpha_0+1}{4}\sigma\\ &\le \frac{5}{8}\sigma.\end{aligned}$$ □

Lemma 3.7. *The inequality*

$$\frac{\bar\beta_3(1)}{\sigma^3(1)} < \frac{8}{\sqrt{7}}\left(\frac{\beta_3}{\sigma^3} + \frac{1}{7}\right) \tag{3.18}$$

holds.

Proof. It is easy to see that

$$\bar\beta_3(1) < |L + a(1)|\sigma^2(1) < 2\left(\frac{\beta_3}{\sigma^2} + \frac{\sigma}{7}\right)\sigma^2(1).$$

Hence, applying (3.16), we obtain

$$\frac{\bar\beta_3(1)}{\sigma^3(1)} < \frac{2\left(\frac{\beta_3}{\sigma^2} + \frac{\sigma}{7}\right)}{\sigma(1)} < \frac{8}{\sqrt{7}}\left(\frac{\beta_3}{\sigma^3} + \frac{1}{7}\right).$$ □

Lemma 3.8. *The following inequality is valid:*

$$\beta_3(2) < 2\beta_3. \tag{3.19}$$

Proof. Obviously,

$$\begin{aligned}\frac{\beta_3(2)}{2} &\le \alpha_0\mathbb{E}\{|X|^3;|X|\le L\} + \mathbb{E}\{|X|^3;|X|>L\}\\ &= \alpha_0\beta_3 + (1-\alpha_0)\mathbb{E}\{|X|^3;|X|>L\} < \beta_3.\end{aligned}$$ □

Lemma 3.9. *If*

$$\frac{\beta_3}{\sigma^3\sqrt{n}} \le \frac{1}{32} \tag{3.20}$$

then

$$\frac{\bar\beta_3(2)}{\sigma^3(2)} < 3.3\frac{\beta_3}{\sigma^3} + 1.6. \tag{3.21}$$

Proof. First of all, we observe that

$$\bar\beta_3(2) < \beta_3(2) + 3b^2(2)|a(2)| + 3\mathbb{E}|X(2)|a^2(2) + |a^3(2)|. \tag{3.22}$$

Clearly,

$$\frac{b^2(2)}{\sigma^2(2)} = 1 + \frac{a^2(2)}{\sigma^2(2)}.$$

By Lemma 3.2, under the condition (3.20), we have

$$|a(2)| < 0.286\sigma. \tag{3.23}$$

It follows from (3.23), (3.15), and (3.20) that

$$\sigma^2(2) > \frac{93}{112}\sigma^2 - 0.082\sigma^2 > 0.748\sigma^2. \tag{3.24}$$

Combining (3.23) and (3.24), we obtain

$$\frac{a^2(2)}{\sigma^2(2)} < 0.11. \tag{3.25}$$

Hence,

$$\frac{b^2(2)a(2)}{\sigma^3(2)} = \frac{a(2)}{\sigma(2)} + \frac{a^3(2)}{\sigma^3(2)} < 0.37. \tag{3.26}$$

It ensues from (3.17), (3.24), and (3.25) that

$$\frac{1}{\sigma^3(2)}\mathbb{E}|X(2)|a^2(2) < 0.147. \tag{3.27}$$

Finally, by (3.25),

$$\frac{|a(2)|^3}{\sigma^3(2)} = 0.037. \tag{3.28}$$

The claim of the lemma is immediate from (3.22) and (3.26)–(3.28). □

Lemma 3.10. *The following relation holds:*

$$\mathrm{Var}_k A_n(k) = \frac{n}{4}\big(a(1) - a(2)\big)^2. \tag{3.29}$$

Proof. It is easy to see that

$$\mathrm{Var}_k A_n(k) = \mathrm{Var} \sum_1^n \zeta_j,$$

where ζ_j are i.i.d. r.v.'s taking two values $a(1)$ and $a(2)$ with probability 1/2.

On the other hand,

$$\mathrm{Var}\,\zeta_1 = \frac{\big(a(1) - a(2)\big)^2}{4}.$$

This completes the proof. □

Lemma 3.11. *Under the condition* (3.20),

$$\mathbb{E}_k A_n^2(k) < 0.09n\sigma^2. \tag{3.30}$$

Proof. Obviously,

$$\mathbb{E}_k A_n^2(k) = \mathrm{Var}_k A_n(k) + \mathbb{E}_k^2 A_n(k). \tag{3.31}$$

Furthermore,

$$\mathbb{E}_k A_n(k) = \frac{n}{2}\big(a(1) + a(2)\big).$$

By Lemmas 3.1 and 3.2,

$$\frac{a(1) + a(2)}{2} = \theta\frac{\beta_3}{\sigma^2 n}, \quad |\theta| \le 1.$$

Hence, by (3.20), we have

$$\big|\mathbb{E}_k A_n(k)\big| = \theta\frac{\beta_3}{\sigma^2} < \frac{\sigma\sqrt{n}}{32}. \tag{3.32}$$

Lemmas 3.1 and 3.2 and (3.20) imply that

$$|a(1) - a(2)| < 0.58\sigma. \tag{3.33}$$

Combining (3.29) and (3.31)–(3.33), we obtain the claim of the lemma. □

Lemma 3.12. *Under the condition* (3.20),

$$\mathbb{E}_k \frac{1}{k^2} < \frac{4.12}{(n+1)(n+2)}. \tag{3.34}$$

Proof. Since

$$\binom{n}{k} < \frac{n^k}{k!},$$

we have the bound

$$2^{-n}\binom{n}{k} < \frac{2^{k\ln_2 n - n}}{k!}.$$

If

$$k\ln_2 n < \frac{n}{2} \tag{3.35}$$

then

$$2^{-n}\binom{n}{k} < \frac{2^{-n/2}}{k!}. \tag{3.36}$$

Under the condition (3.20), we have $n \ge 2^8$. Therefore, the inequality (3.35) holds for $k \le 16$, i.e.,

$$2^{-n}\sum_1^{16}\binom{n}{k}k^{-2} < 2^{-n/2}e. \tag{3.37}$$

Furthermore,

$$\frac{1}{k^2} = \frac{1}{(k+1)(k+2)} + \frac{3k+2}{k^2(k+1)(k+2)}. \tag{3.38}$$

For $k \geq 17$, we have

$$\frac{3k+2}{k^2(k+1)(k+2)} < \frac{53 \cdot 20}{289(k+1)(k+2)(k+3)}. \tag{3.39}$$

In view of (3.27), we infer that

$$\begin{aligned}\sum_{17}^{n} \binom{n}{k} 2^{-n} \frac{1}{k^2} &= \sum_{17}^{n} \binom{n}{k} 2^{-n} \frac{1}{(k+1)(k+2)} + 3.67 \sum_{17}^{n} \binom{n}{k} 2^{-n} \frac{1}{(k+1)(k+2)(k+3)} \\ &= \Sigma_1 + \Sigma_2. \end{aligned} \tag{3.40}$$

It is easy to see that

$$\Sigma_1 < \frac{4}{(n+1)(n+2)}. \tag{3.41}$$

By (3.39),

$$\Sigma_2 < \frac{8 \cdot 3.667}{(n+1)(n+2)(n+3)} < \frac{0.1}{(n+1)(n+2)}. \tag{3.42}$$

Using (3.37) and (3.40)–(3.42), we arrive at the claim of the lemma. □

Lemma 3.13. *Under the condition* (3.20),

$$\mathbb{E}\big|\overline{X}(2)\xi(2)\big| < 2\beta_3 + \frac{9}{4}\sigma^3. \tag{3.43}$$

Proof. Lemmas 3.2, 3.4, and 3.8 imply that

$$\begin{aligned}\mathbb{E}\big|\overline{X}(2)\xi(2)\big| &< \mathbb{E}\big|\overline{X}(2)X^2(2)\big| + \mathbb{E}\big|\overline{X}(2)\big|b^2(2) \\ &< \mathbb{E}\big|X(2)\big|^3 + b^3(2) + b^2(2)\big|a(2)\big| \\ &< 2\beta_3 + \frac{9}{4}\sigma^3. \quad \square\end{aligned}$$

Lemma 3.14. *The following relation holds:*

$$\mathbb{E}\eta_{n,k}^2 \leq \frac{2k}{n^{3/2}}\sigma\beta_3. \tag{3.44}$$

Proof. Taking the inequality $\big|X(2)\big| < \sigma\sqrt{n}$ into account and employing Lemma 3.8, we infer that

$$\mathbb{E}\eta_{n,k}^2 < \frac{k}{n^2}\mathbb{E}\xi^2(2) < \frac{k}{n^2}\mathbb{E}X^4(2) < \frac{2k}{n^{3/2}}\sigma\beta_3. \quad \square$$

Lemma 3.15. *The estimate*

$$\mathbb{E}\,\overline{S}_{n-k}^2(2)\eta_{n,k}^2 < 3\frac{k(n-k)}{n^{3/2}}\sigma^3\beta_3 \tag{3.45}$$

takes place.

Proof. Since $\overline{S}_{n-k}^2(2)$ and $\eta_{n,k}^2$ are independent, we have

$$\mathbb{E}\,\overline{S}_{n-k}^2(2)\eta_{n,k}^2 = \mathbb{E}\,\overline{S}_{n-k}^2(2)\mathbb{E}\,\eta_{n,k}^2. \tag{3.46}$$

Using Lemma 3.4, we obtain

$$\mathbb{E}\,\overline{S}_{n-k}^2(2) = (n-k)\sigma^2(2) < \frac{3}{2}(n-k)\sigma^2. \tag{3.47}$$

Combining (3.44), (3.46), and (3.47) we come to the desired result. □

Lemma 3.16. *The following relation is valid:*

$$\mathbb{E}\big|\overline{S}_{n-k}(2)\big|\eta_{n,k}^2 < \sqrt{6}\frac{k(n-k)^{1/2}}{n^{3/2}}\sigma^2\beta_3. \tag{3.48}$$

Proof. In view of independence of $\overline{S}_{n-k}(2)$ and $\eta_{n,k}$, we have

$$\mathbb{E}\big|\overline{S}_{n-k}(2)\big|\eta_{n,k}^2 = \mathbb{E}\big|\overline{S}_{n-k}(2)\big|\mathbb{E}\,\eta_{n,k}^2 \leq \mathbb{E}^{1/2}\,\overline{S}_{n-k}^2(2)\mathbb{E}\,\eta_{n,k}^2. \tag{3.49}$$

From (3.44), (3.47), and (3.49) we derive the bound (3.48). □

Lemma 3.17. *Under the condition* (3.20),

$$\mathbb{E}_k k^{-2}\mathbb{E}\,\overline{S}_{n-k}^2(2)\eta_{n,k}^2 < 3.05\frac{\beta_3\sigma^3}{n^{3/2}}. \tag{3.50}$$

Proof. By Lemma 3.15,

$$\mathbb{E}_k k^{-2}\mathbb{E}\,\overline{S}_{n-k}^2(2)\eta_{n,k}^2 < 3n^{-3/2}\mathbb{E}_k\frac{n-k}{k}\sigma^3\beta_3. \tag{3.51}$$

Lemma 3.12 yields the inequalities

$$\mathbb{E}_k\frac{n-k}{k} < \mathbb{E}_k^{1/2}(n-k)^2\mathbb{E}_k^{1/2}\frac{1}{k^2} < \left(\frac{4.12}{4}\right)^{1/2} < 1.015. \tag{3.52}$$

Combining (3.51) and (3.52), we obtain (3.50). □

Lemma 3.18. *Under the condition* (3.20),

$$\mathbb{E}_k k^{-2}\mathbb{E}|\bar{S}_{n-k}(2)|\eta_{n,k}^2 < 3.5\frac{\beta_3\sigma^2}{n^2}. \tag{3.53}$$

Proof. Using Lemma 3.16, we have

$$\mathbb{E}_k k^{-2}\mathbb{E}|\bar{S}_{n-k}(2)|\eta_{n,k}^2 < \sqrt{6}\, n^{-3/2}\mathbb{E}_k\frac{(n-k)^{1/2}}{k}\sigma^2\beta_3.$$

On the other hand, Lemma 3.12 implies that

$$\mathbb{E}_k\frac{(n-k)^{1/2}}{k} < \mathbb{E}_k^{1/2}(n-k)\mathbb{E}_k^{1/2}\frac{1}{k^2} < \left(\frac{4.12}{2}\right)^{1/2} n^{-1/2} < 1.43n^{-1/2}. \tag{3.54}$$

From these inequalities we derive the bound (3.53). □

Lemma 3.19. *Under the condition* (3.20),

$$\mathbb{E}_k\frac{1}{k^{3/2}}\mathbb{E}|\bar{S}_{n-k}(2)|\eta_{n,k}^2 \le 2.5\frac{\sigma^2\beta_3}{n^{3/2}}. \tag{3.55}$$

Proof. Applying Lemma 3.16 yields the relation

$$\mathbb{E}_k k^{-3/2}\mathbb{E}|\bar{S}_{n-k}(2)|\eta_{n,k}^2 \le \sqrt{6}\, n^{-3/2}\mathbb{E}_k\left(\frac{n-k}{k}\right)^{1/2}\sigma^2\beta_3. \tag{3.56}$$

By (3.52), we have

$$\mathbb{E}_k\left(\frac{n-k}{k}\right)^{1/2} \le \mathbb{E}_k^{1/2}\frac{(n-k)}{k} < \left(\frac{4.12}{4}\right)^{1/4} < 1.008. \tag{3.57}$$

The inequalities (3.56) and (3.57) imply the bound (3.55). □

Lemma 3.20. *Under the condition* (3.20),

$$\mathbb{E}\mathbb{E}_k\frac{A_n(k)}{k^{3/2}}\eta_{n,k}^2 < 0.86\sigma^2\frac{\beta_3}{n^{3/2}}. \tag{3.58}$$

Proof. According to Lemma 3.14,

$$\mathbb{E}\mathbb{E}_k\frac{A_n(k)}{k^{3/2}}\eta_{n,k}^2 < 2\frac{\sigma\beta_3}{n^{3/2}}\mathbb{E}_k\frac{A_n(k)}{k^{1/2}}.$$

From Lemmas 3.11 and 3.12 we deduce

$$\mathbb{E}_k\frac{A_n(k)}{k^{1/2}} < \mathbb{E}_k^{1/2}\frac{1}{k}\mathbb{E}_k^{1/2}A_{n,k}^2 < (4.12)^{1/4}0.3\sigma < 0.43\sigma. \tag{3.59}$$

It remains to combine these inequalities. □

Lemma 3.21. *Under the condition* (3.20),

$$\mathbb{E}_k\frac{1}{k^4} < \frac{17}{\Pi_{j=1}^4(n+j)}. \tag{3.60}$$

Proof. For $k \ge 17$, we have

$$\frac{1}{k^4} - \frac{1}{\Pi_{j=1}^4(k+j)} = \Pi_{j=1}^4(k+j)\left(\frac{\Pi_{j=1}^4(k+j)}{k^4} - 1\right) < \frac{15.84}{\Pi_{j=1}^5(k+j)}.$$

Hence,

$$\begin{aligned}\frac{1}{2^n}\sum_{17}^{n}\frac{\binom{n}{k}}{k^4} &< \frac{1}{2^n}\left(\sum_{17}^{n}\frac{\binom{n}{k}}{\Pi_{j=1}^4(k+j)} + 15.84\sum_{17}^{n}\frac{\binom{n}{k}}{\Pi_{j=1}^5(k+j)}\right)\\ &< 16\left(\frac{1}{\Pi_{j=1}^4(n+j)} + \frac{2}{\Pi_{j=1}^5(n+j)}\right)\\ &< \frac{16.9}{\Pi_{j=1}^4(n+j)}.\end{aligned}$$

To complete the proof, it remains to combine this inequality with (3.37). □

Lemma 3.22. *Under the condition* (3.20),

$$\mathbb{E}_k A_n^4(k) < 0.011n^2\sigma^4. \tag{3.61}$$

Proof. Denoting $\bar{A}_n(k) = A_n(k) - \mathbb{E}_k A_n(k)$, we obtain

$$\begin{aligned}\mathbb{E}_k A_n^4(k) &= \mathbb{E}_k\bar{A}_n^4(k) + 4\mathbb{E}_k\bar{A}_n^3(k)\mathbb{E}_k A_n(k)\\ &\quad + 6\mathbb{E}_k\bar{A}_n^2(k)\mathbb{E}_k^2 A_n(k) + \mathbb{E}_k^4 A_n(k).\end{aligned} \tag{3.62}$$

It is easy to check that

$$\mathbb{E}_k\bar{A}_n^3(k) = n\frac{(a(1)-a(2))^3}{8},$$

$$\mathbb{E}_k\bar{A}_n^4(k) = n^2\frac{(a(1)-a(2))^4}{16}.$$

Hence, by (3.33),

$$|\mathbb{E}_k\bar{A}_n^3(k)| < 0.025n\sigma^2, \quad |\mathbb{E}_k\bar{A}_n^4(k)| < 0.007n^2\sigma^4 \tag{3.63}$$

Combining (3.30), (3.32), (3.33), and (3.61)–(3.63), we come to the desired result. □

Lemma 3.23. *Under the condition* (3.20),

$$\mathbb{E}_k \frac{A_n(k)}{k^2}\eta_{n,k}^2 < 1.22\frac{\sigma^2\beta_3}{n^2}. \tag{3.64}$$

Proof. In view of Lemma 3.14,

$$\mathbb{E}_k \frac{A_n(k)}{k^2}\eta_{n,k}^2 < 2\frac{\sigma\beta_3}{n^{3/2}}\mathbb{E}_k \frac{A_n(k)}{k}.$$

Applying Lemmas 3.11 and 3.12 yields the relations

$$\mathbb{E}_k \frac{A_n(k)}{k} < \mathbb{E}_k^{1/2}\frac{1}{k^2}\mathbb{E}_k^{1/2}A_n^2(k) < (4.12)^{1/2}(0.09)^{1/2}\frac{\sigma}{\sqrt{n}} < 0.61\frac{\sigma}{\sqrt{n}}. \tag{3.65}$$

These inequalities imply the desired result. □

Lemma 3.24. *Under the condition* (3.20),

$$\mathbb{E}_k \frac{A_n^2(k)}{k^2}\eta_{n,k}^2 < 0.44\frac{\sigma^3\beta_3}{\sqrt{n}}. \tag{3.66}$$

Proof. By Lemma 3.14, we have

$$\mathbb{E}_k \frac{A_n^2(k)}{k^2}\eta_{n,k}^2 < 2\frac{\sigma\beta_3}{n^{3/2}}\mathbb{E}_k \frac{A_n^2(k)}{k}.$$

Applying Lemmas 3.12 and 3.22, we conclude that

$$\mathbb{E}_k \frac{A_n^2(k)}{k} < \mathbb{E}_k^{1/2}\frac{1}{k^2}\mathbb{E}_k^{1/2}A_n^4(k) < (4.12)^{1/2}(0.011)^{1/2}\sigma^2 < 0.22\sigma^2.$$

It remains to combine these inequalities. □

Lemma 3.25. *Under the condition* (3.20),

$$\mathbb{E}_k \frac{1}{k^2}\mathbb{E}\eta_{n,k}^2\left(\bar{S}_{n-k}(2) + A_n(k)\right)^2 < 5.6\frac{\sigma^3\beta_3}{n^{3/2}}. \tag{3.67}$$

Proof. According to Lemma 3.16,

$$\mathbb{E}_k\mathbb{E}\eta_{n,k}^2|\bar{S}_{n-k}(2)|\frac{A_n(k)}{k^2} < \sqrt{6}\,\sigma^2\beta_3 n^{-3/2}\mathbb{E}_k\frac{(n-k)^{1/2}}{k}A_n(k).$$

Applying Lemmas 3.11 and 3.21, we obtain

$$\begin{aligned}\mathbb{E}_k\frac{(n-k)^{1/2}}{k}A_n(k) &< \mathbb{E}_k^{1/2}(n-k)\mathbb{E}^{1/2}A_n^2(k)\\ &< \mathbb{E}_k^{1/4}(n-k)^2\mathbb{E}_k^{1/4}\frac{1}{k^4}\mathbb{E}^{1/2}A_n^2(k)\\ &< \left(\frac{17}{4}\right)^{1/4}0.3\sigma\\ &< 0.43\sigma.\end{aligned}$$

It follows from these inequalities that

$$\mathbb{E}_k\mathbb{E}\eta_{n,k}^2|\bar{S}_{n-k}(2)|\frac{A_n(k)}{k^2} < 1.06\frac{\beta_3\sigma^3}{n^{3/2}}.$$

Combining this bound and Lemmas 3.17 and 3.24 leads to (3.67). □

Lemma 3.26. *Under the condition* (3.20),

$$D_{n,k} + D'_{n,k} > \sqrt{\frac{45}{56}}\,\sigma. \tag{3.68}$$

Proof. First, we find

$$D_{n,k} + D'_{n,k} > \sqrt{\frac{k}{n}b^2(1) + \frac{n-k}{n}b^2(2)} + \sqrt{\frac{k}{n}b^2(2)}.$$

By Lemma 3.4, under the condition (3.20),

$$b^2(2) > \frac{45}{56}\sigma^2.$$

For every a^2 and b^2 such that $2a^2 \geq b^2$ and $0 \leq x \leq 1$, we have

$$\sqrt{a^2x} + \sqrt{a^2x + b^2(1-x)} \geq \sqrt{(2a^2 - b^2)x + b^2} > |b|.$$

Hence, putting $x = \frac{k}{n}$, $b^2 = \frac{45}{46}$, and $a^2 = \frac{1}{2}$ and taking (3.13) into account, we obtain (3.68). □

Return to (3.6). Clearly,

$$G_{n,k}(r) = \mathbb{E}\mathbb{P}\left(\frac{S_k(1) + S_{n-k}(2)}{\sqrt{U_k(1) + U_{n-k}(2)}} < r \;\middle|\; S_{n-k}(2), U_{n-k}(2)\right).$$

Suppose that r satisfies the condition

$$r < 0.2n^{3/4}\left(\frac{b(1)}{L}\right)^{1/2}.$$

Without loss of generality we may assume that

$$\frac{\beta_3}{\sigma^3\sqrt{n}} < \frac{1}{36}, \quad \frac{1}{\sqrt{n}} < \frac{1}{45}. \tag{3.69}$$

Therefore, for $\gamma = 0.3$,

$$2\gamma\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2} < 0.17. \tag{3.70}$$

Here we have used the bound (3.13). Observe that $\gamma\sqrt{\frac{k}{n}} \ge 0.186$ for $\frac{k}{n} \ge \frac{4}{9}$. From Lemma 2.7 with $\nu = 0.043$ we deduce, for $\frac{k}{n} \ge \frac{4}{9}$,

$$\Bigg| \mathbb{P}\left(\frac{S_k(1)+S_{n-k}(2)}{\sqrt{U_k(1)+U_{n-k}(2)}} < r \,\Big|\, S_{n-k}(2)=x,\, U_{n-k}(2)=y \right) - \Phi\left(\frac{r\sqrt{\frac{k}{n}b^2(1)+y^2} - x - ka(1)}{\sigma(1)\sqrt{k}} \right) \Bigg|$$

$$< c_0 \frac{1.043L + |a(1)|}{\sigma(1)\sqrt{k\Omega_n}} + \frac{16}{9}\frac{L\beta_3(1)}{b^4(1)k} + \frac{2}{e}\sqrt{\frac{2}{\pi}}\frac{\beta_3(1)}{\sqrt{k}\,\sigma(1)b^2(1)} \frac{1+\frac{\gamma}{8}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2}}{\sqrt{\Omega_n}}$$
$$+ \frac{1}{\sqrt{2\pi e}} \frac{\delta L}{\left(1-\frac{\delta L}{b(1)\sqrt{n}}\right)b(1)\sqrt{n}} + \frac{4}{9}\frac{\sqrt{n}\,\beta_3(1)}{\delta b^3(1)k}$$
$$+ \frac{|z|}{\sigma(1)}\left(\frac{\delta L}{\sqrt{2\pi}\,b(1)\sqrt{nk}} + \frac{1}{\sqrt{\pi e}}\frac{\beta_3(1)}{\sigma(1)b^2(1)k} \frac{1+\frac{\gamma}{8}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2}}{\sqrt{\Omega_n\Omega_n'}} \right), \qquad (3.71)$$

where

$$z = x + ka(1), \quad L = \frac{2\beta_3}{\sigma^2},$$

$$\Omega_n = 1 - 2\gamma\frac{b^2(1)}{\sigma^2(1)}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2}, \quad \Omega_n' = 1 - \gamma\frac{b^2(1)}{\sigma^2(1)}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2}.$$

It follows from (3.7), (3.16), and (3.70) that

$$2\gamma\frac{b^2(1)}{\sigma^2(1)}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2} < 0.2.$$

Hence,

$$\sqrt{\Omega_n} > 0.89, \quad \sqrt{\Omega_n'} > 0.94. \qquad (3.72)$$

Furthermore, by (3.16) and (3.72),

$$c_0\frac{L}{\sigma(1)\sqrt{\Omega_n}} < 2.61\frac{\beta_3}{\sigma^3}$$

and, by (3.7) and (3.16),

$$c_0\frac{|a(1)|}{\sigma(1)\sqrt{\Omega_n}} < 0.38.$$

Thus,

$$c_0\frac{1.043L + |a(1)|}{\sigma(1)\sqrt{\Omega_n}} < 2.72\frac{\beta_3}{\sigma^3} + 0.38. \qquad (3.73)$$

Furthermore, by (3.13) and (3.70), we infer that

$$1 + \frac{\gamma}{8}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2} < 1.01.$$

On the other hand, by (3.16), we have

$$\frac{\beta_3(1)}{\sigma(1)b^2(1)} < \frac{L}{\sigma(1)} = \frac{2\beta_3}{\sigma(1)\sigma^2} < 3.03\frac{\beta_3}{\sigma^3}.$$

Consequently,

$$\frac{\beta_3(1)}{\sigma(1)b^2(1)} \frac{1+\frac{\gamma}{8}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2}}{\sqrt{\Omega_n}} < 3.42\frac{\beta_3}{\sigma^3}. \qquad (3.74)$$

It follows from (3.73), (3.74), and (3.34) that

$$\mathbb{E}_k\left(c_0\frac{1.043L+|a(1)|}{\sigma(1)\sqrt{k\Omega_n}} + \frac{2}{e}\sqrt{\frac{2}{\pi}}\frac{\beta_3(1)}{\sqrt{k}\,\sigma(1)b^2(1)} \frac{1+\frac{\gamma}{8}\left(\frac{L}{b(1)\sqrt{n}}\right)^{1/2}}{\sqrt{\Omega_n}} \right)$$
$$< 6.59\frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{0.55}{\sqrt{n}}. \qquad (3.75)$$

Putting $\delta = 1$ and using (3.13) and (3.34), we obtain

$$\sqrt{n}\,\frac{4}{9}\frac{\beta_3(1)}{\delta b^3(1)}\mathbb{E}_k\frac{1}{k} < 2.56\frac{\beta_3}{\sigma^3\sqrt{n}}. \qquad (3.76)$$

Taking (3.69), (3.34), and (3.13) into account, we infer that, for $\delta = 1$,

$$\frac{1}{\sqrt{2e\pi}}\frac{\delta L}{\left(1-\frac{\delta L}{b(1)\sqrt{n}}\right)b(1)\sqrt{n}} < \sqrt{\frac{1}{e\pi}}\frac{2\beta_3}{\sigma^2\left(\sigma - \frac{\beta_3}{\sigma^2\sqrt{n}}\right)\sqrt{n}} < 0.7\frac{\beta_3}{\sigma^3\sqrt{n}}. \qquad (3.77)$$

According to (3.13) and (3.69), we have

$$\mathbb{E}_k\frac{16}{9}\frac{L\beta_3(1)}{b^4(1)k} < \frac{16}{9}\frac{L^2}{b^2(1)}\mathbb{E}_k\frac{1}{k} < 28.87\frac{\beta_3^2}{\sigma^6 n} < 0.81\frac{\beta_3}{\sigma^3\sqrt{n}}. \qquad (3.78)$$

It is easy to see that

$$\begin{aligned}\mathbb{E}\big|S_{n-k}(2)+ka(1)\big| &< \mathbb{E}^{1/2}\big|S_{n-k}(2)+ka(1)\big|^2\\ &= \Big((n-k)\sigma^2(2)+\big(ka(1)+(n-k)a(2)\big)^2\Big)^{1/2}\\ &\le \sigma(2)\sqrt{n-k}+\big|A_n(k)\big|.\end{aligned}$$

Hence, by Lemmas 3.4, 3.11, and 3.12,

$$\begin{aligned}&\mathbb{E}_k\frac{1}{\sqrt{k}}\mathbb{E}\big|S_{n-k}(2)+ka(1)\big|\\ &\quad< \mathbb{E}_k\left(\sqrt{\frac{n-k}{k}}\,\sigma(2)+\frac{|A_n(k)|}{\sqrt{k}}\right)\\ &\quad< \sqrt{\frac{3}{2}}\,\sigma\mathbb{E}_k^{1/2}(n-k)\mathbb{E}_k^{1/2}\frac{1}{k}+\mathbb{E}^{1/2}A_n^2(k)\mathbb{E}^{1/2}\frac{1}{k}\\ &\quad< 1.24\sigma+0.43\sigma=1.67\sigma.\end{aligned} \tag{3.79}$$

Similarly,

$$\begin{aligned}&\mathbb{E}_k\frac{1}{k}\mathbb{E}\big|S_{n-k}(2)+ka(1)\big|\\ &\quad< \mathbb{E}_k\left(\frac{\sqrt{n-k}}{k}\sigma(2)+\frac{|A_n(k)|}{k}\right)\\ &\quad< \sqrt{\frac{3}{2}}\,\sigma\mathbb{E}_k^{1/2}(n-k)\mathbb{E}_k^{1/2}\frac{1}{k^2}+\mathbb{E}^{1/2}A_n^2(k)\mathbb{E}^{1/2}\frac{1}{k^2}\\ &\quad< \frac{1}{\sqrt{n}}(1.76\sigma+0.61\sigma)=2.37\frac{\sigma}{\sqrt{n}}.\end{aligned} \tag{3.80}$$

It follows from (3.74), (3.72), and (3.16) that

$$\frac{1}{\sqrt{\pi e}}\,\frac{\beta_3(1)}{\sigma^2(1)b^2(1)}\,\frac{1+\frac{\gamma}{8}\Big(\frac{L}{b(1)\sqrt{n}}\Big)^{1/2}}{\sqrt{\Omega_n\Omega_n'}} < \frac{4\cdot 3.42}{\sqrt{6.3\pi e}}\,\frac{\beta_3}{\sigma^4} < 1.87\frac{\beta_3}{\sigma^4}. \tag{3.81}$$

Furthermore, for $\delta=1$,

$$\frac{\delta L}{\sqrt{2\pi}\,b(1)\sigma(1)} < 1.71\frac{\beta_3}{\sigma^4}. \tag{3.82}$$

Collecting (3.79)–(3.82), we arrive at the relation

$$\begin{aligned}&\frac{1}{\sqrt{\pi e}}\,\frac{\beta_3(1)}{\sigma^2(1)b^2(1)}\,\frac{1+\frac{\gamma}{8}\Big(\frac{L}{b(1)\sqrt{n}}\Big)^{1/2}}{\sqrt{\Omega_n\Omega_n'}}\,\mathbb{E}_k\frac{1}{k}\mathbb{E}\big|S_{n-k}(2)+ka(1)\big|\\ &\quad+\frac{\delta L}{\sqrt{2\pi}\,b(1)\sigma(1)}\,\mathbb{E}_k\frac{1}{\sqrt{kn}}\mathbb{E}\big|S_{n-k}(2)+ka(1)\big| < 7.29\frac{\beta_3}{\sigma^3\sqrt{n}}.\end{aligned} \tag{3.83}$$

Combining (3.71), (3.75)–(3.78), (3.83), and the bound

$$\frac{1}{2^n}\sum_{k\le\frac{4}{9}n}\binom{n}{k} < \frac{81}{n} < \frac{1.8}{\sqrt{n}},$$

we conclude that

$$\begin{aligned}&\left|\mathbb{P}(\overline{Z}_n<r)-\mathbb{E}_k\mathbb{E}\,\Phi\left(\frac{r\sqrt{\frac{k}{n}b^2(1)+U_{n-k}(2)}-S_{n-k}(2)-ka(1)}{\sigma(1)\sqrt{k}}\right)\right|\\ &\quad< 18\frac{\beta_3}{\sigma^3\sqrt{n}}+\frac{2.35}{\sqrt{n}}\end{aligned} \tag{3.84}$$

for $r<\frac{1}{5}\Big(\frac{b(1)}{L}\Big)^{1/2}$. Now, estimate $\mathbb{E}_k\mathbb{E}\big|\triangle_{n,1}(k)\big|$, where

$$\triangle_{n,1}(k) := \Phi\left(\frac{rD_{n,k}'-\overline{S}_{n-k}(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right)-\Phi\left(\frac{rD_{n,k}-\overline{S}_{n-k}(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right).$$

It is easy to see that

$$\begin{aligned}\triangle_{n,1}(k) &= \frac{r(D_{n,k}'-D_{n,k})}{\sqrt{2\pi}\,\sigma(1)\sqrt{k}}\exp\left\{-\left(\frac{D_{n,k}r-\overline{S}_{n-k}(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\ &\quad+\frac{r^2(D_{n,k}'-D_{n,k})^2}{2\sigma^2(1)k}\Phi''\left(\frac{D_{n,k}(\theta)r-\overline{S}_{n-k}(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right)\\ &\equiv \Omega_1(k)+\Omega_2(k),\end{aligned} \tag{3.85}$$

where $D_{n,k}(\theta)=D_{n,k}+\theta(D_{n,k}'-D_{n,k})$, $0\le\theta\le 1$.

First, estimate $\mathbb{E}_k\mathbb{E}\Omega_1(k)$. For every $1\le j\le n-k$,

$$\begin{aligned}&\exp\left\{-\frac{1}{2}\left(\frac{D_{n,k}r-\overline{S}_{n-k}(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\ &\quad=\exp\left\{-\frac{1}{2}\left(\frac{D_{n,k}r-S_{n-k}^j(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\ &\qquad+\sqrt{2\pi}\,\frac{\overline{X}_j(2)}{\sigma(1)\sqrt{k}}\Phi''\left(\frac{D_{n,k}r-S_{n-k}^j(2)-\theta\overline{X}_j(2)-A_n(k)}{\sigma(1)\sqrt{k}}\right),\quad |\theta|\le 1,\end{aligned}$$

where

$$S^j_{n-k}(2) = \sum_{l\neq j} \overline{X}_l(2) = \overline{S}_{n-k}(2) - \overline{X}_j(2).$$

Notice that, by Lemma 2.1,

$$D'_{n,k} - D_{n,k} = \frac{\eta_{n,k}}{2D_{n,k}} - \frac{\eta^2_{n,k}}{D_{n,k}(D_{n,k}+D'_{n,k})^2}. \tag{3.86}$$

Thus,

$$\begin{aligned}\mathbb{E}\Omega_1 &= \frac{r}{2\sqrt{2\pi}\,\sigma(1)\sqrt{k}\,nD_{n,k}} \sum_1^{n-k} \mathbb{E}\xi_j(2)\exp\left\{-\left(\frac{D_{n,k}r - S^j_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\ &+ \frac{r}{2\sigma^2(1)knD_{n,k}} \sum_1^{n-k} \mathbb{E}\xi_j(2)\overline{X}_j(2)\,\Phi''\left(\frac{D_{n,k}r - S^j_{n-k}(2) - \theta\overline{X}_j(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\ &+ \frac{r}{\sqrt{2\pi}\,\sigma(1)\sqrt{k}}\mathbb{E}\frac{\eta^2_{n,k}}{D_{n,k}(D_{n,k}+D'_{n,k})^2}\exp\left\{-\left(\frac{D_{n,k}r - \overline{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\ &= \sum_1^3 H_j(k).\end{aligned} \tag{3.87}$$

It is clear that $\xi_j(2)$ and $S^j_{n-k}(2)$ are independent. Therefore,

$$H_1(k) = 0. \tag{3.88}$$

Obviously,

$$\begin{aligned}&\frac{D_{n,k}r}{\sigma(1)\sqrt{k}}\Phi''\left(\frac{D_{n,k}r - S^j_{n-k}(2) - \theta\overline{X}_j(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\ &= \frac{D_{n,k}r - S^j_{n-k}(2) - \theta\overline{X}_j(2) - A_n(k)}{\sigma(1)\sqrt{k}}\Phi''\left(\frac{D_{n,k}r - S^j_{n-k}(2) - \theta\overline{X}_j(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\ &\quad + \frac{S^j_{n-k}(2) + \theta\overline{X}_j(2) + A_n(k)}{\sigma(1)\sqrt{k}}\Phi''\left(\frac{D_{n,k}r - S^j_{n-k}(2) - \theta\overline{X}_j(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\ &= \Omega'_{1,j} + \Omega''_{1,j}.\end{aligned} \tag{3.89}$$

Clearly,

$$|\Omega'_{1,j}| < \frac{1}{\sqrt{2\pi}}\sup_x x^2 e^{-x^2/2} < \frac{1}{e}\sqrt{\frac{2}{\pi}}. \tag{3.90}$$

It follows from (3.90) and (3.43) that

$$\sum_1^{n-k}\mathbb{E}\big|\overline{X}_j(2)\xi_j(2)\Omega'_{1,j}\big| < \frac{1}{e}\sqrt{\frac{2}{\pi}}(n-k)\left(2\beta_3 + \frac{9}{4}\sigma^3\right). \tag{3.91}$$

Using (3.43), (3.12), and the bound

$$|\Phi''(x)| < \frac{1}{\sqrt{2\pi e}}, \tag{3.92}$$

we obtain the following estimates for the corresponding summands in (3.89):

$$\begin{aligned}\sum_1^{n-k}\mathbb{E}\big|\overline{X}^2_j(2)\xi_j(2)\Phi''(\cdots)\big| &< \frac{(n-k)(\sqrt{n}+0.29)\sigma}{\sqrt{2\pi ek}\,\sigma(1)}\mathbb{E}\big|X(2)\xi(2)\big|\\ &< \frac{(n-k)(\sqrt{n}+0.29)\sigma}{\sqrt{2\pi ek}\,\sigma(1)}\left(2\beta_3 + \frac{9}{4}\sigma^3\right),\\ \mathbb{E}\big|S^j_{n-k}(2)\overline{X}_j(2)\xi_j(2)\Phi''(\cdots)\big| &< \frac{\mathbb{E}\big|S^j_{n-k}(2)\big|\mathbb{E}\big|\overline{X}(2)\xi(2)\big|}{\sqrt{2\pi ek}\,\sigma(1)}\\ &< \frac{\sqrt{n-k-1}\sigma(2)}{\sqrt{2\pi ek}\,\sigma(1)}\left(2\beta_3 + \frac{9}{4}\sigma^3\right).\end{aligned}$$

Taking the last two bounds into account, we conclude that

$$\begin{aligned}&\sum_1^{n-k}\mathbb{E}\big|\overline{X}_j(2)\xi_j(2)\Omega''_{1,j}\big|\\ &\quad< \frac{n-k}{\sqrt{2\pi ek}\,\sigma(1)}\left(\sigma\sqrt{n} + \sigma(2)\sqrt{n-k-1} + A_n(k)\right)\left(2\beta_3 + \frac{9}{4}\sigma^3\right).\end{aligned} \tag{3.93}$$

It follows from (3.87), (3.89), (3.91), and (3.93) that

$$\begin{aligned}H_2(k) &< \frac{n-k}{2\sigma(1)\sqrt{2\pi ek}\,nD^2_{n,k}}\left(2\beta_3 + \frac{9}{4}\sigma^3\right)\\ &\quad\times\left(1.22 + \frac{1}{\sigma(1)\sqrt{k}}\left((\sqrt{n}+0.29)\sigma\sqrt{n-k-1} + A_n(k)\right)\right).\end{aligned} \tag{3.94}$$

Observe that, by Lemmas 3.3 and 3.4,

$$D^2_{n,k} > \frac{1}{2}\sigma^2. \tag{3.95}$$

In view of (3.30), (3.34), and (3.95), we derive from (3.94) that

$$\mathbb{E}_k\left|H_2(k)\right| < \frac{1}{\sqrt{2\pi e}}\left(0.87 + \frac{1}{\sigma(1)}(1.015\sigma + 0.718\sigma(2) + 0.305\sigma)\right)\frac{2\beta_3 + \frac{9}{4}\sigma^3}{\sigma(1)\sigma^2\sqrt{n}}.$$

Hence, using (3.15) and (3.16), we obtain the following estimate as a result of routine calculations:

$$\mathbb{E}_k\left|H_2(k)\right| < \frac{3.07\frac{\beta_3}{\sigma^3} + 3.46}{\sqrt{n}}. \tag{3.96}$$

Estimate $\mathbb{E}\,H_3(k)$. Clearly,

$$\begin{aligned}
&\frac{rD_{n,k}}{\sigma(1)\sqrt{k}}\exp\left\{-\left(\frac{D_{n,k}r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\
&= \frac{D_{n,k}r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\exp\left\{-\left(\frac{D_{n,k}r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\
&\quad + \frac{\bar{S}_{n-k}(2) + A_n(k)}{\sigma(1)\sqrt{k}}\exp\left\{-\left(\frac{D_{n,k}r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)^2\right\}\\
&= \Omega'_{2,k} + \Omega''_{2,k}
\end{aligned} \tag{3.97}$$

By (3.92), we have

$$|\Omega'_{2,k}| < \frac{1}{\sqrt{e}}.$$

Consequently, by Lemmas 3.14 and 3.26 and the bound (3.95), we infer that

$$\frac{1}{\sqrt{2\pi}\,D^2_{n,k}}\mathbb{E}\,\eta^2_{n,k}\frac{|\Omega'_{2,k}|}{(D_{n,k} + D'_{n,k})^2} < \frac{2.49}{\sqrt{2\pi e}\,\sigma^4}\mathbb{E}\,\eta^2_{n,k} < 1.21\frac{k}{n^{3/2}}\frac{\beta_3}{\sigma^3}.$$

Hence,

$$\frac{1}{\sqrt{2\pi}}\mathbb{E}_k\frac{1}{D^2_{n,k}}\mathbb{E}\,\eta^2_{n,k}\frac{|\Omega'_{2,k}|}{(D_{n,k} + D'_{n,k})^2} < 0.61\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{3.98}$$

Furthermore, by Lemma 3.26 and the bound (3.95), we find

$$\frac{1}{\sqrt{2\pi}D^2_{n,k}}\mathbb{E}\,\eta^2_{n,k}\frac{|\Omega''_{2,k}|}{(D_{n,k} + D'_{n,k})^2} < \frac{2.49}{\sqrt{2\pi}\,\sigma(1)\sigma^4\sqrt{k}}\mathbb{E}\,\eta^2_{n,k}\left(|\bar{S}_{n-k}(2)| + A_n(k)\right).$$

Using Lemma 3.16 and the bound $(n-k)k < \frac{n^2}{4}$, we obtain

$$\mathbb{E}_k\frac{1}{\sqrt{k}}\mathbb{E}\,\eta^2_{n,k}\left|\bar{S}_{n-k}(2)\right| < \frac{\sqrt{6}}{n^{3/2}}\sigma^2\beta_3\mathbb{E}_k(n-k)^{1/2}k^{1/2} < 1.23\frac{\sigma^2\beta_3}{\sqrt{n}}.$$

On the other hand, by Lemmas 3.14 and 3.11,

$$\mathbb{E}_k k^{-1/2}A_n(k)\mathbb{E}\,\eta^2_{n,k} < \frac{2\sigma\beta_3}{n^{3/2}}\mathbb{E}_k^{1/2}k\mathbb{E}^{1/2}A_n^2(k) < 0.43\sigma^2\beta_3 n^{-1/2}.$$

As a result, we obtain the bound

$$\frac{1}{\sqrt{2\pi}}\mathbb{E}_k\frac{1}{D^2_{n,k}}\mathbb{E}\,\eta^2_{n,k}\frac{|\Omega''_{2,k}|}{(D_{n,k} + D'_{n,k})^2} < 2.5\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{3.99}$$

We use Lemma 3.5 to estimate $\sigma(1)$. It follows from (3.97)–(3.99) that

$$\mathbb{E}_k\left|H_3(k)\right| < 3.11\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{3.100}$$

Combining (3.87), (3.88), (3.96), and (3.100), we conclude that

$$\mathbb{E}_k\mathbb{E}\,\Omega_1(k) < 5.92\frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{3.16}{\sqrt{n}}. \tag{3.101}$$

We now turn to estimating $\mathbb{E}_k\mathbb{E}\,\Omega_2(k)$. Obviously,

$$\begin{aligned}
&\frac{D^2_{n,k}(\theta)r^2}{\sigma^2(1)k}\Phi''\left(\frac{D_{n,k}(\theta)r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\
&= \frac{\left(D_{n,k}(\theta)r - \bar{S}_{n-k}(2) - A_n(k)\right)^2}{\sigma^2(1)k}\Phi''\left(\frac{D_{n,k}(\theta)r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\
&\quad + \frac{2\left(D_{n,k}(\theta)r - \bar{S}_{n-k}(2) - A_n(k)\right)\left(\bar{S}_{n-k}(2) + A_n(k)\right)}{\sigma^2(1)k}\\
&\quad\quad \times\Phi''\left(\frac{D_{n,k}(\theta)r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right) + \frac{\left(\bar{S}_{n-k}(2) + A_n(k)\right)^2}{\sigma^2(1)k}\\
&\quad\quad \times\Phi''\left(\frac{D_{n,k}(\theta)r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right)\\
&= \sum_1^3 Q_j(k).
\end{aligned} \tag{3.102}$$

By (2.2),

$$D'_{n,k} - D_{n,k} = \frac{\eta_{n,k}}{D'_{n,k} + D_{n,k}}. \tag{3.103}$$

It is easy to see that

$$\left|Q_1(k)\right| < \frac{1}{\sqrt{2\pi}}\sup_{x>0}x^3e^{-x^2/2} < \left(\frac{3}{e}\right)^{3/2}\frac{1}{\sqrt{2\pi}}.$$

Hence, by Lemma 3.14,

$$\mathbb{E}\eta_{n,k}^2 |Q_1(k)| < 0.93 \frac{k}{n^{3/2}} \sigma\beta_3.$$

and, by (3.51),

$$n\mathbb{E}_k k^{-1} \mathbb{E}\eta_{n,k}^2 |Q_1(k)| < 0.93 \frac{\sigma\beta_3}{\sqrt{n}}. \tag{3.104}$$

Furthermore,

$$|Q_2(k)| < \frac{2}{\sqrt{2\pi}} \sup_{x>0} x^2 e^{-x^2/2} \frac{|\bar{S}_{n-k}(2) + A_n(k)|}{\sigma(1)\sqrt{k}}.$$

Thus,

$$\mathbb{E}\eta_{n,k}^2 |Q_2(k)| < \frac{4}{e\sqrt{2\pi}} \frac{\mathbb{E}|\bar{S}_{n-k}(2)|\eta_{n,k}^2 + \mathbb{E}A_n(k)\eta_{n,k}^2}{\sigma(1)\sqrt{k}}.$$

Hence, employing (3.55), (3.58), and (3.16), we obtain

$$n\mathbb{E}_k k^{-1} \mathbb{E}\eta_{n,k}^2 |Q_2|(k) < 3 \frac{\sigma\beta_3}{\sqrt{n}}. \tag{3.105}$$

Finally,

$$|Q_3(k)| < \frac{1}{\sqrt{2\pi}} \sup_{x>0} x e^{-x^2/2} |\bar{S}_{n-k}(2) + A_n(k)|^2 \frac{1}{\sigma^2(1)k}.$$

Therefore, by (3.16),

$$\mathbb{E}\eta_{n,k}^2 |Q_3(k)| < \frac{1}{\sigma^2\sqrt{2\pi e}k} \frac{16}{7} \mathbb{E}\eta_{n,k}^2 |\bar{S}_{n-k}(2) + A_n(k)|^2.$$

Applying now Lemma 3.25 implies

$$n\mathbb{E}_k k^{-1} \mathbb{E}\eta_{n,k}^2 |Q_3(k)| < 3.11 \frac{\sigma\beta_3}{\sqrt{n}}. \tag{3.106}$$

It follows from (3.102)–(3.106), the bound $D_{n,k}^2(\theta) > \frac{k\sigma^2}{2n}$, and Lemma 3.26 that

$$\mathbb{E}_k\mathbb{E}|\Omega_2(k)| < \frac{56n}{45\sigma^4} \sum_{j=1}^{3} \mathbb{E}_k k^{-1} \mathbb{E}\eta_{n,k}^2 |Q_j(k)| < 7.04 \frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{3.107}$$

Combining (3.85), (3.101), and (3.107), we obtain

$$\mathbb{E}_k\mathbb{E}|\Delta_{n,1}(k)| < 12.54 \frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{3.16}{\sqrt{n}}. \tag{3.108}$$

Letting

$$\Delta_{n,2}(k) = \mathbb{E}\left(\Phi\left(\frac{D_{n,k}r - \bar{S}_{n-k}(2) - A_n(k)}{\sigma(1)\sqrt{k}}\right) - \Phi\left(\frac{D_{n,k}r - A_n(k)}{b_n(k)\sqrt{n}}\right)\right)$$

and applying the Berry–Esseen bound and Lemma 3.9, we derive

$$|\Delta_{n,2}(k)| < 2.53 \frac{\beta_3}{\sigma^3\sqrt{n-k}} + \frac{1.23}{\sqrt{n-k}}.$$

Hence, by Lemma 3.12,

$$\mathbb{E}_k|\Delta_{n,2}(k)| < 3.6 \frac{\beta_3}{\sigma^3\sqrt{n}} + \frac{1.76}{\sqrt{n}}. \tag{3.109}$$

Estimate

$$\Delta_{n,3}(k) := \Phi\left(\frac{D_{n,k}r - A_n(k)}{\sqrt{n}b_n(k)}\right) - \Phi\left(\frac{D_{n,k}r - A_n(k)}{\sqrt{n}\bar{\sigma}}\right).$$

It is easy to see that

$$\Delta_{n,3}(k) = \frac{1}{\sqrt{2\pi n}}\left(\frac{1}{b_n(k)} - \frac{1}{\bar{\sigma}}\right) \times (D_{n,k}r - A_n(k)) \exp\left\{-\frac{(D_{n,k}r - A_n(k))^2}{2n\bar{\sigma}_n^2(\theta)}\right\}, \tag{3.110}$$

where

$$\bar{\sigma}_n(\theta) = b_n(k)\theta + (1-\theta)\bar{\sigma}, \quad 0 < \theta < 1.$$

Notice that, by (3.16) and (3.24),

$$\bar{\sigma}^2 > \frac{1}{2}\left(0.748 + \frac{7}{16}\right)\sigma^2 > 0.592\sigma^2,$$
$$b_n^2(k) > \sigma^2(1) \wedge \sigma^2(2) > \frac{7}{16}\sigma^2. \tag{3.111}$$

Using (3.111), we obtain

$$\left|\frac{1}{b_n(k)} - \frac{1}{\bar{\sigma}}\right| = \left|\frac{\bar{\sigma}^2 - b_n^2(k)}{\bar{\sigma}b_n(k)(\bar{\sigma} + b_n(k))}\right| < 1.91\left|\left(\frac{1}{2} - \frac{k}{n}\right)(\sigma^2(1) - \sigma^2(2))\right|\sigma^{-3}. \tag{3.112}$$

Obviously,

$$\sigma^2(1) - \sigma^2(2) = b^2(1) - b^2(2) + a^2(2) - a^2(1).$$

By definition,

$$\frac{1}{2}(b^2(1) - b^2(2)) = (1 - 2\alpha_0)\mathbb{E}\{X^2; |X| \le L\} - \mathbb{E}\{X^2; |X| > L\}. \quad (3.113)$$

According to (3.3) and (3.12),

$$2(1 - 2\alpha_0)\mathbb{E}\{X^2; |X| \le L\} + a^2(2) < \left(\frac{2}{7} + 0.092\right)\sigma^2.$$

On the other hand, by (3.7),

$$2\mathbb{E}\{X^2; |X| > L\} + a^2(1) < \frac{53}{49}\sigma^2.$$

Finally,

$$\big|\sigma^2(1) - \sigma^2(2)\big| < 1.1\sigma^2. \quad (3.114)$$

It follows from (3.110), (3.112), and (3.114) that

$$\big|\Delta_{n,3}(k)\big| < \frac{2.1}{\sqrt{2\pi e}}\frac{\sigma_n(\theta)}{\sigma}\left|\frac{k}{n} - \frac{1}{2}\right|.$$

Notice that, by (3.13) and (3.15),

$$\bar{\sigma}_n(\theta) < b_n(k) \vee \bar{\sigma} < b(1) \vee b(2) < \sqrt{\frac{3}{2}}\sigma.$$

Therefore,

$$\mathbb{E}_k\big|\Delta_{n,3}(k)\big| < 0.63\mathbb{E}_k\left|\frac{k}{n} - \frac{1}{2}\right| < \frac{0.32}{\sqrt{n}}. \quad (3.115)$$

Put

$$\Delta_{n,4}(k) = \Phi\left(\frac{D_{n,k}r - A_n(k)}{\sqrt{n}\,\bar{\sigma}}\right) - \Phi\left(\frac{\bar{b}r - A_n(k)}{\sqrt{n}\,\bar{\sigma}}\right).$$

It is clear that

$$\begin{aligned}\Delta_{n,4}(k) &= \frac{(D_{n,k} - \bar{b})r}{\sqrt{2\pi n}\bar{\sigma}} \exp\left\{-\frac{1}{2}\left(\frac{r\bar{b}(1-\theta) + rD_{n,k}\theta - A_n(k)}{\sqrt{n}\,\bar{\sigma}}\right)^2\right\} \\ &= \frac{(D_{n,k} - b)\big((1-\theta)\bar{b}r + \theta D_{n,k}r - A_n(k)\big)}{\sqrt{2\pi n}\,\bar{\sigma}\big((1-\theta)\bar{b} + \theta D_{n,k}\big)} \\ &\quad \times \exp\left\{-\frac{1}{2}\left(\frac{r\bar{b}(1-\theta) + rD_{n,k}\theta - A_n(k)}{\sqrt{n}\,\bar{\sigma}}\right)\right\} \\ &\quad + \frac{A_n(k)(D_{n,k} - \bar{b})}{\sqrt{2\pi n}\,\bar{\sigma}\big((1-\theta)\bar{b} + \theta D_{n,k}\big)} \\ &\quad \times \exp\left\{-\frac{1}{2}\left(\frac{r\bar{b}(1-\theta) + rD_{n,k}\theta - A_n(k)}{\sqrt{n}\,\bar{\sigma}}\right)^2\right\} \\ &= \omega_{1,n}(k) + \omega_{2,n}(k), \quad 0 < \theta < 1. \end{aligned} \quad (3.116)$$

From (2.2) we derive

$$D_{n,k} - \bar{b} = \frac{\left(\frac{k}{n} - \frac{1}{2}\right)\left(b^2(1) - b^2(2)\right)}{D_{n,k} + \bar{b}}. \quad (3.117)$$

By (3.13) and (3.15),

$$\bar{b}^2 > \left(\frac{1}{2} + \frac{45}{56}\right)\frac{\sigma^2}{2} > 0.65\sigma^2. \quad (3.118)$$

It follows from (3.95) and (3.118) that

$$D_{n,k} + \bar{b} > 1.51\sigma, \quad (1-\theta)\bar{b} + \theta D_{n,k} > \frac{\sigma}{\sqrt{2}}. \quad (3.119)$$

Combining (3.117) and (3.119), we infer that

$$\begin{aligned}\big|\omega_{1,n}(k)\big| &< \frac{\big|D_{n,k} - \bar{b}\big|}{\sqrt{2\pi e}\big((1-\theta)\bar{b} + \theta D_{n,k}\big)} \\ &< 0.23\frac{\left|\frac{k}{n} - \frac{1}{2}\right|\left|b^2(1) - b^2(2)\right|}{\sigma^2}. \end{aligned} \quad (3.120)$$

Furthermore, by (3.111), (3.117), and (3.119), we obtain

$$\begin{aligned}\big|\omega_{2,n}(k)\big| &< \frac{\big|D_{n,k} - b\big|\big|A_n(k)\big|}{\sqrt{2\pi n}\big((1-\theta)\bar{b} + \theta D_{n,k}\big)\bar{\sigma}} \\ &< 0.5\frac{\left|\frac{k}{n} - \frac{1}{2}\right|\left|b^2(1) - b^2(2)\right|}{\sigma^3}\big|A(k)\big|. \end{aligned} \quad (3.121)$$

In view of (3.113) and (3.3),

$$\big|b^2(1) - b^2(2)\big| \le \sigma^2. \quad (3.122)$$

By (3.120) and (3.122),

$$\mathbb{E}_k\big|\omega_{1,n}(k)\big| < \frac{0.12}{\sqrt{n}}.$$

Employing Lemma 3.11 and (3.122), we obtain from (3.121) that

$$\mathbb{E}_k\big|\omega_{2,n}(k)\big| < \frac{0.5}{\sqrt{n}\,\sigma}\mathbb{E}^{1/2}\left(\frac{k}{n} - \frac{1}{2}\right)^2\mathbb{E}^{1/2}A_k^2(n) < \frac{0.75}{\sqrt{n}}.$$

Thus,

$$\mathbb{E}_k\big|\Delta_{n,4}(k)\big| < \mathbb{E}_k\big|\omega_{1,n}(k)\big| + \mathbb{E}_k\big|\omega_{2,n}(k)\big| < \frac{0.2}{\sqrt{n}}. \quad (3.123)$$

Denote

$$\Delta_{n,5}(k)=\Phi\left(\frac{\bar{b}r-A_n(k)}{\bar{\sigma}\sqrt{n}}\right)-\Phi\left(\frac{\bar{b}r-\mathbb{E}_kA_n(k)}{\bar{b}\sqrt{n}}\right).$$

Estimate $\Delta_{n,5}$ on using CLT. Recall that we may treat $A_n(k)$ as the sum of i.i.d. ξ_i, $i=1,n$, taking values $a(1)$ and $a(2)$ with the same probability 1/2 (see the proof of Lemma 3.10). The corresponding Lyapunov ratio is equal to 1. Applying the Berry–Esseen bound, we obtain

$$\mathbb{E}_k\big|\Delta_{n,5}(k)\big|<\frac{0.8}{\sqrt{n}}. \tag{3.124}$$

We have taken it into account here that

$$\bar{\sigma}^2+\frac{1}{n}\mathrm{Var}_kA_n(k)=\bar{b}^2-\theta\frac{\big((a(1)-a(2)\big)^2}{4}=\bar{b}^2-\theta\left(\frac{\beta_3}{\sigma^2n^2}\right)^2.$$

Denoting

$$\Delta_{n,6}(k)=\Phi\left(\frac{\bar{b}r-\mathbb{E}_kA_n(k)}{\bar{b}\sqrt{n}}\right)-\Phi\left(\frac{r}{\sqrt{n}}\right),$$

we have, by (3.32) and (3.118),

$$\mathbb{E}_k\big|\Delta_{n,6}(k)\big|<\frac{\beta_3}{\sqrt{2\pi}\,\sigma^2\bar{b}\sqrt{n}}<0.5\frac{\beta_3}{\sigma^3\sqrt{n}}. \tag{3.125}$$

Collecting the bounds (3.4), (3.84), (3.109), (3.115), and (3.123)–(3.125), we conclude that, for $0<r<0.2n^{3/4}\left(\frac{b(1)}{L}\right)^{1/2}$,

$$\left|\mathbb{P}(Z_n<r)-\Phi\left(\frac{r}{\sqrt{n}}\right)\right|<36\frac{\beta_3}{\sigma^3\sqrt{n}}+\frac{9}{\sqrt{n}}. \tag{3.126}$$

It remains to consider the case $r=r_n>0.2n^{3/4}\left(\frac{b(1)}{L}\right)^{1/2}$. Applying the Berry–Esseen bound, we obtain

$$\begin{aligned}\mathbb{P}\left(S_n>r_nV_n,\,V_n>\frac{\sigma}{\sqrt{2}}\right)&<\mathbb{P}\left(S_n>\frac{r_n\sigma}{\sqrt{2}}\right)\\&<1-\Phi\left(\frac{r_n}{\sqrt{2n}}\right)+c_0\frac{\beta_3}{\sigma^3\sqrt{n}}.\end{aligned} \tag{3.127}$$

By (2.42) and (3.13), we find

$$1-\Phi\left(\frac{r_n}{\sqrt{2n}}\right)<25\sqrt{\frac{2}{\pi e}}\left(\frac{L}{b(1)\sqrt{n}}\right)<34.24\frac{\beta_3}{\sigma^3\sqrt{n}}.$$

Hence, taking (3.127) into account, we obtain

$$\begin{aligned}\left|\mathbb{P}(S_n<rV_n)-\Phi\left(\frac{r}{\sqrt{n}}\right)\right|&\le\mathbb{P}(S_n>rV_n)\vee\left(1-\Phi\left(\frac{r}{\sqrt{n}}\right)\right)\\&<1-\Phi\left(\frac{r_n}{\sqrt{n}}\right)+c_0\frac{\beta_3}{\sigma^3\sqrt{n}}+\mathbb{P}\left(V_n<\sigma\sqrt{\frac{n}{2}}\right)\\&<33.4\frac{\beta_3}{\sigma^3\sqrt{n}}+\mathbb{P}\left(V_n<\sigma\sqrt{\frac{n}{2}}\right).\end{aligned} \tag{3.128}$$

Estimate $\mathbb{P}\left(V_n<\frac{\sigma}{\sqrt{2}}\right)$. Applying Lemma 2.9 yields

$$\mathbb{P}\left(V_n^2<\frac{\sigma^2n}{2}\right)<\exp\left(-\frac{\sigma^6n}{27\beta_3^2}\right)<\frac{27}{e}\frac{\beta_3^2}{n\sigma^6}<\frac{3}{4e}\frac{\beta_3}{\sqrt{n}\sigma^3}, \tag{3.129}$$

provided that $\frac{\beta_3}{\sigma^3\sqrt{n}}<\frac{1}{36}$. It follows from (3.128) and (3.129) that, for $r>0.2n^{3/4}\left(\frac{b(1)}{L}\right)^{1/2}$,

$$\left|\mathbb{P}(Z_n>r)-\Phi\left(\frac{r}{\sqrt{n}}\right)\right|<35\frac{\beta_3}{\sigma^3\sqrt{n}}.$$

This completes the proof of Theorem 1. □

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