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On the Accuracy of Gaussian Approximation in Hilbert Space

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Abstract. The bound on remainder term in CLT for a sum of independent random variables taking values in Hilbert space is obtained. This bound sharpens the result due to Bentkus and Götze.

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1. Introduction

Let H be a separable Hilbert space with the norm $|\cdot|$ and the inner product $(\cdot,\cdot)$, $X, X_1, X_2, \dots$ be H-valued i.i.d. random variables with the covariance operator T and $\mathbf{E}X = 0$. Denote by $\sigma_1^2 \geqslant \sigma_2^2 \geqslant \cdots$ the eigenvalues of T, and by $e_1, e_2, \dots$ the corresponding eigenvectors. Put

$$\Lambda_l = \prod_{j=1}^{l} \sigma_j^2, \quad \sigma^2 = \mathbf{E}|X|^2, \quad \beta_\mu = \mathbf{E}|X|^\mu,$$

$$\Gamma_{\mu,l} = \beta_\mu \sigma^\mu / \Lambda_l^{\mu/l}.$$

Let $\Phi(\cdot)$ be the Gaussian distribution with the covariance operator T. Denote

$$S_n = n^{-1/2} \sum_{j=1}^{n} X_j.$$

Let $B(a,r)$ be the ball $\{x : x \in H,\ |x-a| < r\}$. Denote

$$\Delta_n(a;r) = \big|\mathbf{P}\big(S_n \in B(a,r)\big) - \Phi\big(B(a,r)\big)\big|, \quad \Delta_n(a) = \sup_{r>0} \Delta_n(a;r).$$

It follows from the results of Bentkus and Götze [4] that

$$\Delta_n(\alpha) < \frac{C}{n}\left(\frac{\beta_4}{\sigma^4} + \frac{\beta_3^2}{\sigma^6}\right)$$

with

$$C \leqslant \exp\left\{c\sigma^2/\sigma_{13}^2\right\},$$

where c is an absolute constant.

Let us formulate our result in this direction. Denote

$$L_l = \max_{1\leqslant j\leqslant l} \frac{\beta_{3j}}{\sigma_j^3},$$

where

$$\beta_{3,j} = \mathbf{E}\left|(X, e_j)\right|^3.$$

THEOREM. *There exists an absolute constant c such that*

$$\Delta_n(0) < \frac{c}{n}\left(\Gamma_{4,13} + \Gamma_{3,13}^2 + \left(\frac{\sigma^2}{\Lambda_9^{1/9}}\right)^2 L_9^2\right). \tag{1.1}$$

Using (1.1), we can obtain simpler bounds.

EXAMPLE. Notice that

$$L_l \leqslant \frac{\beta_3}{\sigma_l^3}.$$

Therefore, we may replace the third summand in the bound (1.1) by

$$\left(\frac{\sigma^2\beta_3}{\sigma_9^2\Lambda_9^2}\right)^2 < \frac{\sigma^{11}\beta_4}{\Lambda_{15}^{1/2}}.$$

On the other hand,

$$\Gamma_{4,13} < \frac{\beta_4\sigma^{11}}{\Lambda_{15}^{1/2}}, \qquad \Gamma_{3,13}^2 < \frac{\beta_4\sigma^{11}}{\Lambda_{15}^{1/2}}.$$

As a result, we obtain the bound

$$\Delta_n(0) < \frac{c\beta_4\sigma^{11}}{n\Lambda_{15}^{1/2}}.$$

The latter is much simpler than (1.1) but, of course, rougher.

We shall use the following notations: X' – independent copy of X, $X^s = X - X'$, $\tilde{X} = X - \mathbf{E}X$. Let

$$B(x; L) = \mathbf{E}\big[(Z^s, x)^2; |Z| \vee |Z'| \leqslant L\big],$$

where Z is an arbitrary H-valued random variable, and $\sigma_j^2(L) \geqslant \sigma_{j+1}^2(L),\ j = \overline{1, \infty}$, are the eigenvalues of $B(x; L)$.

Denote

$$\Lambda_l(L) = \prod_1^l \sigma_k^2(L), \quad \delta_l^2(L) = \frac{L^2}{\Lambda_l^{2/l}(L)} \sum_i^l \sigma_j^2(L), \quad f_x^Y(t) = \mathbf{E}e^{it(Y,x)}.$$

We shall use the notation $I(A)$ to denote the indicator of the set A. The symbols $c(l)$ indicates a constant depending on l. Absolute constants will be denoted by c.

In the present paper, the results and the methods of the papers [1 – 7] are applied. On the other hand, some new ideas are used. In this connection, we draw the reader's attention to the key Lemmas 3.1, 3.3–3.6, 3.12, 3.13. It should be noticed that we do not use the specific methods of the number theory in contrast with [4].

2. Bounds on Characteristic Functions in the Neighborhood of Zero

Let $X, X_1, \ldots, X_n, \ldots$ be a sequence of i.i.d. H-valued random vectors. In contrast to the introduction, we do not suppose in this section that $\mathbf{E}X = 0$. We shall use notations $U_{k_1,k_2} = \sum_{k_1}^{k_2} X_j$, $U_m = U_{1,m}$.

LEMMA 2.1. *For every* $1 < m < n$, $L > 0$, $t \in \mathbb{R}$

$$\mathbf{E}\big[\big|f_{U_m^s}^{U_{m+1,n}}(2t)\big|; |U_m| \vee |U_m'| < \sqrt{3}/8|t|L\big] \leqslant c(l)\left[\big(1 + \Lambda_l(L)\big(|t|\sqrt{m(n-m)}\big)^l\big)^{-1} + \left(\frac{\delta_l(L)}{\sqrt{m}}\right)^l\right]. \tag{2.1}$$

LEMMA 2.2. *Let X and Y be independent H-valued random vectors. Then, for every $r > 0$, $a \in H$, $t \in \mathbb{R}$,*

$$\left|\mathbf{E}\exp\left\{it|X + Y - a|^2\right\}\right| < \mathbf{P}(|X| > r) \sup_{b \in H}\left|\mathbf{E}\exp\left\{it|Y - b|^2\right\}\right| + \mathbf{E}^{1/2}\big[\big|f_{X^s}^Y(2t)\big|; |X| \vee |X'| \leqslant r\big]. \tag{2.2}$$

Both lemmas are proved in [7].

Denote $c_0 = \sqrt{3}/8$.

LEMMA 2.3. *Let*

$$B_n^{-1/2} \leqslant |t| \leqslant \frac{c_0}{2\sigma L} \tag{2.3}$$

and $\{l_i\}_1^k$ *be any sequence such that*

$$1 \leqslant l_i \leqslant \frac{c_0}{2\sigma L|t|}, \tag{2.4}$$

$$\sum_i^k l_i^{-2} < 2\left(\frac{|t|\sigma L}{c_0}\right)^2 n. \tag{2.5}$$

Then

$$\sup_{a\in H} \left|\mathbf{E}\exp\left\{it|U_n - a|^2\right\}\right| < c(l)|t|^{l/2} D(l) \sum_i^k l_j^{l/2}/4^j + 4^{-k}, \tag{2.6}$$

where

$$D(l) = (\sigma L)^{l/2}\left(B^{-l/2} + (\sigma L)^{l/2}\right)/\Lambda_l^{1/2}(L). \tag{2.7}$$

Proof. Define the sequence $\{m_i\}_1^k$ by the formula

$$m_i = \left[\left(c_0/2l_i\sigma L|t|\right)^2\right]. \tag{2.8}$$

Notice that in view of (2.4) $m_i \geqslant 1$. Let $\mu_j = \sum_1^j m_i$. Denote

$$q_j(t) = \sup_{a\in H}\left|\mathbf{E}\exp\left\{it|U_{\mu_j+1,n} - a|^2\right\}\right|.$$

Putting in Lemma 2.2 $r = 2\sigma\sqrt{m_j}$, $X = U_{\mu_{j-1},\mu_j} - \mathbf{E}U_{\mu_{j-1},\mu_j} Y = U_{\mu_j+1,n} - \mathbf{E}U_{\mu_j+1,n}$, we obtain the inequalities

$$q_{j-1}(t) \leqslant \frac{1}{4} q_j(t) + Q_j, \quad j = 1, \ldots, k, \tag{2.9}$$

where

$$Q_j = \mathbf{E}\left\{\left|f_{U^s_{m_j}}^{\tilde U_{\mu_j+1,n}}(2t)\right|; \left|\tilde U_{m_j}\right| \vee \left|\tilde U'_{m_j}\right| < 2\sigma\sqrt{m_j}\right\}; \quad \tilde U = U - \mathbf{E}U.$$

We used here the bound $\mathbf{P}(|x| > r) < \mathbf{E}|X|^2/r^2 = \frac{1}{4}$ and the equality $U_{\mu_{j-1}+1,\mu_j} - \mathbf{E}U_{\mu_{j-1}+1,\mu_j} \overset{d}{=} U_{m_j} - \mathbf{E}U_{m_j}$.

Since by (2.4) and (2.8) $r < \sqrt{3}/8|t|L$, we may apply Lemma 2.1 to estimate Q_i. As a result we have

$$Q_i < c(l)(a_i + b_i), \tag{2.10}$$

where

$$a_i = \left(1 + \Lambda_l(L)\left(|t|\sqrt{m_i(n-\mu_i)}\right)^l\right)^{-1/2},$$

$$b_i = \left(\frac{\delta_l(L)}{\sqrt{m_i}}\right)^{l/2}.$$

In view of (2.1)

$$\delta_l^2(L) < \frac{2L^2\sigma^2}{\Lambda_l^{2/l}(L)}.$$

Therefore

$$b_i < \left(\frac{2L\sigma}{\Lambda_l^{1/l}(L)\sqrt{m_i}}\right)^{l/2} < c(l)(l_i|t|)^{l/2}\tilde{b}_l, \tag{2.11}$$

where

$$\tilde{b}_l = \left(L\sigma/\Lambda_l^{1/2l}\right)^l. \tag{2.12}$$

By (2.5) and (2.8)

$$\mu_i < \mu_k \leqslant n/2. \tag{2.13}$$

Hence, taking into account (2.3) and (2.8), we deduce

$$a_i < \left(\frac{cL\sigma l_i}{\Lambda_l^{1/l}(L)\sqrt{n}}\right)^{l/2} < c(l)(l_i t)^{l/2}\tilde{a}_l, \tag{2.14}$$

where

$$\tilde{a}_l = \left(\frac{L\sigma}{B\Lambda_l^{1/l}(L)}\right)^{l/2}. \tag{2.15}$$

It follows from (2.10), (2.11), (2.14), that

$$q_{i-1}(t) < \frac{1}{4}q_i(t) + c(l)(\tilde{a}_l + \tilde{b}_l)(l_i|t|)^{l/2}.$$

Subsequently applying this inequality for $i = 1, \ldots, k$ we conclude that

$$q_0(t) < \frac{1}{4^k}q_k(t) + c(l)|t|^{l/2}(\tilde{a}_l + \tilde{b}_l)\sum_1^k l_j^{l/2}\big/4^j.$$

It remains to use the trivial bound $q_k(t) \leqslant 1$ and to insert instead $\tilde{a}_l$ and $\tilde{b}_l$ the expressions (2.15) and (2.12).

LEMMA 2.4. *If*

$$Bn^{-1/2} \leqslant |t| \leqslant \frac{c_0}{8\sigma L}, \tag{2.16}$$

then

$$\sup_{a\in H}\left|\exp\left\{it|U_n - a|^2\right\}\right| < c(l)D(l)|t|^{l/2}. \tag{2.17}$$

Proof. Put

$$l_i^{l/2} = 4^{i/2}\big(2/4^{2/l} - 1\big)^{l/4}\big(c_0/2|t|\sigma L\sqrt{n}\big)^{l/2} + 1, \tag{2.18}$$

$$k_0 = \max\left\{i\colon\ l_i \leqslant \frac{c_0}{2\delta L|t|}\right\}. \tag{2.19}$$

Then

$$l_i^2 4^{2i/l} 2/\big(4^{2/l} - 1\big)\big(c_0/2|t|\sigma L\sqrt{n}\big)^2.$$

Hence

$$\sum_1^\infty l_i^{-2} < 2\left(\frac{|t|\delta L}{c_0}\right)^2 n. \tag{2.20}$$

Further, by (2.18)

$$\sum_1^\infty \frac{l_i^{l/2}}{4^i} < \left(\frac{c_0}{2|t|\sigma L\sqrt{n}}\right)^{l/2} + 1. \tag{2.21}$$

According to (2.19),

$$l_{k_0} > \frac{c_0}{2\sigma L|t|} - 1. \tag{2.22}$$

It follows from (2.16), (2.18) and (2.22) that

$$\frac{c_0}{4\sigma L|t|} < \frac{c_0}{2\sigma L|t|} - 2 < l_{k_0} < 4^{k_0/l}\left(\frac{2}{4^{2/l}} - 1\right)^{1/2}\frac{c_0}{2|t|\sigma L\sqrt{n}}.$$

Hence

$$4^{-k_0} < \left(\frac{2\sqrt{2}}{4^{2/l} - 1}\right)^{l/2} n^{-l/2}. \tag{2.23}$$

Putting $k = k_0$ in Lemma 2.3 and taking into account (2.19), (2.20) and (2.23) we obtain the bound

$$\begin{aligned}\sup_{a\in H}\big|\mathbf{E}\exp\{it|U_n - a|\}\big| &< c(l)\left(|t|^{l/2}D(l)\left(\left(\frac{1}{2|t|\sigma L\sqrt{n}}\right)^{l/2} + 1\right) + n^{-l/2}\right)\\ &< c_1(l)\big(|t|^{l/2}D(l) + n^{-l/2}\big) < c_2(l)|t|^{l/2}D(l).\end{aligned}$$

This completes the proof of Lemma 2.4.

LEMMA 2.5. *If* $|t| < c_0(2^{l/2}L\sigma(L)\sqrt{2n})^{-1}$, *then*

$$\mathop{\mathbf{E}}_{a\in H}\exp\{it|U_n - a|^2\} < \frac{c(l)}{\sqrt{1 + \Lambda_l(L)(|t|n)^l}}. \tag{2.24}$$

3. Some Preparation Bounds

LEMMA 3.1. *Let A_k be the set of vectors $a = (a_1, a_2, \ldots, a_N)$ such that*

$$|a_j| < \lambda_j/2N, \quad j \neq k, \quad \lambda_k < |a_k| < (1+\varepsilon)\lambda_k. \tag{3.1}$$

Then, for every $x = (x_1, x_2, \ldots, x_N)$,

$$\sum_1^N \inf_{a\in A_k} (x, a)^2 > \frac{1}{4N} \sum_1^N \lambda_j^2 x_j^2, \tag{3.2}$$

$$\max_k \sup_{a\in A_k} (x, a)^2 < N(1+\varepsilon)^2 \sum_1^N \lambda_j^2 x_j^2. \tag{3.3}$$

Proof. Let k be fixed and

$$\lambda_k|x_k| \geqslant \lambda_j|x_j|, \quad j \neq k. \tag{3.4}$$

Then by (3.1), for every $a \in A_k$,

$$|(x, a)| > |x_k a_k| - \sum_{j\neq k} |x_j a_j| > \lambda_k|x_k| - \frac{1}{2N} \sum_{j\neq k} \lambda_j|x_j| > \lambda_k|x_k|/2.$$

Hence,

$$\sum_j \inf_{a\in A_j} (x, a)^2 > \inf_{a\in A_k} (x, a)^2 > \lambda_k^2 x_k^2/4 > \frac{1}{4N} \sum_1^N \lambda_j^2 x_j^2.$$

Obviously, for every x there exists k for which (3.4) holds. It means that (3.2) is valid for every x.

Now prove (3.3). By (3.1) for every $a \in \bigcup_1^N A_j$

$$|(x, a)| < (1+\varepsilon) \sum_1^N \lambda_k|x_k|.$$

On the other hand,

$$\left(\sum_1^N \lambda_k|x_k|\right)^2 < N \sum_1^N \lambda_k^2 x_k^2.$$

Thus

$$\sup_j \sup_{a\in A_j} (x, a)^2 < (1+\varepsilon)^2 N \sum_1^N \lambda_k^2 x_k^2.$$

This completes the proof.

The following inequality is proved in [6].

LEMMA 3.2. *Let random variable* $\xi \geqslant 0$ *and* $F(r)$ *be the distribution of* ξ. *If* $F(r) < Qr^l$ *for* $r \geqslant \varepsilon > 0$, *then*

$$\mathbf{E}\exp\big\{-\xi^2 f^2\big\} < \big(c(l)t^{-l} + r^l\big)Q,$$

where $c(l) \leqslant \Gamma(l/2+1)$, $\Gamma(p)$ *being the Euler function.*

LEMMA 3.3. *Let* F *be the discrete positive measure concentrated on the union of two finite disjoint sets* $A = (x_1, x_2, \ldots, x_m)$ *and* $B = (y_1, \ldots, y_m)$. *Then there exists the nonnegative matrix* ε_{ij}, $i = \overline{1, m}$, $j = \overline{1, n}$, *such that* F *is represented as the mixture*

$$F = \sum_{i=1}^{m}\sum_{j=1}^{n} \varepsilon_{ij} F_{ij}, \tag{3.5}$$

where F_{ij} *are two-point probability measures,*

$$F_{ij}(x_i) = \frac{a}{a+1}, \qquad F_{ij}(y_j) = \frac{1}{a+1}, \quad a = \frac{F(A)}{F(B)},$$

and, in addition,

$$\sum_{i,j} \varepsilon_{ij} F_{ij}(x_i) = F(A), \qquad \sum_{i,j} \varepsilon_{ij} F_{ij}(y_j) = F(B).$$

Proof. Denote $p_i = F(x_i)$, $q_j = F(y_j)$. Without loss of generality $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_m$, $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n$. If $p_1 < aq_1$, then put $\varepsilon_{11} = (a+1)q_1$. As a result

$$F = \varepsilon_{11} F_{11} + F_1, \tag{3.6}$$

where $F_1(x_1) = 0$, $F_1(y_1) = q_1 - ap_1$, $F_1(x_i) = F(x_i)$, $i > 1$, $F_1(y_j) = F(y_j)$, $j > 1$. So F_1 is concentrated on the union of the sets $A_1 := (x_2, x_3, \ldots, x_m)$ and B.

If $p_1 > aq_1$, then $\varepsilon_{11} = (a+1)q_i$, $F_1(x_1) = p_1 - q_1/a$, $F_1(y_1) = 0$, $F_1(x_i) = F(x_i)$, $i > 1$, $F_1(y_j) = f(y_j)$, $j > 1$, i.e., F_1 in (3.6) is concentrated on the $A \cup B_1$, $B_1 = (y_2, \ldots, y_m)$. In the case $p_1 = aq_1$ we have $F_1(x_1) = F_1(y_1) = 0$ and F_1 is concentrated on $A_1 \cup B_1$. Repeating this procedure at most $m+n$ steps we come to representation (3.5).

LEMMA 3.4. *Let random vector* X *take two values* x_1 *and* x_2, $x_1 \neq x_2$, $\mathbf{P}(X = x_1) = p$, $\mathbf{P}(X = x_2) = q$. *Then*

$$\mathbf{E}\big\{\big((X+X')^s, y\big)^2 / X^s = (X^s)' = 0\big\} = \frac{8p^2q^2}{(p^2+q^2)^2}(x_1 - x_2, y)^2, \tag{3.7}$$

$$\mathbf{E}\big\{\big((X+X')^s, y\big)^2 / X^s = (X^s)' = x_2 - x_1\big\} = 0. \tag{3.7$'$}$$

Proof. It is easily seen that

$$\mathbf{P}\big(X + X' = 2x_1 / X^s = 0\big) = \frac{p^2}{p^2 + q^2},$$

$$\mathbf{P}\big(X + X' = 2x_2 / X^s = 0\big) = \frac{q^2}{p^2 + q^2}.$$

Therefore

$$\begin{aligned}\mathbf{P}\big((X + X')^s &= 2(x_1 - x_2) \,/\, X^s = (X^s)' = 0\big) \\ &= \mathbf{P}\big((X + X')^s = 2(x_2 - x_1) \,/\, X^s = (X^s)' = 0\big) \\ &= \frac{p^2 q^2}{(p^2 + q^2)^2}. \qquad (3.8)\end{aligned}$$

Obviously (3.8) implies equality (3.7). It is sufficient to remark that $X^s = (X^s)' = x_2 - x_1$ implies $(X + X')^s = 0$ in order to prove (3.7′).

LEMMA 3.5. *Let* (X_k, Y_k), $k = \overline{1, n}$, *be the sequence of independent random variables taking values in* $H \times H$. *Then, for every Borel function* $\varphi(\cdot)$ *on* H,

$$\mathbf{E} \prod_1^n \big|\mathbf{E}\big(\varphi(X_k)/Y_k\big)\big| = \prod_1^n \mathbf{E}\big|\mathbf{E}\big(\varphi(X_k)/Y_k\big)\big|.$$

Proof. Random variables

$$\xi_k = \big|\mathbf{E}\big(\varphi(X_k)/Y_k\big)\big|$$

are independent. Therefore

$$\mathbf{E} \prod_1^n \xi_k = \prod_1^n \mathbf{E}\xi_k.$$

Call the random vector X' by a conditionally independent copy of X relative to σ-algebra $\mathfrak{B}$ if, for every Borel functions $\varphi_1(\cdot)$ and $\varphi_2(\cdot)$,

$$\mathbf{E}\big(\varphi_1(X)\varphi_2(X')/\mathfrak{B}\big) = \mathbf{E}\big(\varphi_1(X)/\mathfrak{B}\big)\mathbf{E}\big(\varphi_2(X)/\mathfrak{B}\big).$$

Let the sequence (X_k, Y_k) be the same as in Lemma 3.5. As above, we use the notation $V_{k,m} = \sum_k^m X_j$, $\widetilde{V}_{k,m} = V_{k,m} - \mathbf{E}V_{k,m}$, $V_m = V_{1,m}$.

Let $\mathfrak{B}_{k,m}$ be σ-algebra generated by random variables Y_j, $j = \overline{k, m}$, $\mathfrak{B}_m = \mathfrak{B}_{1,m}$. Denote

$$\varphi_k(t; \mathfrak{B}_k) = \sup_{b \in H} \big|\mathbf{E}\big\{\exp(it|V_k - b|^2)/\mathfrak{B}_k\big\}\big|,$$

$$\varphi_k(t) = \mathbf{E}\varphi_k(t; \mathfrak{B}_k),$$

$$f_k(t;x,y) = \mathbf{E}\big\{\exp(it(X_k,x))/Y_k = y\big\}.$$

LEMMA 3.6. *For any* $r>0$, $t\in R$ *and* $1\leqslant k\leqslant n$

$$\varphi_n(t) < \mathbf{P}\big(|\tilde V_{k+1,n}| > r\big)\varphi_k(t) + \\ +\mathbf{E}^{1/2}\left\{\prod_1^k \mathbf{E}_{Y_j}|f_j(2t;V^s_{k+1,n},Y_j)|I\big(|\tilde V_{k+1,n}|\vee|\tilde V'_{k+1,n}|<r\big)/\mathfrak{B}_{k+1,n}\right\}. \quad (3.9)$$

Proof. Using Lemma 2.2, we have

$$\varphi_n(t;\mathfrak{B}_n) < \mathbf{P}\big(|\tilde V_{k+1,n}| > r/\mathfrak{B}_{k+1,n}\big)\varphi_k(t;\mathfrak{B}_k) + \\ +\mathbf{E}^{1/2}\left\{\prod_{j=1}^k |f_j(2t;V^s_{k+1,n},Y_j)|I\big(|\tilde V_{k+1,n}|\vee|\tilde V'_{k+1,n}|<r\big)/\mathfrak{B}_{k+1,n}\right\} \\ \equiv D_1 + D_2. \quad (3.10)$$

By Lemma 3.5,

$$\mathbf{E}D_2 < \mathbf{E}^{1/2}D_2^2 \\ < \mathbf{E}^{1/2}\left\{\prod_{j=1}^k \mathbf{E}_{Y_j}|f_j(2t;V^s_{k+1,n},Y_j)|I\big(|\tilde V_{k+1,n}|\vee|\tilde V'_{k+1,n}|<r\big)/\mathfrak{B}_{k+1,n}\right\}. \quad (3.11)$$

On the other hand

$$\mathbf{E}D_1 = \mathbf{P}\big(|\tilde V_{k+1,n}| > r\big)\varphi_k(t). \quad (3.12)$$

Combining (3.10)–(3.12) we obtain the bound (3.9). Lemma 3.6 is proved.

LEMMA 3.7. *For every real random variable* ξ *with* $\mathbf{E}\xi=0$, $\mathbf{E}|\xi|^3<\infty$

$$\mathbf{E}(1-\cos\xi) \geqslant \mathbf{E}\xi^2/2 - \mathbf{E}|\xi|^3/4\sqrt{3}$$

(*see* [7, Lemma 1]).

LEMMA 3.8. *If* $|t| < \sqrt{3}/2L|x|$ *then, for every* $L>0$,

$$\big|f_x^X(t)\big|^2 < 1 - \frac{B(x;L)}{4}t^2. \quad (3.13)$$

Proof. First

$$\big|f_x^X(t)\big|^2 = \mathbf{E}\big\{\exp\big(X^s,x)it\big);|X|\vee|X'|\leqslant L\big\} + \\ +\mathbf{E}\big\{\exp\big(X^s,x)it\big);|X|\vee|X'|>L\big\} \equiv E_1 + E_2. \quad (3.14)$$

Applying Lemma 3.7, we have

$$\begin{aligned} E_1 &< \mathbf{P}\big(|X| \vee |X'| \leqslant L\big) - B(x; L)t^2/2 + \\ &\quad + \mathbf{E}\big(|(X^s, x)|^3; |X| \vee |X'| \leqslant L\big)|t|^3/4\sqrt{3} \\ &< \mathbf{P}^2\big(|X| \leqslant L\big) - B(x; L)\bigg(\frac{t^2}{2} - L|x||t|^3/2\sqrt{3}\bigg). \end{aligned}$$

Therefore, for $|t| < \sqrt{3}/2L|x|$,

$$E_1 < \mathbf{P}^2\big(|X| \leqslant L\big) - B(x; L)t^2/4. \tag{3.15}$$

On the other hand,

$$E_2 \leqslant 1 - \mathbf{P}^2\big(|X| \leqslant L\big). \tag{3.16}$$

Combining (3.14)–(3.16), we obtain the bound (3.13). Lemma 3.8 is proved.

Denote

$$f(t; x, y) = \mathbf{E}\big\{\exp(it(X, x))/Y = y\big\},$$

$$B(x, y; L) = \mathbf{E}\big\{(X^s, x)^2 I(|X| \vee |X'| < L)/Y = y\big\}.$$

LEMMA 3.9. *For every* $|t| < \sqrt{3}/2L|x|$,

$$\mathbf{E}|f(t; x, Y)| < \exp\big\{-t^2\mathbf{E}B(x, Y; L)/8\big\}. \tag{3.17}$$

Proof. Using the Cauchy inequality and the bound (3.13) we have

$$\begin{aligned} \mathbf{E}^2|f(t; x, Y)| &< \mathbf{E}|f(t; x, Y)|^2 \\ &< 1 - \mathbf{E}B(x, Y; L)\frac{t^2}{4} < \exp\bigg\{-\mathbf{E}B(x, Y; L)\frac{t^2}{4}\bigg\}. \end{aligned}$$

Taking the square root, we obtain the bound (3.17).

Denote by $\bar{\sigma}_1^2(k_1, k_2; L) > \bar{\sigma}_2^2(k_1, k_2; L) > \cdots$ eigenvalues of the form

$$\bar{B}(k_1, k_2; x; L) = \sum_{j=k_1}^{k_2} \mathbf{E}B(x, Y_j; L).$$

Put

$$\bar{\Lambda}_l(k_1, k_2; L) = \prod_1^l \bar{\sigma}_j^2(k_1, k_2; L), \quad \bar{\sigma}_j^2(k; L) = \bar{\sigma}_j^2(1, k; L),$$

$$\bar{\Lambda}_l(k; L) = \bar{\Lambda}_l(1, k; L).$$

LEMMA 3.10. *Let $V_1, V_2, \dots, V_m$ be R^l-valued random variables which conditionally independent with respect to σ-algebra $\mathfrak{F}$, $X = \sum_1^m V_j$, B_l be $l \times l$ nonnegative symmetric matrix with eigenvalues $b_1^2 \geqslant \cdots \geqslant b_l^2$. Then, for every $L > 0$ and $r > 0$,*

$$\mathbf{EP}\big(B_l(X^s) < r/\mathfrak{F}\big) < c(l)(r + \varepsilon_l)^l / \bar{\Lambda}_l^{1/2}(m; L) \prod_1^l b_j, \tag{3.19}$$

where

$$\varepsilon_l^2 = 32L^2 \sum_1^l b_k^2.$$

Lemma 3.10 generalizes Lemma 2 from [7]. The proof is quite similar and therefore is omitted.

The following bound generalizes Lemma 5 from [7].

LEMMA 3.11. *For every real t*

$$\mathbf{E}\Bigg[\prod_1^k \mathbf{E}_{Y_j}|f_j(2t; U_{k+1,n}^s, Y_j)| I\big(|\tilde{U}_{k+1,n}| \vee |\tilde{U}'_{k+1,n}| < \sqrt{3}/8|t|L\big)/\mathfrak{B}_{k+1,n}\Bigg]$$
$$< c(l)\left(\frac{1}{|t|^l \sqrt{\bar{\Lambda}_l(k; L)\bar{\Lambda}_l(k+1, n; L)}} + \bar{\delta}_l^l(k, n; L)\right), \tag{3.20}$$

where

$$\bar{\delta}_l^2(k, n; L) = \frac{L^2 \sum_{k+1}^n \bar{\sigma}_j^2(k; L)}{(\bar{\Lambda}_l(k; L)\bar{\Lambda}_l(k+1, n, L)}.$$

Proof. Using Lemma 3.9, we conclude that for $|t| < \sqrt{3}/4L|x|$

$$\prod_1^k \mathbf{E}_{Y_j}\big|f_j(2t; x, Y_j)\big| < \exp\left\{-\frac{t^2}{8}\sum_1^k \mathbf{E}B(x, Y_j; L)\right\}.$$

Without loss of generality, one may assume that

$$\sum_1^k \mathbf{E}B(x, Y_j; L) = \sum_1^\infty \bar{\sigma}_j^2(k; L)x_j^2 > \sum_1^l \bar{\sigma}_j^2(k; L)x_j^2,$$

where $x_j = (x, e(j; L))$, $e(j; L)$ are eigenvectors corresponding to $\bar{\sigma}_j^2(k; L)$.

Putting $b_j^2 = \bar{\sigma}_j^2(k; L)$ in Lemma 3.10, we obtain

$$\mathbf{EP}\left(\sum_1^l \bar{\sigma}_j^2(k; L)\big(U_{k+1,n}^s, e(k; L)\big)^2 < r^2/\mathfrak{B}_{k+1,n}\right)$$
$$< c(l)\frac{(r + \varepsilon_l)^l}{\sqrt{\bar{\Lambda}_l(k; L)\bar{\Lambda}_l(k+1, n; L)}},$$

where

$$\varepsilon_l^2 = 32L^2 \sum_{j=1}^{l} \bar{\sigma}_j^2(k; L).$$

Letting

$$\xi^2 = \sum_1^l \bar{\sigma}_j^2(k; L)\big(U_{k+1,n}^s, e(k; L)\big)^2,$$

we obtain the bound on left side of (3.24):

$$\mathbf{E} \exp\left\{ -\frac{t^2}{8}\xi^2 \right\}.$$

It remains to apply Lemma 3.2 with $\xi^2 = \sum_1^l \bar{\sigma}_j^2(k; L)(U_{k+1,n}, e(k; L))^2$.

Denote $\bar{B}_j(x; L) = \mathbf{E}B(x, Y_j; L)$.

LEMMA 3.12. *Let the sets* A_k *be the same as in Lemma* 3.1, $N = l$,

$$Q_n(\gamma; k) = \big\{j\colon\ 1 \leqslant j \leqslant n,\ \bar{B}_j(x; L) = \gamma(a_j, x)^2,\ a_j \in A_k\big\},$$
$$n_k = \operatorname{card} \Omega_n(\gamma; k), \quad n_0 = \min_{1\leqslant k\leqslant l} n_k.$$

If

$$n_0 > \eta n, \quad 0 < \eta < 1/l, \tag{3.21}$$

and

$$\sum_1^l \lambda_i^2 < \gamma_1 \sigma^2, \tag{3.22}$$

λ_i *satisfying* (3.7), *then for*

$$\frac{c_0 n}{4\sigma L|t|} > |t| > Bn^{-1/2}$$

$$\varphi_n(t) < c(l)\left(\frac{(1+\varepsilon)^2\gamma_1}{\gamma\eta^2}\right)^{l/4} \bar{D}(l)|t|^{l/2}, \tag{3.23}$$

where

$$\bar{D}(l) = (\sigma L)^{l/2}\big(B^{-l/2} + (\sigma L)^{l/2}\big)/\prod_1^l \lambda_j.$$

Proof. Let l_i be defined by (2.17) and, in addition,

$$l_i \leqslant \frac{c_0\eta}{4\sigma L|t|}. \tag{3.24}$$

Further let m_i satisfy (2.8) and $\mu_j = \sum_1^j m_i$.

Denote

$$m_{ik} = \operatorname{card}\big(\Omega_n(\gamma; k) \cap \{j : \mu_i < j \leqslant \mu_{i+1}\}\big).$$

Notice that $\varphi_n(t)$ does not depend on the order of summands X_i.

Therefore, taking into acount (3.21), we may assume without loss of generality that $m_{ik} \geqslant [\eta m_i]$, $k = \overline{1, l}$. It follows from (2.7) and (3.24) that

$$m_i \geqslant 2/\eta.$$

Hence, $[\eta m_i] \geqslant 2$. It means in turn that $[\eta m_i] \geqslant \eta m_i/2$, i.e.,

$$m_{ik} > \eta m_i, \quad k = \overline{1, l}. \tag{3.25}$$

Denote

$$\bar{q}_i(t) = \varphi_{n-\mu_i}(t).$$

Taking $n - \mu_{i-1}$ instead of n in Lemma 3.6 and putting $k = n - \mu_i$, $r = 2\sigma\sqrt{m_i}$, we conclude that

$$\bar{q}_{i-1} < \frac{1}{4}\bar{q}_i + \bar{Q}_i, \tag{3.26}$$

where

$$\bar{Q}_i = \mathbf{E}^{1/2} \prod_{j=1}^{n-\mu_i} \mathbf{E}_{Y_j}\big|f_j(2t; U_{\mu_{i-1}+1,\mu_i}, Y_j)\big|,$$

$$\big(\big|\tilde{U}_{\mu_{i-1}+1,\mu_i}\big| \vee \big|\tilde{U}_{\mu_{i-1}+1,\mu_i}\big| < 2\sigma\sqrt{m_i}/\mathfrak{B}_{\mu_{i-1},\mu_i}\big).$$

Applying Lemma 3.12, we have

$$\bar{Q}_i < c(l)\left[\frac{1}{|t|^{l/2}\bar{\Lambda}_l^{l/4}(\mu_{i-1}, \mu_i; L)\bar{\Lambda}_l^{l/4}(n-\mu_i; L)} + \bar{\delta}^{l/2}(n-\mu_i; L)\right]. \tag{3.27}$$

Here we took into account that by (2.17), $l_i \geqslant 1$ and, consequently, $\sigma\sqrt{m_i} < \sqrt{3}/8|t|L$.

Using (3.25) and Lemma 3.1, we get the inequalities

$$\frac{\gamma\eta m_i}{2}\sum_1^l \lambda_j^2 x_j^2 < \sum_{\mu_{i-1}+1}^{\mu_i} \bar{B}_j(x; L) < \gamma m_i(1+\varepsilon)^2 \sum_1^l \lambda_j^2 x_j^2.$$

Hence

$$\frac{1}{2}\gamma\eta m_i\lambda_j^2 < \bar{\sigma}_j^2(\mu_{i-1}+1,\mu_i;L) < \gamma m_i l(1+\varepsilon)^2\lambda_j^2, \quad j=\overline{1,l}.$$

Consequently

$$\left(\frac{\gamma\eta m_i}{2}\right)^l \prod_1^l \lambda_j^2 < \bar{\Lambda}_l\big(\mu_{i-1}+1,\mu_i;L\big) < (\gamma m_i l)^l(1+\varepsilon)^2\prod_1^l \lambda_j^2. \tag{3.28}$$

Similarly

$$\frac{1}{2}\gamma\eta(n-\mu_i)\lambda_j^2 < \bar{\sigma}_j^2(n-\mu_i;L) < \gamma(n-\mu_i)l(1+\varepsilon)^2\lambda_j^2, \tag{3.29}$$

$$\left(\frac{\gamma\eta(n-\mu_i)}{2}\right)^l < \bar{\Lambda}_l(n-\mu_i;L) < \big(\gamma l(n-\mu_i)\big)^l(1+\varepsilon)^{2l}\prod_1^l \lambda_j^2. \tag{3.30}$$

It follows from (3.28)–(3.30) that

$$\bar{\delta}^2(n-\mu_i;L) < c(l)\frac{(1+\varepsilon)^2L^2\sum_1^l\lambda_j^2}{\gamma\eta^2\prod_1^l\lambda_j^4 m_i}. \tag{3.31}$$

combining (3.27), (3.28), (3.30) and (3.31) we get

$$\bar{Q}_i < c(l)(a_i+b_i), \tag{3.32}$$

where

$$a_i = \frac{1}{|t|^{l/2}m_i^{l/4}(n-\mu_i)^{l/4}(\gamma\eta)^{l/2}\prod_1^l\lambda_j},$$

$$b_i = \left(\frac{(1+\varepsilon)^2L^2\sum_1^l\lambda_i^2}{m_i\gamma\eta^2\prod_1^l\lambda_j^4}\right)^{l/4}.$$

It follows from (2.7), (2.12) and (2.15)

$$a_i < c\left(\frac{\sigma L l_i}{\gamma\eta\sqrt{n}}\right)^{l/2}\left(\prod_1^l\lambda_j\right)^{-1} < c(l)(\gamma\eta)^{-l/2}(l_i t)^{l/2}\tilde{a}_l, \tag{3.33}$$

where

$$\tilde{a}_l = (L\sigma/B)^{l/2}\left(\prod_1^l\lambda_j\right)^{-1}.$$

Further, by (2.7) and (3.26)

$$b_i < c(l)\frac{(1+\varepsilon)^2\gamma_1}{\gamma\eta^2}(l_i|t|)^{l/2}\tilde{b}_l, \tag{3.34}$$

where

$$\tilde{b}_l = (\sigma L)^l/\prod_1^l \lambda_j.$$

Combining (3.26) and (3.32)–(3.34), we obtain the inequality

$$\bar{q}_{i-1} < \frac{1}{4}\bar{q}_i + c(l)\big(\tilde{a}_l + \tilde{b}_l\big)(l_i|t|)^{l/2}.$$

Hence, by (3.24),

$$\bar{q}_0 < \frac{1}{4^{k_0}} + c(l)|t|^{l/2}\big(\tilde{a}_l + \tilde{b}_l\big),$$

where

$$k_0 = \max\left\{k\colon\ l_k \leqslant \frac{c_0\eta}{4\sigma L|t|}\right\}.$$

Subsequent arguments coincide with those proving Lemma 2.4.

LEMMA 3.13. *Let the conditions of Lemma* 3.13 *hold and*

$$|t| < c_0/2^{l/2}\sigma L\sqrt{n}.$$

Then

$$\varphi_n(t) < c(l)\left(\frac{\gamma_1(1+\varepsilon)^2}{(\gamma\eta)^2}\right)^{l/4}\left(nt\prod_1^l \lambda_j\right)^{-l/2}.$$

4. Estimation of a Characteristic Function in the Neighborhood of the Unit

Let $X_1, X_2, \ldots, X_n, \ldots$ be independent random variables, taking values in H, $U_n = \sum_1^n X_j$. Denote $X_j^0 = X_j + X_j'$, $U_n^0 = \sum_1^n X_j^0$. Let $\mathfrak{A}_n$ be a σ-algebra generated by random variable X_j^s, $j = \overline{1, n}$.

Define

$$\Psi_n(t;\tau;a) = \big|\mathbf{E}\exp\big\{it|U_n - a|\big\}\big|\,\big|\mathbf{E}\exp\big\{i(t+\tau)|U_n - a|^2\big\}\big|,$$
$$\Psi(\tau;b;\mathfrak{A}_n) = \big|\mathbf{E}\big\{\exp\big\{i\tau\big|U_n^0 - b\big|^2\big\}/\mathfrak{A}_n\big\}\big|.$$

LEMMA 4.1. *For any $t \in R$, $a \in H$ and $\tau > 0$,*

$$\Psi_n(t;\tau;a) < \mathbf{E}\sup_{b\in H}\Psi(\tau/4;b;\mathfrak{A}_n). \tag{4.1}$$

(*See the proof in* [1].)

LEMMA 4.2. *Let μ_n be the number of successes in Bernoulli trials, p being the probability of a success, $q = 1-p$, $H(p,\varepsilon) = \varepsilon\ln\frac{\varepsilon}{p} + (1-\varepsilon)\ln\frac{1-\varepsilon}{1-p}$. Then*

$$(1)\ \mathbf{P}(\mu_n \geqslant n(1-\varepsilon)) \leqslant \exp\{-nH(q,\varepsilon)\}, \quad \textit{if}\ 0<\varepsilon<q, \tag{4.2}$$

$$(2)\ \mathbf{P}(\mu_n \leqslant n\varepsilon) \leqslant \exp\{-nH(p,\varepsilon)\}, \quad \textit{if}\ 0<\varepsilon<p. \tag{4.3}$$

Define

$$M_l(t;A) = \begin{cases} (|t|A^2)^{-l/2}, & \text{if } |t| < A^{-1},\\ |t|^{l/2}, & \text{if } |t| \geqslant A^{-1}. \end{cases}$$

LEMMA 4.3. *Let $\varphi(t)$ be a continuous nonnegative function defined on* $[0,\infty)$, $\varphi(0)=1$, *and for every $\tau > 0$*

$$\sup_{t\geqslant 0}\big(\varphi(t)\varphi(t+\tau)\big) \leqslant \chi_l M_l(\tau,A),$$

where $\chi_l \geqslant 1$ does not depend on τ. Then, for $l \geqslant 9$,

$$\int_{A^{-4/l}}^{1}\frac{\varphi(t)}{t}\,\mathrm{d}t \leqslant \frac{\chi_l}{A^2}.$$

This bound is obtained in [3].

Let $X, X_1, X_2, \ldots$ be i.i.d. random vectors satisfying the conditions of the theorem. Without loss of generality, we may assume X takes a finite number of values. Define for every $1 \leqslant k \leqslant l$

$$A_k^+ = \big\{x\colon \sigma_k < (x,e_k) < 2\sigma_k,\ 0<(x,e_j)<\sigma_j/2l,\ j\neq k\big\},$$
$$A_k^- = \big\{-2\sigma_k < (x,e_k) < -\sigma_k,\ -\sigma_j/2l < (x,e_j) < 0,\ j\neq k\big\}.$$

Denote $\tilde{A}_k = A_k^+ \vee A_k^-$. Let F_k be the restriction of F to $\tilde{A}_k$, $k=\overline{1,l}$, and F_0 be the restriction of F to $H - \bigcup_1^l \tilde{A}_k$.

Thus

$$F = \sum_0^l F_k = \sum_0^l \varepsilon_k P_k, \tag{4.4}$$

where

$$\varepsilon_k = F_k(H), \quad P_k = F_k/\varepsilon_k.$$

According to Lemma 3.3

$$F_k = \sum_{i,j} \varepsilon_{ij}^k F_{ij}^k, \tag{4.5}$$

where F_{ij}^k are distributions concentrated in two points $x_i^+ \in A_k^+$, $x_j^- \in A_k^-$,

$$F_{ij}^k(x_i^+) = \frac{\mathbf{P}(X \in A_k^+)}{\mathbf{P}(X \in A_k^+) + \mathbf{P}(X \in A_k^-)},$$

$$F_{ij}^k(x_j^-) = \frac{\mathbf{P}(X \in A_k^-)}{\mathbf{P}(X \in A_k^+) + \mathbf{P}(X \in A_k^-)}.$$

Let $\mathfrak{F}$ be the set of the distributions F_{ij}^k, $k = \overline{1, l}$.

It follows from (4.4) and (4.5) that F^{n*} can be represented as the expectation of a convolution of random distributions

$$F^{n*} = \mathbf{E} G_1 * G_2 * \cdots * G_n, \tag{4.6}$$

where

$$\mathbf{P}(G_j \in \mathfrak{F}_k) = \mathbf{P}(\tilde{A}_k), \quad k = \overline{1, n},$$

$$\mathbf{P}(G_j = F_0) = \mathbf{P}\biggl(H - \bigcup_1^l \tilde{A}_k\biggr).$$

Denote $p_k = \mathbf{P}(X \in \tilde{A}_k)$, $\bar{p} = \min_{1 \leqslant k \leqslant l} p_k$,

$$N_k = \operatorname{card}\bigl\{G_j\colon\ G_j \in \mathfrak{F}_k,\ 1 \leqslant j \leqslant n\bigr\},$$
$$N_0 = \operatorname{card}\bigl\{G_j\colon\ G_j = F_0,\ 1 \leqslant j \leqslant n\bigr\}.$$

By (4.2), for every $0 < \varepsilon < \bar{p}$

$$\begin{aligned} \mathbf{P}\Bigl(\min_{1 \leqslant k \leqslant l} N_k > n\varepsilon\Bigr) &\geqslant 1 - \sum_1^l \exp\bigl\{-nH(p_k, \varepsilon)\bigr\} \\ &\geqslant 1 - l \exp\bigl\{-nH(\bar{p}, \varepsilon)\bigr\}. \end{aligned} \tag{4.7}$$

Denote

$$\mathfrak{M}_n = \left\{\prod_1^n G_k\colon \min_{1 \leqslant k \leqslant l} N_k > \bar{p}n/2\right\}.$$

By (4.7)

$$\mathbf{P}(\mathfrak{M}_n) > 1 - l \exp\bigl\{-c(\bar{p})n\bigr\}, \tag{4.8}$$

where $c(\bar{p}) = H(\bar{p}, \bar{p}/2)$.

Estimate now $\Psi_n(t;\tau;a)$ corresponding to $\prod_1^n G_j \in \mathfrak{M}_n$ with the aid of Lemmas 4.1, 3.13, 3.14. Notice that

$$\sigma_k < \left|(x^+ - x^-, e_k)\right| < 2\sigma_k, \quad \left|(x^+ - x^-, e_j)\right| < \sigma_j/2l,$$
$$j \neq k, \quad \text{if } x^+ \in A_k^+, \quad x^- \in A_k^-.$$

Therefore we may assume that

$$\left\{x^+ - x^-\colon\, x^+ \in A_k^+,\ x^- \in A_k^-\right\} \subset A_k,$$

where A_k is defined by (3.1) with $\lambda_j = \sigma_j$, $N = l$, $\varepsilon = 1$.

Obviously $\Psi_n(t;\tau;a)$ does not depend on the order of summands. Therefore, taking into account Lemma 3.4 and the bound (4.8), we may assume without loss of generality that the condition (3.25) holds with

$$\gamma = 8 \min_j \frac{p_j^2 q_j^2}{(p_j^2 + q_j^2)^2},$$

where

$$p_j = \frac{\mathbf{P}(X \in A_j^+)}{\mathbf{P}(X \in \tilde{A}_j)}, \quad q_j = 1 - p_j = \frac{\mathbf{P}(X \in A_j^-)}{\mathbf{P}(X \in \tilde{A}_j)},$$

and $\eta = \bar{p}/3$. The condition (3.26) is satisfied for $\gamma_1 = 1$.

As a result, applying Lemmas 3.12 and 3.13, we obtain the bound

$$\left|\Psi_n(t;\tau;a)\right| < c(l)\left(\frac{1}{\gamma \bar{p}^2}\right)^{l/4} w(l) M(t, A), \tag{4.9}$$

where

$$A = \sigma\sqrt{n}, \quad w(l) = \frac{(\sigma L)^l}{\Lambda_l^{1/2}}.$$

Let V_j be independent random variables having distribution G_j (see formula (4.6)) and Ω_n be the set of random sequences $\omega_n = \{\xi_1, \xi_2, \ldots, \xi_n\}$ such that $\xi_j = i$ if $G_j \in \mathfrak{F}_i$. Say $w_n \in \mathfrak{M}_n$, if $G_j \in \mathfrak{M}_n$. Denote

$$g_n(t; w_n) = \mathbf{E}\left(\exp\left\{\left|\sum_1^n V_j\right|^2\right\}\Big/ w_n\right).$$

Estimate

$$I_n(w_n) = \int_\tau^\varepsilon \left|\frac{g_n(t; w_n)}{t}\right| \mathrm{d}t,$$

where $\tau < \varepsilon$ are arbitrary constants and $w_n \in \mathfrak{M}_n$.

Evidently

$$I_n(w_n) = \int_{\tau/\varepsilon}^{1} \left| \frac{g_n(\varepsilon t;\, w_n)}{t} \right| \mathrm{d}t.$$

Applying the bound (4.9) we get

$$I_n(w_n) < c(l)w(l) \int_{\tau/\varepsilon}^{1} \frac{M(\varepsilon t;\, A)}{t}\, \mathrm{d}t. \tag{4.10}$$

It is easily to check that

$$M(\varepsilon t;\, A) = \varepsilon^{l/2} M(t;\, \varepsilon A). \tag{4.11}$$

If $\tau/\varepsilon = (A\varepsilon)^{-4/l}$ then it follows from (4.10), (4.11) and Lemma 4.3 that under the condition $w_n \in \mathfrak{M}_n$

$$I_n(w_n) < c(l)w(l)\varepsilon^{l/2}(A\varepsilon)^{-2}.$$

Hence,

$$\int_{\tau}^{\varepsilon} \frac{|g_n(t)|}{t}\, \mathrm{d}t \leqslant \mathbf{E} \int_{\tau}^{\varepsilon} \left| \frac{g_n(t,\, w_n)}{t} \right| \mathrm{d}t < c(l)w(l)\varepsilon^{l/2}(A\varepsilon)^{-2} + \mathbf{P}(w_n \notin \mathfrak{M}_n) \ln \frac{\varepsilon}{\tau}, \tag{4.12}$$

where $A = \sigma L\sqrt{n}$, $w(l) = (\sigma L)^l / \Lambda_l^{1/2}$, $l \geqslant 9$.

By (4.8)

$$\mathbf{P}(w_n \notin \mathfrak{M}_n) < l \exp\big\{-nH(\bar{p},\, \bar{p}/2)\big\}. \tag{4.13}$$

5. Termination of the Proof

Denote $m_n = [\frac{n}{4}] + 1$

$$Y = \frac{1}{\sqrt{n_0}} \sum_{1}^{n_0} \bar{X}_j, \qquad \bar{X}_j = \begin{cases} X_j, & |X_j| \leqslant \sigma\sqrt{m_n}, \\ 0, & |X_j| > \sigma\sqrt{m_n}, \end{cases}$$

where n_0 will be chosen later on.

Let Y_j are i.i.d. random vectors, $Y \overset{D}{=} Y_j$. Clearly

$$\sum_{1}^{n} \bar{X}_j = \sqrt{n_0} \sum_{1}^{n/n_0} Y_j \tag{5.1}$$

if

$$n \equiv 0 \pmod{n_0}.$$

In what follows we suppose, for simplicity, that this condition holds.

Further, let c_0 be an absolute constant in the Berry–Esseen version of a multidimensional CLT. Put

$$n_0 = \min\left\{n\colon \frac{c_0 L_l}{\sqrt{n}} < q(l)/4\right\}, \tag{5.2}$$

the constant $q(l)$ being defined in Lemma 5.5.

Our aim is to apply the results of Sections 2 and 3 to the sum $\sum_1^{n/n_0} Y_j$.

First we prove several lemmas.

LEMMA 5.1. *For every k*

$$|\mathbf{E}(Y, e_k)| < 2\sqrt{\frac{n_0}{n}}\sigma_k.$$

Proof. It is easily seen that

$$\mathbf{E}(Y, e_k) = -\sqrt{n_0}\,\mathbf{E}\big\{(X, e_k);\, |X| > \sigma\sqrt{m_n}\big\}.$$

It remains to notice that by Cauchy inequality,

$$\mathbf{E}^2\big\{(X, e_k);\, |X| > \sigma\sqrt{m_n}\big\} < \mathbf{P}\big(|X| > \sigma\sqrt{m_n}\big)\sigma_k^2 < \frac{\sigma_k^2}{m_n}.$$

LEMMA 5.2. *For every k*

$$\mathbf{E}(Y, e_k)^2 < \left(1 + 4\left(\frac{n_0}{n}\right)\right)\sigma_k^2.$$

Proof. It is easily seen that

$$\mathbf{E}(Y, e_k)^2 \leqslant \sigma_k^2 + \mathbf{E}^2(Y, e_k).$$

It remains to refer to Lemma 5.1.

LEMMA 5.3. *The bound*

$$\mathbf{E}|Y|^2 < \left(1 + 4\frac{n_0}{n}\right)\sigma^2$$

is valid.

Proof. It is easily seen that

$$\mathbf{E}|Y|^2 \leqslant \sigma^2 + n_0\mathbf{E}^2\big\{|X|;\ |X| > m_n\sigma\big\}.$$

Hence, using the bound

$$\mathbf{E}^2\big\{|X|;\ |X| > m_n\sigma\big\} < \frac{\sigma^2}{m_n},$$

we obtain the desired result.

LEMMA 5.4. *For every k*

$$\mathbf{E}\big\{(Y, e_k);\ |Y| < a\sigma\big\} < \left(2\sqrt{\frac{n_0}{n}} + \frac{1}{a}\sqrt{1 + 4\frac{n_0}{n}}\right)\sigma_k, \quad a > 0.$$

Proof. Clearly

$$\mathbf{E}\big\{(Y, e_k);\ |Y| < a\sigma\big\} = \mathbf{E}(Y, e_k) - \mathbf{E}\big\{(Y, e_k);\ |Y| > a\sigma\big\}.$$

Hence, using the bound

$$\mathbf{E}^2\big\{(Y, e_k);\ |Y| > a\sigma\big\} < \mathbf{P}\big(|Y| > a\sigma\big)\mathbf{E}(Y, e_k)^2$$

and Lemmas 5.1–5.3, we obtain the desired result.

LEMMA 5.5. *If $|(x, e_j)| < \sigma_j/4l$, $j = \overline{1, l}$, then for every k, $1 \leqslant k \leqslant l$,*

$$\Phi(\tilde{A}_k + x) > q(l) := 2\left(\Phi_0\left(\frac{1}{4l}\right) - \Phi_0\left(-\frac{1}{4l}\right)\right)^{l-1} \times \\ \times \left(\Phi_0\left(2 - \frac{1}{4l}\right) - \Phi_0\left(1 - \frac{1}{4l}\right)\right),$$

where Φ_0 is the standard Gaussian law.

Proof. Obviously

$$\tilde{A}_k + x \supset C_k := \Big\{x\colon\ |(x, e_j)| < \sigma_j/4l, \\ j \neq k,\ \left(1 - \frac{1}{4l}\right)\sigma_j < |(x, e_k)| < 2 - \frac{1}{4l}\Big\}.$$

It is easily seen that

$$\Phi(C_k) > q(l).$$

Returning to the previous inclusion, we obtain the desired result.

LEMMA 5.6. *Let Y be a random vector in* R^N *and the sets A_k satisfy* (3.1). *Then*

$$\mathbf{E}(Y, x)^2 > \min_k \mathbf{P}(Y \in A_k) \sum_1^N x_m^2 \lambda_m^2/4N.$$

Proof. It is easily seen that $A_k A_l = \varnothing$, $k \neq l$. Therefore, by Lemma 3.1,

$$\mathbf{E}(Y, x)^2 \geqslant \mathbf{E}\left\{(Y, x)^2;\ Y \in \bigcup_1^N A_k\right\} = \sum_1^N \mathbf{E}\big\{(Y, x)^2;\ Y \in A_k\big\} \\ > \sum_1^N \inf_{a_l \in A_k} (a_l, x)^2 \mathbf{P}(Y \in A_k) > \min_k \mathbf{P}(Y \in A_k) \sum_1^N x_m^2 \lambda_m^2/4N.$$

Lemma 5.6 is proved.

LEMMA 5.7. *If* $L > a\sigma$, $n_0/n < q(l)/20$,

$$2\sqrt{\frac{n_0}{n}} + \frac{1}{a}\sqrt{1 + 4\frac{n_0}{n}} < \frac{1}{8l},$$

then

$$\mathbf{E}\big\{(Y^s, x)^2;\ |Y| \vee |Y'| < L\big\} > \frac{q(l)}{2}\sum_1^l \sigma_i^2(x, e_j)^2. \tag{5.3}$$

Proof. It is easily seen that

$$\mathbf{E}\big\{(Y^s, x)^2;\ |Y| \vee |Y'| < L\big\} = 2p_L\mathbf{E}\big\{(Y - a_L, x)^2;\ |Y| < L\big\}, \tag{5.3$'$}$$

where

$$a_L = \mathbf{E}\{Y/|Y| < L\}, \quad p_L = \mathbf{P}(|Y| < L).$$

If $L > 2\sigma$, then

$$\tilde{A}_k \subset \{x\colon |x| < L\}.$$

By Lemma 5.6 for such L,

$$\mathbf{E}\big\{(Y - a_L, x)^2;\ |Y| < L\big\} > \min_{1\leqslant k\leqslant l} \mathbf{P}\big(Y - a_L \in \tilde{A}_k\big)\sum_1^l \sigma_j^2(x, e_j)^2. \tag{5.4}$$

Elaborate now the bound on $\mathbf{P}(Y \in \tilde{A}_k + a_L)$. Denote

$$P_0 = \mathbf{P}\left(\frac{1}{\sqrt{n_0}}\sum_1^{n_0} X_j \in \tilde{A}_k + a_L\right).$$

According to the Berry–Esseen bound and definition (5.2),

$$\big|\Phi\big(\tilde{A}_k + a_L\big) - P_0\big| < q(l)/4.$$

On the other hand, by Lemma 5.3,

$$\big|\mathbf{P}\big(Y \in \tilde{A}_k + a_L\big) - P_0\big| < n_0\mathbf{P}\big(|Y| > m_n\sigma\big) < 4\frac{n_0}{n}\left(1 + 4\frac{n_0}{n}\right) < q(l)/4.$$

Thus

$$\mathbf{P}\big(Y \in \tilde{A}_k + a_L\big) > \Phi\big(\tilde{A}_k + a_L\big) - q(l)/2. \tag{5.5}$$

Under conditions of the lemma, in according to Lemma 5.3,

$$p_L > \frac{1}{2}. \tag{5.5$'$}$$

On the other hand, by Lemma 5.4,

$$|\mathbf{E}(Y, e_k)| < \frac{\sigma_k}{8l}.$$

Therefore,

$$|(a_L, e_k)| < \frac{\sigma_k}{4l}. \tag{5.6}$$

It follows from (5.5), (5.6) and Lemma 5.5 that

$$\mathbf{P}\big(Y - a_L \in \tilde{A}_k\big) > q(l)/2.$$

Returning to (5.3′) and (5.4) and taking into account (5.5′), we obtain the bound (5.3).

Without loss of generality, we may assume the conditions of Lemma 5.7 to be satisfied. Then, according to (5.3),

$$\sigma_j^2(L) \geqslant \frac{q(l)}{2}\sigma_j^2. \tag{5.7}$$

Put

$$\bar{\Delta}_n(0) = \sup_r \left|\mathbf{P}\left(\frac{1}{\sqrt{n}}\sum_1^n \bar{X}_j \in B(0, r)\right) - \Phi\big(B(0, r)\big)\right|,$$

$$\bar{g}_n(t) = \mathbf{E}\exp\left\{it\left|\sum_1^n X_j\right|^2\right\},$$

$$g(t) = \int_H \mathrm{e}^{\,it|x|^2}\Phi(\mathrm{d}x).$$

By the Esseen bound for every $t_0 > 0$,

$$\bar{\Delta}_n < J/\pi + 2\pi(t_0\sigma_1\sigma_2)^{-1}, \tag{5.8}$$

where

$$J = \int_{|t|<t_0}\left|\frac{\bar{g}_n(t) - g(\sqrt{n}t)}{t}\right|\mathrm{d}t.$$

Let

$$\tau_1 = n^{-1}\Lambda_l^{-1/l}\left|\frac{\sqrt{n}}{\Gamma_{3,l}}\right|^{1/3}.$$

It is proved in [6] that for $l \geqslant 7$

$$J_1 := \int_{0<|t|<\tau_1}\left|\frac{\bar{g}_n(t) - g(\sqrt{n}t)}{t}\right|\mathrm{d}t < \frac{c(l)}{n}\big(\Gamma_{3,l}^2 + \Gamma_{4,l}\big). \tag{5.9}$$

Estimate now the integral

$$J_2 = \int_{\tau_1<|t|<\tau_2} \left|\frac{\bar{g}_n(t)}{t}\right| \mathrm{d}t,$$

where

$$\tau_2 = c_0/2^{l/2}\sigma L\sqrt{n}, \quad c_0 = \sqrt{3}/8.$$

Notice that

$$\bar{g}_n(t) = \tilde{g}_n(n_0 t), \tag{5.10}$$

where

$$\tilde{g}_n(t) = \exp\left\{\left|\sum_1^{n/n_0} Y_j\right|^2 it\right\}.$$

Hence, applying Lemma 2.5 and taking into account (5.7), we get the bound

$$J_2 < \frac{c(l)}{\Lambda_l^{1/2} n^{l/2}} \int_{\tau_1}^{\infty} \frac{\mathrm{d}t}{t^{l/2+1}} < \frac{2c(l)}{l\Lambda_l^{1/2}}\left(\frac{\Gamma_{3,l}}{\sqrt{n}}\right)^{l/6}. \tag{5.11}$$

Finally, put

$$t_0 = \Lambda_l^{1/l}/(\sigma L)^2 n_0,$$
$$\tau_3 = t_0(A_n n_0 t_0)^{-4/l},$$
$$A_n = \sigma L\sqrt{n/n_0}.$$

It is easily seen that

$$J_4 = \int_{\tau_3<|t|<t_0} \left|\frac{\bar{g}_n(t)}{t}\right| \mathrm{d}t = \int_{\tau<|t|<\varepsilon} \left|\frac{\tilde{g}_n(t)}{t}\right| \mathrm{d}t,$$

where

$$\varepsilon = n_0 t_0 = \frac{\Lambda_l^{1/l}}{(\sigma L)^2}, \quad \tau = n_0\tau_3 = \varepsilon(A_n\varepsilon)^{-4/l}. \tag{5.12}$$

Using (4.12) and (4.13), we have

$$J_4 < c(l)\omega(l)\varepsilon^{l/2}(A_n\varepsilon)^{-2} + l\exp\left\{-\frac{n}{n_0}H(\bar{p}, \bar{p}/2)\right\}, \quad l \geqslant 9. \tag{5.13}$$

Here

$$\bar{p} = \min_{1\leqslant j\leqslant l} \mathbf{P}\big(Y \in \tilde{A}_j\big) > \frac{3}{4}q(l). \tag{5.14}$$

This bound follows from (5.2).

We are to take into account that

$$\inf_{1\leqslant j\leqslant l} \mathbf{P}\big(Y \in A_j^{\pm}\big) > \frac{3}{4}q(l)$$

as well.

It is easy to check that

$$\omega(l)\varepsilon^{l/2}(A_n\varepsilon)^{-2} = \frac{n_0}{n}\left(\frac{\sigma L}{\Lambda_l^{1/l}}\right)^2. \tag{5.15}$$

It follows from (5.13)–(5.15) that

$$J_4 < c(l)\frac{n_0}{n}\left(\frac{\sigma L}{\Lambda_l^{1/l}}\right)^2, \quad l \geqslant 9. \tag{5.16}$$

Using Lemma 2.4 and (5.10), we have

$$J_3 := \int_{\tau_2\leqslant t<\tau_3} \left|\frac{\bar{g}_n(t)}{t}\right| \mathrm{d}t < \int_{|t|<\tau} \left|\frac{\tilde{g}_n(t)}{t}\right| \mathrm{d}t < c(l)\frac{(\sigma L)^l}{\Lambda_l^{1/2}} \int_0^\tau t^{l/2-1}\,\mathrm{d}t.$$

By (5.12) and (5.15)

$$\frac{(\sigma L)^l}{\Lambda_l^{1/l}}\tau^{l/2} = \frac{n_0}{n}\left(\frac{\sigma L}{\Lambda_l^{1/l}}\right)^2.$$

Consequently,

$$J_3 < c(l)\left(\frac{\sigma L}{\Lambda_l^{1/l}}\right)^2\frac{n_0}{n}. \tag{5.17}$$

Notice that

$$J_5 := \int_{\tau_1}^{t_0} \left|\frac{g(\sqrt{n}t)}{t}\right| \mathrm{d}t < c(l)\left(\frac{\Gamma_{3,l}}{\sqrt{n}}\right)^{l/6}. \tag{5.18}$$

It is clear that

$$J \leqslant \sum_1^5 J_5.$$

Hence, collecting the bounds (5.9), (5.11), (5.16)–(5.18), we get

$$J \leqslant \frac{c(l,l')}{n}\left(\Gamma_{3,l}^2 + \Gamma_{4,l} + n_0\left(\frac{\sigma L}{\Lambda_{l'}^{1/l'}}\right)^2 L_{l'}^2\right), \quad l \geqslant 13,\ l' \geqslant 9. \tag{5.19}$$

It follows from (5.8), (5.2) and (5.19) that

$$\bar{\Delta}_n(0) < \frac{c(l,l')}{n}\left(\Gamma_{3,l}^2 + \Gamma_{4,l} + \left(\frac{\sigma L}{\Lambda_{l'}^{1/l'}}\right)^2 L_{l'}^2\right), \quad l \geqslant 13,\ l' \geqslant 9.$$

Hence, putting $L = a\sigma$ and taking into account that

$$\Delta_n(0) < \bar{\Delta}_n(0) + \frac{n\beta_4}{\sigma^4 m_n^4},$$

we obtain the desired result.

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