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LIMIT THEOREMS FOR THE TOTAL NUMBER OF DESCENDANTS FOR THE GALTON–WATSON BRANCHING PROCESS*

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(*Translated by E. Sokolova*)

Abstract. The main results of the present paper deal with the asymptotic behavior of the conditional distribution for the whole number of descendants S_n of a single particle in the Galton–Watson process with respect to the condition that the process degenerates at time n and the expectation for the number of particles generated by one particle tends to 1 as $n \to \infty$.

Key words. the Galton–Watson branching processes, processes close to critical ones, degeneracy, asymptotic behavior of the total number of descendants

1. Introduction. The Galton–Watson process is one of the simplest models of random reproduction. Some important facts of the theory of branching processes have been established first for the Galton–Watson process, or for its analogue, Markov's branching process with continuous time. In this connection, it suffices to mention the papers [1]–[4]. These and some other works of A. N. Kolmogorov and his successors have greatly influenced the further development of the theory of branching processes. The term "branching processes" itself was introduced for the first time by A. N. Kolmogorov and N. A. Dmitriev in their paper [2].

Further, we consider the Galton–Watson process Z_n, $n = 0, 1, \dots$, with one type of particles starting with a single particle. We interpret the value of the random variable Z_n as a number of particles in the nth generation. According to our condition we have $Z_0 = 1$. Let $N = \min\{n\colon Z_n = 0\}$, $p_k = \mathbf{P}(Z_1 = k)$, $A = \sum_1^\infty kp_k$, $f_n(x) = \mathbf{E}\,(x^{Z_n})$, $|x| \leqq 1$, $f(x) = f_1(x)$. We shall use the following notation: $B = f''(1)$, $L = f'''(1)$. We assume that $p_0 > 0$ and $0 < B < \infty$ (the last only for the case $A \leqq 1$).

For a long time the main object of investigations was the asymptotic behavior of the probability $\mathbf{P}(N > n)$ and of the conditional distribution $\mathbf{P}(Z_n < u \mid N > n)$ (see, for example, [1], [4]). But in due course, some more complicated functionals of trajectories of branching processes have created interest.

In 1971, A. G. Pakes published the article [5], where the asymptotic behavior for the total number of descendants $S_n = \sum_0^n Z_i$ (we shall use the same notation in the formulations of our results as well) of a single particle was investigated. More precisely, Pakes found the limit $\lim_{n\to\infty} \mathbf{P}(S_n/a_n < u \mid N > n)$, where $a_n = \mathbf{E}\,\{S_n \mid N > n\}$ for a fixed parameter A: $A < 1$, $A = 1$, $A > 1$.

Later Pakes turned his attention to the distribution of S_n with respect to the alternative condition $N = n$, namely, in his work [6] he established the following relation for $A < 1$ and $L < \infty$:

$$(1.1)\qquad \lim_{n\to\infty} \mathbf{P}\left(\big(S_N - (1+\gamma)N\big)/TN^{1/2} < x \mid N = n + m\right) = \Phi(x),$$

where γ and T are some constants that can be explicitly calculated by means of $f(x)$, and $\Phi(x)$ is the standard normal distribution.

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The case $A = 1$ was considered by Kesten (see [7]) who proved by means of Durret's principle that there the limit $\lim_{n\to\infty} \mathbf{P}(S_n/m_n < x \mid N = n) = G(x)$ exists. Here and later on we assume that $m_n = \mathbf{E}\{S_n \mid N = n\}$. In [7] no explicit expression for $G(x)$ was obtained.

In our paper [8] the asymptotic behavior of the distribution function $G_n(x) = \mathbf{P}(S_n/m_n < x \mid N = n)$ is considered in the most complicated case, when $A \to 1$ as $n \to \infty$ (processes close to critical). It was established that, in this case,

$$\lim_{n\to\infty} G_n(x) = G(x,r), \qquad \text{if} \quad \lim_{n\to\infty} n(1-A) = -\log r.$$

Here $G(x,r)$ is a distribution function depending on the parameter r, and the explicit expression for its Laplace transform is given. This result is presented in Theorem 5 (see § 3). For comparison recall that

$$\lim_{\substack{n\to\infty \\ A\to 1}} \mathbf{P}\left(Z_n/a_n > x \mid N > n\right) = e^{-x},$$

where $a_n = \mathbf{E}\left(Z_n \mid N > n\right)$ independent of the rate of convergence of A to 1 (see [9], [10]).

The present paper differs from [8] only by its introduction and bibliography. Unfortunately, when the preprint [8] was published the articles [6], [7] were not yet known to us. This explains why we give there our own proof of relation (1.1) without citing [6]. Nevertheless, the present paper contains our proof of the relation mentioned, since it is based on considerations different from those used by Pakes (see Theorem 6, § 3). The only thing the two proofs have in common is the use of the Laplace transform. Moreover, we use a more complete version of the continuity theorem (compare our Lemma 26 with Theorem 1 in [6]), though we could refer to the statement proved by Pakes. We think that the form of the continuity theorem proposed is of independent interest, that is why we present it in our paper.

Our main task is to investigate the asymptotics for $G_n(x)$. In addition, we also touch upon some other subjects. In the paper by Kesten, Ney, and Spitzer [11] it was proved that, for a critical process, $n^2\mathbf{P}(N = n) = 2B^{-1}(1 + o(1))$ as $n \to \infty$. The remainder term for this relation is estimated in Theorem 1 of the present paper under the following condition:

(Γ_δ): $\qquad \mathbf{E}\,|Z_1|^{2+\delta} < \infty, \qquad 0 \leqq \delta < 1.$

In Theorems 1, 2, and 4 we establish the asymptotics for $\mathbf{E}\left(S_n \mid N = n\right)$. In Theorem 3 we give the estimate for small deviations of the random variable S_n with respect to the condition $N = n$.

Since we investigate transitional phenomena, we consider the class K of distributions on the lattice of non-negative integers satisfying the following conditions:

A) $\sum_2^\infty l(l-1)p_l(F) > b_0 > 0$ for some b_0 and any $F \in K$;

B) $\lim_{n\to\infty} \sup_{F\in K} \sum_n^\infty l^2 p_l(F) = 0$;

C) $p_0(F) > \alpha_0 > 0$ for some α_0 and any $F \in K$.

Here $p_l(F)$ is an atom of the distribution F concentrated at the point l. The class K was introduced in [10].

In view of condition (B), there exists $b_1 < \infty$ such that, for any $F \in K$,

$$\sum_2^\infty l(l-1)\,p_l(F) < b_1. \tag{1.2}$$

It is not difficult to see that if the distribution of the random variable (r.v.) Z_1 belongs to K, then

$$f''(x) \geqq \frac{b_0}{2}\, x^{n_0}, \qquad x > 0. \tag{1.3}$$

Here n_0 is such that $\sup\{\sum_{n_0}^{\infty} l^2 p_l(F)\colon\ F \in K\} \leqq b_0/2$.

In what follows we always assume that the distribution of the random variable Z_1 does not leave the class K in case $A \to 1$, and is fixed in case $A = \mathrm{const}$.

Now let us introduce some notations. Let λ be the minimum root of the equation $s = f(s)$, $0 \leqq s \leqq 1$. Let us put $A_0 = f'(\lambda)$, $B_0 = f''(\lambda)$, $L_0 = f'''(\lambda)$. If $A \leqq 1$, then $\lambda = 1$, and hence $A_0 = A$, $B_0 = B$, $L_0 = L$. If $A > 1$, then $\lambda < 1$ and $A_0 < 1$, $B_0 < \infty$, $L_0 < \infty$. Let $\gamma = \lambda B_0/A_0(1 - A_0)$. By the symbol $c(\cdot)$ we shall denote a positive constant depending only on the distribution of Z_1 and the argument inside the braces, and by $\overline{c}(\cdot)$ the constant depending only on the class K and the argument inside the braces. We put $K_A = \{F \in K\colon\ \sum_1^{\infty} l p_l(F) = A\}$. Let $\sigma = (1/n,\ 1 - A,\ 1 - s)$, $\sigma_1 = (1/n,\ 1 - A)$, $\sigma_2 = (1 - A,\ 1 - s)$, $\sigma_3 = (1/n,\ 1 - s)$. By $Q_n(s)$, Q_n, $Q(s)$, Q we denote infinitesimals (depending on F) such that

$$\begin{aligned}
&\sup\big\{Q_n(s)\colon\ F \in K_A\big\} \to 0 && \text{as } \sigma \to 0,\\
&\sup\big\{Q_n\colon\ F \in K_A\big\} \to 0 && \text{as } \sigma_1 \to 0,\\
&\sup\big\{Q(s)\colon\ F \in K_A\big\} \to 0 && \text{as } \sigma_2 \to 0,\\
&\sup\big\{Q\colon\ F \in K_A\big\} \to 0 && \text{as } A \to 1.
\end{aligned}$$

We agree to call such a convergence uniform with respect to $F \in K_A$. Also we need the following infinitesimals: $Q_n^0(s) \to 0$ as $\sigma_3 \to 0$, $Q_n^0 \to 0$ as $n \to \infty$, $Q^0(s) \to 0$ as $s \to 1$. For convenience of notations, let us identify different infinitesimals belonging to one of the classes introduced before. Let us explain this. Let $\mathfrak{N}_i$, $i = 1, 2$, be the classes of such infinitesimals. Instead of $\varphi(x, y)$, where $x, y \in \mathfrak{N}_i$, φ a certain function, we shall write $\varphi(x, x)$. If $\varphi(x) \in \mathfrak{N}_i$, where $x \in \mathfrak{N}_j$, φ a certain function, we shall write $\varphi(x) = y$, $y \in \mathfrak{N}_i$. In investigating transition phenomena, we distinguish two cases:

1) $\limsup_{\sigma_1 \to 0} n|1 - A| \leqq c$,
2) $\lim_{\sigma_1 \to 0} n|1 - A| = \infty$.

For the sake of conciseness, to show that the first (second) case takes place, we shall write $i = 1$ $(i = 2)$. Let $\theta = n \log A_0$. Note that $\theta \leqq 0$. We consider the generating functions

$$u_n(s) = \mathbf{E}\left(s^{s_n};\ Z_n = 0\right), \qquad |s| \leqq 1, \quad n = 0, 1 \ldots, \tag{1.4}$$

$$g_n(s) = \mathbf{E}\left(s^{s_n};\ N = n\right), \qquad |s| \leqq 1, \quad n = 0, 1 \ldots. \tag{1.5}$$

2. Auxiliary statements. It is well known (see [5]) that

$$u_0(s) = 0, \qquad u_{n+1}(s) = s f\big(u_n(s)\big). \tag{2.1}$$

LEMMA 1. *The following recurrent relations hold*:

$$g_0(s) = 0,$$

$$g_{n+1}(s) = u_{n+1}(s) - u_n(s), \tag{2.2}$$

$$g_{n+1}(s) = s f\big(u_n(s)\big) - u_n(s), \tag{2.3}$$

$$g_{n+1}(s) = s\Big(f\big(u_n(s)\big) - f\big(u_n(s) - g_n(s)\big)\Big). \tag{2.4}$$

Proof. The first relation follows from (1.5). Since $\{N = n+1\} = \{Z_{n+1} = 0\}\backslash\{Z_n = 0\}$ and $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$, we have (2.2) because of (1.4), (1.5). Relations (2.3), (2.4) may be obtained from (2.2) with the help of the recurrent relation (2.1).

LEMMA 2. *The following representation holds*:

$$u'_{n+1}(1) = \sum_{k=1}^{n} \prod_{l=k}^{n} f'\big(f_l(0)\big) f_k(0) + f_{n+1}(0). \tag{2.5}$$

Proof. Differentiating both sides of (2.1) at the point 1, we obtain $u'_{n+1}(1) = f(u_n(1)) + f'(u_n(1))u'_n(1)$. Since

$$u_n(1) = \mathbf{P}(Z_n = 0) = f_n(0), \tag{2.6}$$

the relation

$$u'_{n+1}(1) = f_{n+1}(0) + f'\big(f_n(0)\big)u'_n(1)$$

holds. By induction this yields (2.5).

LEMMA 3. *Let η be a random variable with positive integer values, $\mathbf{E}\eta^\delta < \infty$, $0 \leqq \delta < 1$, $|x| \leqq 1$, $|x_0| \leqq 1$, $\alpha \geqq 0$. Then we have*

$$\mathbf{E}|x^\eta - x_0^\eta| \leqq |x - x_0|^\delta \Big(\mathbf{E}\eta^\delta \alpha^{1-\delta} + 2\mathbf{E}\big(\eta^\delta;\ \eta \geqq \alpha|x - x_0|^{-1}\big)\Big).$$

Proof. In view of the inequality $|x^\eta - x_0^\eta| \leqq |x - x_0|\eta$ we have the estimate

$$\mathbf{E}\left(|x^\eta - x_0^\eta|;\ |x - x_0|\eta \leqq \alpha\right) \leqq |x - x_0|^\delta \mathbf{E}\eta^\delta \alpha^{1-\delta}.$$

The same inequality and the estimate $|x^\eta - x_0^\eta| \leqq 2|x^\eta - x_0^\eta|^\delta$ yield

$$\mathbf{E}\left(|x^\eta - x_0^\eta|;\ |x - x_0|\eta > \alpha\right) \leqq 2|x - x_0|^\delta \mathbf{E}\big(\eta^\delta;\ \eta > \alpha|x - x_0|^{-1}\big).$$

COROLLARY 1. *Let the condition (Γ_δ) hold. Then, for a fixed x_0,*

$$f''(x) - f''(x_0) = o(|x - x_0|^\delta), \tag{2.7}$$

$$f'(x) = f'(x_0) + f''(x_0)(x - x_0) + o(|x - x_0|^{1+\delta}), \tag{2.8}$$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2/2 + o(|x - x_0|^{2+\delta}). \tag{2.9}$$

Proof. According to the definition of a generating function, we have

$$f''(x) = \sum_2^\infty k(k-1)p_k x^{k-2}.$$

Let us use the following notation:

$$\rho_k = k(k-1)p_k \Big/ \sum_2^\infty k(k-1)p_k.$$

Suppose η is a random variable such that $\mathbf{P}(\eta = k) = \rho_k$. Then $\mathbf{E}\eta^\delta < \infty$, and the conditions of Lemma 3 are satisfied. Supposing $\alpha = |x - x_0|^{1/2}$, we obtain $\mathbf{E}\,|x^\eta - x_0^\eta| =$

$o(|x-x_0|^\delta)$. Since $\mathbf{E}\,x^\eta = \sum_2^\infty \rho_k x^k$, the last relation leads to (2.7). Formulas (2.8), (2.9) may be obtained from (2.7) by means of the Taylor formula with Lagrange remainder.

LEMMA 4. *Let $A=1$ and assume condition (Γ_δ) holds. Then we can write*

$$f_n(0) = 1 - 2(Bn)^{-1}\big(1+o(n^{-\delta})\big), \tag{2.10}$$

$$f'\big(f_n(0)\big) = 1 - 2n^{-1}\big(1+o(n^{-\delta})\big), \tag{2.11}$$

as $n\to\infty$.

Proof. Because of (2.9),

$$f_{n+1}(0) = f\big(f_n(0)\big) = f_n(0) + B\big(1-f_n(0)\big)^2/2 + o\Big(\big(1-f_n(0)\big)^{2+\delta}\Big)$$

as $f_n(0)\to 1$. Let us put $x_n = 1-f_n(0)$, $y_n = 1/x_n$. We write the equality in the following form: $x_{n+1} = x_n - Bx_n^2/2 + o(x_n^{2+\delta})$. Hence, we have

$$y_{n+1} = y_n/\big(1 - Bx_n/2 + o(x_n^{1+\delta})\big) = y_n + B/2 + o(y_n^{-\delta}).$$

It is well known that $1/y_n = 1-f_n(0) \sim 2/Bn$ as $n\to\infty$ (see, for example, [12, p. 19]). Hence, by induction, we obtain $y_n = Bn/2 + o(n^{1-\delta})$ from the previous relation. Returning to the old notations, we get (2.10). Relation (2.11) may be obtained from (2.10) using formula (2.8).

LEMMA 5. *Let $A=1$ and suppose that the condition (Γ_δ) is satisfied. Then we have*

$$u_n'(1) = n\big(1+o(n^{-\delta})\big)/3 \tag{2.12}$$

as $n\to\infty$.

Proof. We proceed from relation (2.5). Using the expansion $\log(1+x) = x + O(x^2)$, we obtain

$$\prod_l (1+x_l) = \exp\Big\{\log\prod_l(1+x_l)\Big\} = \exp\Big\{\sum_l \big(x_l + O(x_l^2)\big)\Big\}. \tag{2.13}$$

Taking into account (2.11) this yields

$$\prod_{l=k}^n f'(f_l(0)) = \exp\Big\{\sum_{l=k}^n\Big(-2l^{-1} + o\big(l^{-(1+\delta)}\big)\Big)\Big\}.$$

Since $\sum_{l=1}^n l^{-1} = c + \log n + O(n^{-1})$ as $n\to\infty$,

$$\prod_{l=k}^n f'\big(f_l(0)\big) = \exp\Big\{2\log(k/n) + o\big(k^{-\delta}\big)\Big\} = (k/n)^2\Big(1+o\big(k^{-\delta}\big)\Big) \tag{2.14}$$

as $k\to\infty$. Let us point out that if we multiply both sides of the last relation by $f_k(0)$ (see (2.10)), the asymptotics of the right-hand side remains the same. Substituting (2.14) and (2.10) into (2.5) and summing the last relation, we obtain (2.12).

Let us consider the case $A = \text{const} \neq 1$. Using the result of [13], we may write the following asymptotic expansion:

$$f_n(0) = \lambda - RA_0^n - R_1A_0^{2n} + o\big(A_0^{2n}\big), \tag{2.15}$$

as $n \to \infty$, where R, R_1 are values depending only on the form of $f(x)$. Let us denote by $\omega_\delta(n)$ an arbitrary infinitesimal satisfying the conditions $\omega_\delta(n) = o(A^{(1+\delta)n})$ for $A < 1$, $\omega_\delta(n) = O(A_0^{2n})$ for $A > 1$.

If the condition (Γ_δ) holds and $A < 1$, then

$$f'(f_n(0)) = A_0\bigl(1 - RB_0A_0^{n-1} + \omega_\delta(n)\bigr). \tag{2.16}$$

For a subcritical branching process this follows from (2.15) and (2.8), and for a supercritical one from (2.15) and the Taylor expansion, since $L_0 < \infty$.

LEMMA 6. *Let $A = \text{const} \neq 1$, and let the condition (Γ_δ) be satisfied in the case $A < 1$. Then we have*

$$u_n'(1) = \frac{\lambda}{1 - A_0} - R(1+\gamma)\,nA_0^n + A_0^n\chi_\delta(n), \tag{2.17}$$

as $n \to \infty$, where $\chi_\delta(n) = o(n)$ for $A < 1$, $\delta = 0$ and $\chi_\delta(n) = c + o(1)$ in other cases.

Proof. Using (2.13), (2.16), we obtain

$$\prod_{l=k}^{n-1} f'(f_l(0)) = A_0^{n-k}\exp\Bigl\{-RB_0A_0^{k-1}\bigl(1 - A_0^{n-k}\bigr)/(1 - A_0) + \omega_\delta(k)\Bigr\},$$

as $k \to \infty$. Hence it follows that

$$\prod_{l=k}^{n-1} f'(f_l(0)) = A_0^{n-k}\Bigl\{1 - RB_0\bigl(A_0^{k-1} - A_0^{n-1}\bigr)/(1 - A_0) + \omega_\delta(k)\Bigr\}.$$

Since the asymptotics of $f_k(0)$ is known (see (2.15)), we have

$$\prod_{l=k}^{n-1} f'(f_l(0))f_k(0) = \lambda A_0^{n-k} - R(1+\gamma)\,A_0^n + R\gamma A_0^{2n-k} + A_0^{n-k}\omega_\delta(k).$$

Substituting this expression and the representation (2.15) into (2.5), and summing it, we get the representation (2.17), where

$$\chi_\delta(n) = c + \sum_{k=1}^{n-1}\bigl(R\gamma A_0^k + A_0^{-k}\omega_\delta(k)\bigr).$$

It is not difficult to see that $\chi_\delta(n)$ satisfies the condition of the lemma.

Let us consider the equation

$$u = sf(u), \qquad |s| < 1. \tag{2.18}$$

It has a unique solution $u^*(s)$ which is analytic in an open unit circle; in addition, this solution is a generating function of some (may be trivial) distribution, and $0 < u^*(1-) = \lambda \leqq 1$ (see, for example, [14]). Hence, we have

$$u^*(s) \leqq \lambda, \qquad 0 \leqq s \leqq 1. \tag{2.19}$$

Pakes has shown [5] that the sequence of functions $\{u_n(s)\}$, $0 < s < 1$, defined by relations (2.1) strictly increases, and $u_n(s) \to u^*(s)$, $0 \leqq s < 1$, as $n \to \infty$. It is also known that $f_n(0) \to \lambda$ as $n \to \infty$ (see, for example, [15, p. 7]). Hence we see that

$$sp_0 \leqq u_n(s) \leqq u^*(s), \qquad n \geqq 1, \quad 0 \leqq s \leqq 1, \tag{2.20}$$

in view of (2.1) and (2.6). To be concise, in what follows we shall omit the argument s in $u^*(s)$.

LEMMA 7. *Let* $0 < c_0 \leqq s \leqq 1$. *Then we have*

$$\lambda - u^* \leqq \bar{c}(c_0)(1-s)^{1/2}. \tag{2.21}$$

Proof. Since u^* is a solution of (2.18), we have, according to the Taylor formula,

$$u^* = s\big(\lambda - A_0(\lambda - u^*) + f''(\rho)(\lambda - u^*)^2/2\big), \tag{2.22}$$

where $\rho \in (u^*, \lambda)$. And from the last relation it follows because of (2.19) that $sf''(\rho)(\lambda - u^*)^2 \leqq 2(u^* - su^*) \leqq 2\lambda(1-s)$. To complete the proof, it is sufficient to note that $f''(\rho) \geqq f''(c_0\alpha_0) \geqq \bar{c}(c_0)$ in view of (2.20) and (1.3).

LEMMA 8. *Let* $A = 1$, $0 < s \leqq 1$. *Then the following holds*:

$$1 - u^* \geqq \big(2B^{-1}p_0(1-s)\big)^{1/2}. \tag{2.23}$$

Proof. Putting $\lambda = 1$ in (2.22), we obtain the equality $2(1-s)u^* = sf''(\rho)(1-u^*)^2$. This yields (2.23) taking into account (2.19) and (2.20).

LEMMA 9. *Let* $A = 1$. *Then,*

$$g_n(1) \geqq f''(p_0)\big(2B(n-1)\big)^{-2}. \tag{2.24}$$

Proof. According to the Taylor formula we can write $f_{n+1}(0) = f(f_n(0)) = f_n(0) + f''(\rho)(1-f_n(0))^2/2$, where $\rho \in (f_n(0), 1)$. And this together with (2.2) and (2.6) gives the following relation:

$$f''(p_0)\big(1-f_n(0)\big)^2/2 \leqq g_{n+1}(1) = f_{n+1}(0) - f_n(0) \leqq B\big(1-f_n(0)\big)^2/2. \tag{2.25}$$

Let $x_n = 1 - f_n(0)$. It is evident that x_n decreases monotonically; hence, we can choose n_0 so that $x_n \geqq 3/(4B)$ for $n \leqq n_0$, and $x_n < 3/(4B)$ for $n > n_0$. Using the first inequality from (2.25), we get $g_{n+1}(1) \geqq \frac{9}{32} f''(p_0)B^{-2}$ for $n \leqq n_0$ and thus (2.24) holds.

Now let $n > n_0$. Because of the second inequality in (2.25) we have $x_{n+1} \geqq x_n - Bx_n^2/2$. If we put $y_n = 1/x_n$, we arrive at the inequality $y_{n+1} \leqq y_n/(1-Bx_n/2)$. Noting that $0 \leqq Bx_n/2 < \frac{3}{8}$ and using the inequality $(1-x)^{-1} \leqq 1+(1-x)^{-2}x$, $x \geqq 0$, we get $y_{n+1} \leqq y_n + \frac{32}{25}B$. Thus, we have $y_n \leqq y_{n_0} + \frac{32}{25}B(n-n_0)$. Noting that $y_{n_0} \leqq \frac{4}{3}B$, we see that $y_n \leqq \frac{4}{3}Bn$. Hence, we have $1 - f_n(0) \geqq 3/(4Bn)$. This together with (2.25) gives (2.24).

LEMMA 10. *Let* $A = 1$, $0 < c_0 \leqq s \leqq 1$. *Then,*

$$g_n(s) \leqq s^n \exp\big\{-c(c_0)n(1-s)^{1/2}\big\}, \tag{2.26}$$

where $c(c_0) = f''(c_0p_0)\{2p_0/B\}^{1/2}$.

Proof. Using the Lagrange formula, we obtain $g_{n+1}(s) \leqq sf'(u_n(s))g_n(s)$ from (2.4). Since because of (1.5) $g_1(s) < 1$, we can deduce that

$$g_{n+1}(s) \leqq \prod_{k=1}^{n} f'\big(u_k(s)\big) \tag{2.27}$$

by means of induction. According to the Lagrange formula, $f'(u_k(s)) = 1-f''(\rho_k)(1-u_k(s))$, $k \geqq 1$, where $\rho_k \in (u_k(s), 1)$. Consequently,

$$f'\big(u_k(s)\big) \leqq 1 - f''(c_0p_0)\big(1-u_k(s)\big). \tag{2.28}$$

In view of (2.20) and (2.23), $1-u_k(s) \geqq (2B^{-1}p_0(1-s))^{1/2}$. Applying consecutively (2.27), (2.28), and the previous inequality, we obtain the inequality

$$g_{n+1}(s) \leqq s^n\Big\{1-f''(c_0p_0)\big(2B^{-1}p_0(1-s)\big)^{1/2}\Big\}^n,$$

from which (2.26) follows.

In what follows we shall use the notations $a = a(s) = sf'(u^*)$, $b = b(s) = sf''(u^*)/2$.

LEMMA 11. *Let* $0 < c_0 \leqq s \leqq 1$. *Then,*

$$u^8 - u_n(s) \leqq \bar{c}(c_0)n^{-1}. \tag{2.29}$$

Proof. Expanding the right-hand side of (2.1) into a Taylor series, we get

$$u_{n+1}(s) = s\big(f(u^8) - f'(u^*)\big)\,\big(u^* - u_n(s)\big) + f''(\rho_n)\,\big(u^* - u_n(s)\big)^2/2,$$

where $\rho_n \in (u_n(s),\, u^*)$. We put $x_n(s) = u^* - u_n(s)$ and, taking into account (2.20) and (1.3), we obtain the inequality

$$0 \leqq x_{n+1}(s) \leqq ax_n(s) - \bar{c}(c_0)x_n^2(s).$$

The change of variables $y_n(s) = 1/x_n(s)$, easily yields the inequality

$$y_{n+1}(s) \geqq y_n(s)/a + \bar{c}(c_0)\, y_{n+1}(s)/ay_n(s). \tag{2.30}$$

Hence, we get $y_{n+1}(s) \geqq y_n(s)/a$. Substituting the lower bound for $y_{n+1}(s)$ obtained from the last inequality into (2.30), we establish that $y_{n+1}(s) \geqq y_n(s)/a + \bar{c}(c_0)/a^2$. Using induction this yields $y_{n+1}(s) \geqq \bar{c}(c_0)\sum_{k=2}^{n+1} a^{-k}$. Since because of (2.19), $f'(u^*) \leqq f'(\lambda) \leqq 1$ we have $a \leqq 1$. Hence $y_{n+1}(s) \geqq \bar{c}(c_0)n$. Now returning to the notations used before, we obtain (2.29).

LEMMA 12. *Let* $\pi_k(A,s) \geqq 0$ *and*

$$\limsup_{\sigma_2 \to 0}\Big\{\sup\big(\pi_k(A,\, s)\colon\; F \in K_A\big)\Big\} < \infty$$

for any

$$\lim_{\sigma \to 0}\left\{\inf\left(\sum_{k=0}^{n}\pi_k(A,\, s)\colon\; F \in K_A\right)\right\} < \infty.$$

Then,

$$\sum_{k=0}^{n}\pi_k(A,s)\,Q_k(s) = \left(\sum_{k=0}^{n}\pi_k(A,s)\right)Q_n(s).$$

Proof. Let $q_n(s) = \sup\{|Q_k(s)|\colon\; k > n\}$. For any $m < n$ we have

$$\left|\sum_{k=0}^{n}\pi_k(A,s)Q_k(s)\left(\sum_{k=0}^{n}\pi_k(A,s)\right)^{-1}\right| \leqq c\sum_{k=0}^{m}\pi_k(A,s)\left(\sum_{k=0}^{n}\pi_k(A,s)\right)^{-1} + q_m(s).$$

It is not difficult to see that for any fixed m the first summand vanishes as $\sigma \to 0$ uniformly with respect to $F \in K_A$. On the other hand, $\sup\{q_n(s)\colon F \in K_A\} \to 0$ as $\sigma \to 0$. This means that the right-hand side of the inequality vanishes as $\sigma \to 0$ uniformly with respect to $F \in K_A$ which had to be proved.

It is not difficult to see that, in view of condition (B), $f''(x) = f''(x_0) + O(H(x - x_0))$ as $x - x_0 \to 0$, where $H(x)$ is a function (depending on F) such that $\sup\{H(x)\colon F \in K\} \to 0$ as $x \to 0$. For $0 < c_0 \leqq x_0 \leqq 1$, using (1.3), we get

$$f''(x) = f''(x_0)\big(1 + O\big(H(x - x_0)\big)\big). \tag{2.31}$$

According to the Taylor formula we obtain from the last relation

$$f'(x) = f'(x_0) + f''(x_0)(x - x_0)\left(1 + O(H(x - x_0))\right), \tag{2.32}$$

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) \\ &\quad + f''(x_0)(x - x_0)^2 \left(1 + O\big(H(x - x_0)\big)\right)/2. \end{aligned} \tag{2.33}$$

Note that, because of (2.19), (2.20) and condition (C), we have $\lambda \geqq u^* \geqq c$ for $0 < c_0 \leqq s \leqq 1$. Hence, formulas (2.31)–(2.33) hold for $x_0 = u^*$ and $x_0 = \lambda$.

LEMMA 13. *Let $A > 1$. Then,*

$$1 - \lambda = 2B^{-1}(A - 1)(1 + Q). \tag{2.34}$$

Proof. Applying the Taylor formula we get $\lambda = f(\lambda) = 1 - A(1 - \lambda) + f''(\rho)(1 - \lambda)^2/2$, where $\rho \in (\lambda, 1)$. Hence, $1 - \lambda \leqq 2(A - 1)/f''(\lambda)$. Using (1.3) we obtain

$$1 - \lambda = Q. \tag{2.35}$$

Because of (2.33), (2.35) we have $\lambda = 1 - A(1 - \lambda) + B(1 - \lambda)^2(1 + Q)/2$. This yields (2.34).

COROLLARY 2. *Let $A > 1$. Then,*

$$1 - A_0 = (A - 1)\,(1 + Q). \tag{2.36}$$

Proof. From (2.32) and (2.35) it follows that $A_0 = A - B(1 - \lambda)(1 + Q)$. Applying (2.34), we obtain (2.36).

With the help of (2.36) it is not difficult to obtain

COROLLARY 3. 1) *If $i = 1$, then*

$$\limsup_{\sigma_1 \to 0} \left\{ \sup\left(n(1 - A_0)\colon\ F \in K_A\right) \right\} \leqq c,$$

$$\liminf_{\sigma_1 \to 0} \left\{ \inf\left(n(A_0^n)\colon\ F \in K_A\right) \right\} > 0.$$

2) *If $i = 2$, then*

$$\lim_{\sigma_1 \to 0} \left\{ \inf\left(n(1 - A_0^n)\colon\ F \in K_A\right) \right\} = \infty,$$

$$\lim_{\sigma_1 \to 0} \left\{ \sup\left(n(A_0^n)\colon\ F \in K_A\right) \right\} = 0.$$

LEMMA 14. *The following representation takes place:*

$$u^* - u_n(s) = \frac{1 - a}{b(a^{-n} - 1)}\big(1 + Q_n(s)\big). \tag{2.37}$$

Proof. Because of (2.1), (2.33), and (2.29), we have

$$u_{n+1}(s) = s(f(u^*) - f'(u^*)(u^* - u_n(s)) + f''(u^*)(u^* - u_n(s))^2(1 + Q_n)/2). \tag{2.38}$$

Putting $x_n(s) = u^* - u_n(s)$, $y_n(s) = 1/x_n(s)$, we obtain

$$x_{n+1}(s) = ax_n(s) - bx_n^2(s)(1 + Q_n), \tag{2.39}$$

and hence,

$$y_n(s) = ay_{n+1}(s) - \big(bx_n(s)/x_{n+1}(s)\big)(1 + Q_n). \tag{2.40}$$

According to the Lagrange formula we have $a = s(A_0 - f''(\rho)(\lambda - u^*))$, where $\rho \in (u^*, \lambda)$. Hence, taking into account (1.2), (2.21), and (2.36) we establish that $a = 1 + Q(s)$. On the other hand, by virtue of (2.39),

$$x_n(s)/x_{n+1}(s) = a^{-1}\left\{1 + \left(bx_n^2(s)/x_{n+1}(s)\right)(1 + Q_n)\right\}.$$

This yields $x_n(s)/x_{n+1}(s) = 1 + Q_n(s)$ on account of (2.29) and (1.2). Hence, (2.40) admits the following representation:

$$y_{n+1}(s) = a^{-1}y_n(s) + b\big(1 + Q_n(s)\big). \tag{2.41}$$

By induction we obtain

$$y_{n+1}(s) = ba^{-n}\sum_{k=0}^{n} a^k\big(1 + Q_k(s)\big) + y_0(s)\,a^{-(n+1)},$$

where $y_0(s) = (u^*)^{-1} \leqq \bar{c}$ because of (2.20). Using Lemma 12 (for $\pi_k(A, s) = a^k$) we come to the conclusion that $y_{n+1}(s) = b(\sum_{k=0}^{n} a^{-k})(1 + Q_n(s))$. Returning to the old notations, we obtain (2.37).

If we put $s = 1$ in (2.37) and take into account (2.6), we obtain the following result.

COROLLARY 4. *The representation*

$$\lambda - f_n(0) = \frac{2(1 - A_0)}{B_0(A_0^{-n} - 1)}(1 + Q_n) \tag{2.42}$$

holds.

LEMMA 15. *The representation*

$$g_n(s) = \frac{a^n}{b}\left\{\frac{1 - a}{1 - a^n}\right\}^2(1 + q_n(s)) \tag{2.43}$$

holds.

Proof. Substituting $u_{n+1}(s)$ from (2.38) into (2.2), we get

$$g_{n+1}(s) = (1 - a)\big(u^* - u_n(s)\big) + b\big(u^* - u_n(s)\big)^2(1 + Q_n).$$

Applying (2.37) we deduce that

$$g_{n+1}(s) = \left\{a^n(1 - a)^2/b(1 - a^n)^2\right\}\big(1 + Q_n(s) + a^nQ_n(s)\big).$$

Hence (2.43) follows, since $a \leqq 1$ (see the proof of Lemma 11).

LEMMA 16. *Let* $i = 1$. *Then,*

$$u_n'(1) = \left(2e^{\theta}(\sinh\theta - \theta)/\theta(1-e^{\theta})^2\right) n(1+Q_n), \tag{2.44}$$

where $\theta = n \log A_0$.

Proof. In view of (2.42), (2.36), and (1.3),

$$\lambda - f_n(0) = Q_n. \tag{2.45}$$

From (2.32) it follows that

$$f'(f_n(0)) = A_0 + B_0(\lambda - f_n(0))\left\{1 + O(H(\lambda - f_n(0)))\right\}.$$

Using (2.42), (2.45), (2.36), this yields

$$f'(f_n(0)) = A_0(1 - (2(1-A_0)/(A_0^{-n}-1))(1+Q_n)). \tag{2.46}$$

Combining (2.13) with (2.46) we get

$$\Pi_k \equiv \prod_{l=k}^{n-1} f'(f_l(0)) = A_0^{n-k} \exp\left\{-2(1+Q_k)\sum_{l=k}^{n-1} \frac{1-A_0}{A_0^{-l}-1}\right\}. \tag{2.47}$$

Let us use the formula $\int_l^{l+1} g(x)\,dx = g(l) + g'(\rho)/2$, $\rho \in (l, l+1)$. We put $g(x) = (1-A_0)/(A_0^{-x}-1)$. Taking into account Corollary 3, it is not difficult to see that $g'(\rho) = g(l)Q_l$. Hence,

$$\sum_{l=k}^{n-1} g(l) = (1+Q_k)\int_k^n g(x)\,dx. \tag{2.48}$$

After the change of variables $A_0 = \exp(\theta/n)$ has been made in (2.47), the last relation yields the representation

$$\Pi_k = \exp\left\{\theta(1-k/n) + 2\log\left\{(1-\exp(\theta k/n))/(1-e^{\theta})\right\}(1+Q_k)\right\}.$$

Now using the inequality $k/n \leqq (1-\exp(\theta k/n))/(1-e^{\theta}) \leqq 1$, we conclude that for any $\varepsilon > 0$ we have, for $k > n\varepsilon$,

$$\Pi_k = \sinh^{-2}(\theta/2)\sinh^2(\theta k/(2n))(1+Q_k). \tag{2.49}$$

Because of (2.5) we can write

$$u_n'(1) - f_n(0) = \sum_{k>n\varepsilon} \Pi_k f_k(0) + \sum_{k\leqq n\varepsilon} \Pi_k f_k(0) \equiv \Sigma_1 + \Sigma_2. \tag{2.50}$$

Since (2.35) and (2.45) hold, we have

$$f_n(0) = 1 + Q_n. \tag{2.51}$$

It follows from (2.49) and (2.51) that

$$\Sigma_1 = \sinh^{-2}(\theta/2)\sum_{k>n\varepsilon}\sinh^2(\theta k/(2n))(1+Q_n).$$

Let us put $g(x) = \sinh^2(\theta x/(2n))$. It is not difficult to see that $g'(\rho) = g(l)Q_l$. Hence, formula (2.48) holds. Taking into account the fact that $\int \sinh^2 x\,dx = \frac{1}{4}\sinh(2x) - x/2$, we obtain the following equality:

$$\Sigma_1 = \sinh^{-2}(\theta/2)\left\{\theta^{-1}\big(\sinh\theta - \sinh(\theta\varepsilon)\big) - (1-\varepsilon)\right\} n(1+Q_n n)/2. \tag{2.52}$$

Now let us estimate Σ_2. Since $f'(f_k(0)) \leqq 1$ and $f_k(0) \leqq 1$, we have $\Pi_k f_k(0) \leqq 1$. From this we deduce

$$\Sigma_2 \leqq n_\varepsilon. \tag{2.53}$$

Relations (2.50)–(2.53) allow us to complete the proof of the lemma.

LEMMA 17. *Let* $i = 2$. *Then*,

$$u_n'(1) = \frac{\lambda}{1 - A_0} - 2nA_0^n(1+Q_n). \tag{2.54}$$

Proof. Our considerations proceed from (2.50). From Corollary 3 it follows that under the assumptions of the lemma we have $1/(A_0^{-n} - 1) = A_0^n(1+Q_n)$. Hence, because of (1.3) and (2.36) we can rewrite the representation (2.42) in the form

$$f_n(0) = \lambda + A_0^n Q_n. \tag{2.55}$$

In addition, summing over the right-hand side of relation (2.47), we obtain the following for any $\varepsilon > 0$, when $k > n\varepsilon$:

$$\Pi_k = A_0^{n-k}\exp\left\{-2A_0^k(1 - A_0^{n-k})(1+Q_n)\right\}.$$

Thus it follows that

$$\Pi_k = A_0^{n-k} - 2A_0^n + 2A_0^{2n-k} + A_0^n Q_n.$$

Combining (2.55) with the last representation, we deduce that

$$\Sigma_1 = \frac{\lambda A_0}{1 - A_0} - 2\lambda n(1-\varepsilon)A_0^n + O\left\{\frac{A_0^{n(1-\varepsilon)}}{1 - A_0}\right\} + n(1-\varepsilon)A_0^n Q_n.$$

On account of (2.47), we have $\Pi_k \leqq \bar{c}A_0^{n-k}$. Hence, $\Sigma_2 \leqq \bar{c}n\varepsilon A_0^{n(1-\varepsilon)}$. Now, applying (2.50), (2.55), (2.35) and Corollary 3 we obtain (2.54).

Let us set, for any $A_0 < 1$, $W = 2B_0(1-s)/(1-A_0)^2$, $V = (1+W)^{1/2}$.

LEMMA 18. *The following representation holds*:

$$\lambda - u^* = \frac{(1-A_0)(V-0)}{B_0}\big(1 + Q(s)\big). \tag{2.56}$$

Proof. Since u^* is a solution of (2.18), we have, in view of (2.33),

$$u^* = s\left\{\lambda - A_0(\lambda - u^*) + \frac{1}{2}B_0(\lambda - u^*)^2\big(1 + O\big(H(\lambda - u^*)\big)\big)\right\}.$$

Hence, we obtain

$$B_0(\lambda - u^*)^2\Big(1 + O\big(H(\lambda - u^*)\big)\Big)\Big/2 + (1 - sA_0)(\lambda - u^*) - \lambda(1-s) = 0. \tag{2.57}$$

Since $1 - sA_0 = 1 - A_0 + A_0(1-s)$, using (1.2), (2.21), and (2.35) we can rewrite the equation (2.57) in the following form:

$$B_0(\lambda - u^*)^2/2 + (1 - A_0)(\lambda - u^*) - (1-s)\big(1 + Q(s)\big) = 0.$$

Since (2.19) holds, $\lambda - u^*$ is a positive solution of this quadratic equation. This solution is unique and has the following form:

$$\lambda - u^* = \frac{1 - A_0}{B_0}\Big\{\Big(1 + W\big(1 + Q(s)\big)\Big)^{1/2} - 1\Big\}. \tag{2.58}$$

Using the equality $1 + W(1+Q(s)) = (1+W)\{1 + (W/(1+W))Q(s)\}$ and the expansion $(1+x)^{1/2} = 1 + x(1 + o(1))/2$, we obtain

$$\Big(1 + W\big(1 + Q(s)\big)\Big)^{1/2} - 1 = V - 1 + (W/V)\,Q(s).$$

It is easy to see that the inequality $W/V(V-1) \leqq c$ holds. Hence, the representation (2.58) can be written in the form (2.56).

LEMMA 19. *Let* $-\theta(V-1) \leqq c$ *and* $-\theta V \geqq c$. *Then,*

$$g_n(s) = \frac{2\exp(\theta V)(1 - A_0)^2 V^2 (1 + Q_n(s))}{B_0\big(1 - \exp(\theta V)\big)^2}. \tag{2.59}$$

Proof. Combining (2.32), (2.56), and (2.21) we get $f'(u^*) = A_0 - (1 - A_0)(V - 1)(1 + Q(s))$. Hence,

$$a = A_0 - (1 - A_0)(V-1)\big(1 + A_0(1-s)/(1-A_0)(V-1) + Q(s)\big). \tag{2.60}$$

Let us point out that

$$\frac{1-s}{(1-A_0)(V-1)} = \left(\frac{1-s}{2B_0}\right)^{1/2}\frac{W^{1/2}}{V-1} = \frac{1-A_0}{2B_0}\frac{W}{V-1}.$$

If $W \geqq c$, then $W^{1/2}/(V-1) \leqq \bar{c}$, and if $W \leqq \bar{c}$, then $W/(V-1) \leqq \bar{c}$. So, taking into account (1.3) and (2.36), we obtain $(1-s)/(1-A_0)(V-1) = Q(s)$. This together with (2.60) yields

$$a = A_0 - (1 - A_0)(V-1)\big(1 + Q(s)\big). \tag{2.61}$$

Because of (2.36) we have $-(1 - A_0)/A_0 = \log A_0(1 + Q)$. Thus it follows from (2.61) that $a = A_0(1 + (V-1)\log A_0(1 + Q(s)))$. Using the first condition of the lemma we obtain

$$\log\big\{1 + (V-1)\log A_0\big(1 + Q(s)\big)\big\} = (V-1)\log A_0\big(1 + Q_n(s)\big).$$

Hence, $a^n = A_0^n \exp\{\theta(V-1)(1 + Q_n(s))\}$. Using the first condition of the lemma once again, we come to the conclusion that

$$a^n = \exp(\theta V)\big(1 + Q_n(s)\big). \tag{2.62}$$

Hence, using the second condition, we deduce that

$$1 - a^n = \big(1 - \exp(\theta V)\big)\,\big(1 + Q_n(s)\big). \tag{2.63}$$

Combining (2.31) and (2.21) we get

$$b = B_0\bigl(1 + Q(s)\bigr)/2. \tag{2.64}$$

It follows from (2.61) that

$$1 - a = (1 - A_0)V\Bigl(1 + \frac{V-1}{V}Q(s)\Bigr) = (1 - A_0)V\bigl(1 + Q(s)\bigr).$$

Substituting (2.62)–(2.64) and the last representation into (2.43) we obtain (2.59).

Let $W \to \infty$ as $\sigma_2 \to 0$. We point out that, since (1.2), (1.3), and (2.36) hold, this convergence is uniform with respect to $F \in K_A$ and

$$(1 - s)/(1 - A_0)^2 = Q(s). \tag{2.65}$$

In addition, using the expansion $(1 + x)^{1/2} = 1 + x(1 + O(x))/2$ we obtain

$$V - 1 = W\bigl(1 + Q(s)\bigr)/2. \tag{2.66}$$

COROLLARY 5. *If $i = 2$, the conditions of Lemma 19 are satisfied, and $W \to 0$ as $\sigma_2 \to 0$, then*

$$g_n(s) = \frac{2}{B_0}(1 - A_0)^2 A_0^n \exp\left(\frac{\theta W}{2}\right)\bigl(1 + Q_n(s)\bigr). \tag{2.67}$$

Proof. We deduce from (2.66) and the first condition of Lemma 19 that $\exp(\theta V) = A_0^n \exp(\theta W/2)(1 + Q(s))$. In addition, $\exp(\theta W/2) \leqq 1$. Taking into account that, because of Corollary 3 $A_0^n = Q_n$, we can write the representation (2.59) in the form (2.67).

LEMMA 20. *Let $i = 2$ and $L < \bar{c}$. If $W \to 0$ as $\sigma_2 \to 0$, then*

$$g_n(s) = 2B_0^{-1}(1 - A_0)^2 a^n\bigl(1 + Q_n(s)\bigr), \tag{2.68}$$

$$a = A_0\bigl(1 - (1 + \gamma)z + \beta z^2\bigl(1 + Q(s)\bigr)\bigr), \tag{2.69}$$

where $s = e^{-z}$, $\beta = B_0^2/2(1 - A_0)^3$.

Proof. In view of (2.56) and (2.66), we have

$$\lambda - u^* = \frac{(1 - A_0)W}{2B_0}\bigl(1 + Q(s)\bigr) = \frac{1 - s}{1 - A_0}\bigl(1 + Q(s)\bigr). \tag{2.70}$$

On the other hand, taking into account (2.21) we rewrite (2.57) in the following way:

$$(1 - A_0)(\lambda - u^*) = \lambda(1 - s) = B_0(\lambda - u^*)^2\bigl(1 + Q(s)\bigr)/2 - A_0(1 - s)(\lambda - u^*).$$

Substituting the representations for $\lambda - u^*$ from (2.70) into the right-hand side of the last relation, and applying (1.3), (2.36), we obtain

$$\lambda - u^* = \frac{\lambda(1 - s)}{1 - A_0} - \frac{B_0(1 - s)^2}{2(1 - A_0)^3}\bigl(1 + Q(s)\bigr). \tag{2.71}$$

Since $L_0 \leqq L < \bar{c}$, we have, according to the Taylor formula,

$$f'(u^*) = A_0 - B_0(\lambda - u^*) + f'''(\rho)(\lambda - u^*)^2/2, \tag{2.72}$$

where $\rho \in (u^*, \lambda)$, $f'''(\rho) \leqq \bar{c}$. Using (2.71), (1.3), (2.36), and (2.65) we deduce that $(\lambda - u^*)^2 = (B_0^2(1-s)^2/(1-A_0)^3)Q(s)$. Combining (2.72), (2.71) and the previous representation we conclude that

$$f'(u^*) = A_0 - \frac{\lambda B_0(1-s)}{1-A_0} + \frac{B_0^2(1-s)^2}{2(1-A_0)^3}(1+Q(s)).$$

Hence, we have

$$a = A_0\big(1 - (1+\gamma)(1-s) + (\beta/A_0 + \gamma)(1-s)^2(1+Q(s))\big).$$

It is not difficult to see that $z = Q^0(s)$ and $1 - s = z - z^2(1+Q^0(s))/2$. Thus, we can write

$$a = A_0\left(1 - (1+\gamma)z + \left(\beta/A_0 + \frac{1}{2}(3\gamma+1)\right)z^2\big(1+Q(s)\big)\right).$$

Taking into account (1.3) and (2.36) we obtain (2.69).

Now let us deduce (2.68). To do this, we shall use (2.43). Owing to (2.61) and Corollary 3, we have $a^n = Q_n(s)$. From (2.61) and (2.66) it follows that $1 - a = (1-A_0)(1+Q(s))$. Taking all this into account as well as (2.64), we obtain (2.68).

LEMMA 21. *Let $A = \text{const} \neq 1$. Then,*

$$g_n(s) = R(1-a)\,a^{n-1}\big(1+Q_n^0(s)\big), \tag{2.73}$$

where R is the same as in (2.15).

Proof. Let us use the considerations we carried out in deducing formula (2.41) in the proof of Lemma 14. We go out from the representation (2.38), where Q_n is substituted for Q_n^0. Let us also point out that it is not necessary to apply (1.2) and (2.36). As a result we get $y_{n+1}(s) = y_n(s)/a + (b/a^2)(1+Q_n^0(s))$ (in the same notations) and moreover, we can write

$$a = A_0 + Q^0(s). \tag{2.74}$$

Making the change of variable $w_n(s) = a^n y_n(s)$, we obtain the relation $w_{n+1}(s) = w_n(s) + ba^{n-1}(1+Q_n^0(s))$. Hence, $w_{n+1}(s) = b\sum_{k=1}^{n-1} a^k(1+Q_k^0(s))$. Since $\sum_{k=1}^{\infty} a^k = (1-a)^{-1} < \infty$, we have $w_n(s) = c_0(1+Q_n^0(s))$, where c_0 is a constant. Returning to the notations introduced before let us rewrite the last relation in the form

$$u_n(s) = u^* - c_0 a^n\big(1+Q_n^0(s)\big). \tag{2.75}$$

Putting $s = 1$ and using (2.6) and the equality $a(1) = A_0$, we see that $f_n(0) = \lambda - c_0 A_0^n(1+Q_n^0)$. Comparing the relation obtained with (2.15) we conclude that $c_0 = R$. Now combining (2.2) with (2.75) we get (2.73).

LEMMA 22. *Let $A = \text{const} \neq 1$, $n(1-s) \leqq c$. Then,*

$$g_n(s) = R(1-A_0)A_0^{n-1}\exp\{-(1+\gamma)n(1-s)\}\big(1+Q_n^0(s)\big). \tag{2.76}$$

Proof. Since u^* is a solution of (2.18), we have $u^* = s(\lambda - A_0(\lambda - u^*) + O((\lambda - u^*)^2))$ according to the Taylor formula, and hence, $(1 - sA_0)(\lambda - u^*) = \lambda(1-s) + O((\lambda - u^*)^2)$. Therefore, owing to (2.21), we have $\lambda - u^* = O(1-s)$. As a result we obtain

$$\lambda - u^* = \lambda(1-s)/(1-A_0) + O\big((1-s)^2\big). \tag{2.77}$$

From the representation (2.77) and the expansion $f'(u^*) = A_0 - B_0(\lambda - u^*)(1+o(1))$ as $u^* \to \lambda$ it follows that $a = A_0(1-(1+\gamma)(1-s)(1+Q^0(s)))$. Further, using the inequality from the conditions of the lemma we deduce that $a^n = A_0^n \exp\{-(1+\gamma)n(1-s)\}(1+Q_n^0(s))$. Combining (2.73), (2.74) and the last representation we obtain (2.76).

LEMMA 23. *Let $A = \text{const} \neq 1$ and $\lambda < \infty$, if $A < 1$. Then,*

$$g_n(s) = RA_0^{-1}(1-A_0)a^n\big(1+Q_n^0(s)\big), \tag{2.78}$$

$$a = A_0\Big(1-(1+\gamma)z + \big(\beta + (3\gamma+1)/2\big)z^2\big(1+Q^0(s)\big)\Big), \tag{2.79}$$

where $s = e^{-z}$, $\beta = (\lambda B_0(2+\gamma) + \lambda^2 L_0/A_0)/2(1-A_0)^2$.

Proof. As in the case $A \to 1$, the representation (2.57) holds, where $H(\lambda - u^*) = Q^0(s)$ by virtue of (2.21). Let us rewrite it in the form

$$(1-A_0)(\lambda-u^*) = \lambda(1-s) - b_0(\lambda-u^*)^2\big(1+Q^0(s)\big)/2 - A_0(1-s)(\lambda-u^*).$$

Hence, in view of (2.77) we obtain

$$\lambda - u^* = \frac{\lambda(1-s)}{1-A_0} - \frac{\lambda A_0\,(1+\gamma/2)\,(1-s)^2}{(1-A_0)^2}\big(1+Q^0(s)\big). \tag{2.80}$$

According to the Taylor formula we get $f'(u^*) = A_0 - B_0(\lambda-u^*) + \frac{1}{2}L_0(\lambda-u^*)^2(1+o(1))$ as $u^* \to \lambda$. Now applying (2.80) we see that the relation $f'(u^*) = A_0(1-\gamma(1-s)+\beta(1-s)^2(1+Q^0(s)))$ holds. Consequently, we deduce that

$$a = A_0(1-(1+\gamma)(1-s) + (\beta+\gamma)(1-s)^2\Big(1+Q^0(s)\big)\Big).$$

Using the equality $z = Q^0(s)$, $1-s = z - z^2(1+Q^0(s))/2$ we obtain (2.79). On account of (2.73), (2.74) we get (2.78).

LEMMA 24. *Let F be a distribution function of random variables. Let it be concentrated on the positive semi-axis and assume $\varphi(t) = \int_0^\infty e^{-ty}dF(y)$, $t \geqq 0$. Then, $F(\nu) \leqq e^{t\nu}\varphi(t)$, $\nu \geqq 0$.*

Proof. It is not difficult to see that $\varphi(t) \geqq \int_0^\nu e^{-ty}dF(y) \geqq e^{-t\nu}F(\nu)$.

Let F be the distribution function of a certain random variable. Let it be defined on the whole real axis. Let us consider the bilateral Laplace transform

$$\psi(t) = \int_{-\infty}^{\infty} e^{-ty}dF(y), \qquad 0 \leqq t < t_0 \leqq \infty, \tag{2.81}$$

where t_0 depends on F and the integral exists for $t \in [0, t_0)$ and does, in general, not exist for $t \notin [0, t_0)$.

LEMMA 25. *Let $0 \leqq \varepsilon < t_0 - t$, $\nu \in (-\infty, \infty)$. Then,*

$$\int_{-\infty}^{\nu} e^{-ty}dF(y) \leqq e^{\varepsilon\nu}\psi(t+\varepsilon).$$

Proof. The definition immediately yields

$$\psi(t+\varepsilon) \geqq \int_{-\infty}^{\nu} e^{-(t+\varepsilon)y}dF(y) \geqq e^{-\varepsilon\nu}\int_{-\infty}^{\nu} e^{-ty}dF(y).$$

LEMMA 26 (Continuity Lemma). *Let F_n, $n \geqq 1$ be a probability distribution with bilateral Laplace transform $\psi_n(t)$ inside the interval $[0, t_0)$. If $\lim_{n\to\infty} \psi_n(t) = \psi(t) < \infty$, $0 < t < t_0$, then $\lim_{n\to\infty} F_n = F$, where F is a probability distribution (which may be degenerate) with the transform $\psi(t)$ inside the interval $[0, t_0)$ (by definition we have $\psi(0) = \lim_{t\to 0} \psi(t)$). The limit F is nondegenerate if and only if $\psi(0) = 1$.*

Proof. According to the selection theorem (see, for example, [16, p. 267]) the sequence $\{F_n\}$ has a subsequence $\{F_{n_k}\}$ which weakly converges to some limit F. By virtue of the convergence theorem [16, p. 51], we can write, for any $\nu \in (-\infty, \infty)$,

$$\lim_{k\to\infty} \int_\nu^\infty e^{-ty} dF_{n_k}(y) = \int_\nu^\infty e^{-ty} dF(y), \qquad 0 < t < t_0. \tag{2.82}$$

Applying Lemma 25 for $\varepsilon = (t_0 - t)/2$ we obtain

$$\int_{-\infty}^\nu e^{-ty} dF_{n_k}(y) \leqq e^{\varepsilon\nu} a\psi_{n_k}(t + \varepsilon),$$

where the right-hand side may be made as small as required by a choice of ν, since the convergence of $\psi_n(t)$, $0 < t < t_0$, is bounded. Together with (2.82) this implies that

$$\lim_{k\to\infty} \psi_{n_k}(t) = \int_{-\infty}^\infty e^{-ty} dF(y), \qquad 0 < t < t_0.$$

On the other hand, according to the condition of the lemma we have $\lim_{n\to\infty} \psi_{n_k}(t) = \psi(t)$ and, hence, $\psi(t)$ is the bilateral Laplace transform of the distribution F.

Since the integral $\int_{-\infty}^\infty e^{-ty} dF(y)$ converges for $0 < t < t_0$, it converges in the strip $0 < \operatorname{Re} t < t_0$ as well and is analytic in this strip according to the theorem on the bilateral Laplace transform (see [17, p. 238, 57]). If there exists a distribution F_0 such that

$$\int_{-\infty}^\infty e^{-ty} dF_0(y) = \int_{-\infty}^\infty e^{-ty} dF(y) \qquad \text{for} \quad 0 < t < t_0,$$

then using the theorem of uniqueness for analytic functions (see, for example, [18, p. 122]), we conclude that the last relation holds for $0 < \operatorname{Re} t < t_0$. Hence, on account of the theorem of uniqueness for a bilateral Laplace transform of a complex variable [17, p. 243], we have $F_0 \equiv F$. Thus, all convergent subsequences converge to the same limit F. Hence, F_n converges to F. Because of (2.81) we have $\psi(0) = F(+\infty)$, and so, $\psi(0) = 1$ is the necessary and sufficient condition for the limit F to be nondegenerate.

3. Main results.

THEOREM 1. *Let $A = 1$ and let the condition (Γ_δ) hold. Then, as $n \to \infty$,*

1)

$$\mathbf{P}(N = n) = (2/(Bn^2))\left(1 + o(n^{-\delta})\right), \tag{3.1}$$

2)

$$\mathbf{E}(S_n;\ N = n) = \frac{1}{3}\left(1 + o(n^{-\delta})\right), \tag{3.2}$$

3)

$$\mathbf{E}(S_n \mid N = n) = \frac{1}{6} Bn^2\left(1 + o(n^{-\delta})\right).$$

THEOREM 2. *Let $A = \text{const} \neq 1$. Then, as $n \to \infty$,* 1)

$$\mathbf{P}(N=n) = R(1-A_0)A_0^{n-1} + R_1(1-A_0^2)A_0^{2(n-1)} + o(A_0^{2n}); \tag{3.3}$$

and if, in addition, the condition (Γ_δ) *holds in the case* $A < 1$, *then*

2)

$$\mathbf{E}(S_n;\ N=n) = R\,(1-A_0)\,(1+\gamma)nA_0^{n-1} + A_0^n\chi_\delta(n), \tag{3.4}$$

3)

$$\mathbf{E}(S_n \mid N=n) = (1+\gamma)n + \chi_\delta(n), \tag{3.5}$$

where R, R_1 *are the same as in* (2.15), *and* $\chi_\delta(n)$ *is defined in Lemma* 6.

From (1.15) it follows that

$$\mathbf{P}(N=n) = g_n(1), \tag{3.6}$$

$$\mathbf{E}\,(S_n;\ N=n) = g_n'(1). \tag{3.7}$$

Proof of Theorem 1. 1) In view of (2.9),

$$f\big(u_n(1)\big) = u_n(1) + B\,\big(1-g\,f_n(0)\big)^2/2 + o\big(1-u_n(1)\big)^2/2 + o\Big(\big(1-u_n(1)\big)^{2+\delta}\Big)$$

as $u_n(1) \to 1$. Substituting this relation into (2.3) for $s=1$ and taking into account (2.6) we obtain

$$g_{n+1}(1) = B\big(1-f_n(0)\big)^2/2 + o\Big(\big(1-f_n(0)\big)^{2+\delta}\Big)$$

as $f_n(0) \to 1$. Now using the representation (2.10) we get the relation $g_n(1) = (2/Bn^2)(1+o(n^{-\delta}))$ which implies (3.1) by virtue of (3.6).

2) Differentiating both sides of the relation (2.3) at the point 1 we obtain

$$g_{n+1}'(1) = f\big(u_n(1)\big) - \Big(1-f'\big(u_n(1)\big)\Big)\,u_n'(1). \tag{3.8}$$

Consequently, applying (2.6), (2.11), (2.12) we get $(1-f'(u_n(1)))\,u_n'(1) = -\frac{2}{3}+o(n^{-\delta})$. On the other hand, because of (2.6) and (2.10) we have $f(u_n(1)) = f_{n+1}(0) = 1 + O(n^{-1})$. Hence, using (3.7) and (3.8) we obtain (3.2).

Proof of Theorem 2. 1) Substituting (2.6) into (2.2) we get $g_{n+1}(1) = f_{n+1}(0) - f_n(0)$. Using now (2.15) we arrive at (3.3).

2) Differentiating both parts of the relation (2.2) at the point 1 we obtain $g_{n+1}'(1) = u_{n+1}'(1) - u_n'(1)$. Substituting now the representation of the right-hand side from (2.17) into the relation obtained, we get (3.4).

Let us put $v(x) = \{f''(p_0x)\}^2 p_0x/(2B)$.

THEOREM 3. *Let* $A=1$. *Then, for any* $0 < c_0 < 1$,

$$\mathbf{P}(S_n < \nu \,|\, N=n) \leqq \frac{4B^2}{f''(p_0)} \exp\left\{\frac{-v(c_0)n^2}{\nu - n} + 2\log n\right\}$$

for $\nu \geqq (1 + (-v(c_0)/\log c_0)^{1/2})n$.

Proof. Due to Lemma 24 we have

$$\mathbf{P}\,\big(S_N < \nu;\ N=n\big) \leqq e^{t\nu} g_n(e^{-t}), \qquad t \geqq 0.$$

Using (2.26) and the inequality $1 - e^{-t} \geqq e^{-t}t$, we obtain

$$g_n(e^{-t}) \leqq e^{-nt} \exp\{ - c(c_0)n(c_0t)^{1/2}\},$$

if $0 < c_0 \leqq e^{-t}$, where $c(c_0) = f''(c_0p_0)(2p_0/B)^{1/2}$. Thus, we can write

$$\mathbf{P}(S_n < \nu;\ N = n) \leqq \exp\{t\nu - nt - c(c_0)n(c_0t)^{1/2}\}.$$

The right-hand side attains its minimum at $t = \min\{c_0(c(c_0)n/2(\nu - n))^2, -\log c_0\}$. Hence, we have

$$\mathbf{P}(S_n < \nu;\ N = n) \leqq \exp\left\{-c_0c^2(c_0)n^2/4(\nu - n)\right\}$$

for $c_0(c(c_0)n/2(\nu - n))^2 \leqq -\log c_0$, i.e., for ν satisfying the condition of the theorem. It remains to apply (3.6) and (2.24).

We set $h(x) = (x(1 + e^x) + 2(1 - e^x))/(e^x - 1)^3$.

Remark 1. It is not difficult to see that $h(0) = \frac{1}{6}$ and $h(x) \geqq c > 0$, $x \geqq 0$.

THEOREM 4. *The following holds as* $\sigma_1 \to 0$:

1)

$$\mathbf{P}(N = n) = (2A_0^n/B_0)\{(1 - A_0)/(1 - A_0^n)\}^2(1 + Q_n), \tag{3.9}$$

2)

$$\mathbf{E}(S_N;\ N = n) = \begin{cases} 2A_0^n h(n\log A_0)(1 + Q_n), & i = 1,\\ 2(1 - A_0)nA_0^n(1 + Q_n), & i = 2,\end{cases} \tag{3.10}$$

3)

$$\mathbf{E}(S_n \mid N = n) = \begin{cases} B_0((1 - A_0^n)/(1 - A_0))^2 h(n\log A_0)(1 + Q_n), & i = 1,\\ (B_0n/(1 - A_0))(1 + Q_n), & i = 2.\end{cases} \tag{3.11}$$

Remark 2. In case $i = 2$,

$$\mathbf{P}(N = n) = \left(2B_0^{-1}(1 - A_0)^2A_0^n\right)(1 + Q_n) \tag{3.12}$$

(see Corollary 3).

Remark 3. Let $\lim_{\sigma_1\to 0} n(1 - A) = 0$. In view of (2.36),

$$A_0^n = \exp\{ - n|1 - A|(1 + Q)\} = 1 - n|1 - A|(1 + Q_n).$$

Using Lemma 12 (for $s = 1$, $\pi_k(A, s) = k$) we obtain

$$\sum_{k=0}^{n-1} A_0^k = n - n^2|1 - A|(1 + Q_n)/2.$$

Hence, $(1 - A_0^n)/(1 - A_0) = n(1 + Q_n)$. Further, using (2.36) and Remark 1, we deduce that $h(n\log A_0) = (1 + Q_n)/6$. As a result, the representation (3.11) (for $i = 1$) takes the following form:

$$\mathbf{E}(S_n \mid N = n) = B_0n^2(1 + Q_n)/6. \tag{3.13}$$

Proof. 1) Putting $s = 1$ in (2.43) and using (3.6) we obtain (3.9).

2) Because of (3.7), (3.8) we have

$$\mathbf{E}(S_n;\ N=n) = f\big(u_n(1)\big) - \Big(1 - f'\big(u_n(1)\big)\Big)u_n'(1). \tag{3.14}$$

Let $i=1$. Consequently, applying (2.6), (2.46), (2.44), and (2.36) we get

$$\begin{aligned}(1 - f'(u_n(1)))\,u_n'(1) &= \Big\{2e^\theta(1+e^\theta)(\sinh\theta - \theta)/(e^\theta - 1)^3\Big\}(1+Q_n)\\ &= \Big(1 - 2e^\theta h(\theta)\Big)(1+Q_n).\end{aligned}$$

Hence, using Remark 1 we deduce that

$$\begin{aligned}\Big(1 - f'(u_n(1))\Big)u_n'(1) &= 1 - 2e^\theta h(\theta)\Big[1 - \{(1-2e^\theta h(\theta))/e^\theta h(\theta)\}Q_n\Big]\\ &= 1 - 2e^\theta h(\theta)(1+Q_n),\end{aligned}$$

where $e^\theta h(\theta) \geqq \bar{c} > 0$ in view of Corollary 3. From (2.6) and (2.51) it follows that $f(u_n(1)) = f_{n+1}(0) = 1 + Q_n$. Now applying (3.14) we obtain (3.10).

Let $i=2$. Using (2.6), (2.46), (2.54), and Corollary 3 we have

$$\Big(1 - f'(u_n(1))\Big)u_n'(1) = \lambda - 2(1-A_0)nA_0^n(1+Q_n),$$

where $(1-A_0)n \to \infty$ as $\sigma_1 \to 0$. Further, owing to (2.6) and (2.55), we can write $f(u_n(1)) = \lambda + A_0^n Q_n$. It remains to use (3.14).

3) For the case $i=1$, (3.11) evidently follows from (3.9) and (3.10). For the case $i=2$, we use Remark 2.

Let $m_n = \mathbf{E}(S_N \mid N=n)$, $\varphi_n(t) = \mathbf{E}(\exp(-tS_n/m_n) \mid N=n)$.

Evidently, we have $m_N = \mathbf{E}(S_N \mid N)$.

To prove subsequent theorems, we shall use the continuity theorems for unilateral (see, for example, [16]) and bilateral (Lemma 26) Laplace transforms.

THEOREM 5. 1) *Let* $\lim_{\sigma_1\to 0} n|1-A| = -\log r < \infty$. *Then,*

$$\lim_{\sigma_1\to 0}\mathbf{P}(S_N/m_N < x \mid N=n) = G(x,r), \qquad x \geqq 0,$$

where

$$\int_0^\infty e^{-ty}\,d_xG(x,r) = g(t,r) \equiv \begin{cases} 3t/\sinh^2((3t)^{1/2}), & r=1,\\ \dfrac{(1-r)^2 r^{d(t,r)}(d(t,r))^2}{r(1-r^{d(t,r)})^2}, & r<1,\end{cases}$$

$$d(t,x) = \Big(1 + 2t/(1-x)^2 h(\log x)\Big)^{1/2}, \qquad 0 \leqq t < \infty.$$

2) *Let* $i=2$. *Then, for any* $\varepsilon > 0$,

$$\lim_{\sigma_1\to 0}\mathbf{P}\big(|S_N/m_N - 1| > \varepsilon \mid N=n\big) = 0. \tag{3.15}$$

In addition, if $L < \bar{c}$, *then*

$$\lim_{\sigma_1\to 0}\mathbf{P}\Big((1-A_0)^{3/2}\big(S_N - (1+\gamma)N\big)/B_0N^{1/2} < x \mid N=n\Big) = \Phi(x), \tag{3.16}$$

where $\Phi(x)$ *is the standard normal distribution.*

Remark 4. It is not difficult to check that $\lim_{r\to 1} g(t,r) = g(t,1)$.

Proof. Let $t > 0$. From (1.5) we obtain

$$\varphi_n(t) = g_n\big(\exp(-t/m_n)\big)/\mathbf{P}(N = n). \tag{3.17}$$

Let $s = \exp(-t/m_n)$. Because of (3.11), (2.36), and (1.3) we have $1/m_n = Q_n$. Hence, we can write

$$1 - s = (t/m_n)(1 + tQ_n). \tag{3.18}$$

Without restricting generality, we can assume that $A \to 1$ but does not attain this value. As in Lemma 19, we shall use the notation $W = 2B_0(1-s)/(1-A_0)^2$, $V = (1+W)^{1/2}$. From (2.36) it follows that

$$n(1 - A_0) = -\theta(1 + Q), \tag{3.19}$$

$$\theta = -n|1 - A|\,(1 + Q). \tag{3.20}$$

1) If $r = 1$, then taking into account (3.18), (3.13) and (2.36), we deduce that $W = 12t(1+tQ_n)/(n(1-A_0))^2 \to \infty$ as $\sigma_1 \to 0$. Hence, in view of (3.19) we obtain $V = (-2(3t)^{1/2}/\theta)(1+tQ_n)$. For $r < 1$, applying (3.18) and (3.11) ($i = 1$), we see that $W = 2t(1+tQ_n)/(1-A_0^n)^2 h(n \log A_0)$. Hence, we can write

$$V = \big(1 + 2t(1+tQ_n)/(1-e^{\theta})^2 h(\theta)\big)^{1/2} = d\big(t(1+tQ_n),\, e^{\theta}\big).$$

It is not difficult to see that the conditions of Lemma 19 are satisfied in both cases. Using (3.17), (2.59) and (3.9) we arrive at the following representation:

$$\varphi_n(t) = \frac{\exp(\theta V)(1-e^{\theta})^2 V^2 (1+Q_n)}{e^{\theta}\,(1-\exp(\theta V))^2}. \tag{3.21}$$

Note that, because of (3.20), $\lim_{\sigma_1\to 0} e^{\theta} = r$. Now substituting the expressions for V obtained above into the right-hand side of (3.21) and passing to the limit as $\sigma_1 \to 0$ we conclude that $\lim_{\sigma_1\to 0}\varphi_n(t) = g(t,r)$. It remains to apply the continuity theorem.

2) Taking into account (3.18), (3.11) (for $i = 2$) and (3.19), it is easy to show that, in the case under consideration, $W = (-2t/\theta)(1+tQ_n) \to 0$ as $\sigma_1 \to 0$ and the conditions of Corollary 5 are satisfied. Combining (3.17), (2.67) and (3.12) we obtain $\varphi_n(t) = \exp(\theta W/2)(1+Q_n)$. Hence, we have $\lim_{\sigma_1\to 0}\varphi_n(t) = e^{-t}$ which is equivalent to (3.15) because of the continuity theorem.

Now let $L < \bar{c}$. With the help of (1.5) we deduce that

$$\mathbf{E}\Big(\exp\big(-\big(S_n - (1+\gamma)N\big)z\big) \,\big|\, N = n\Big) = \frac{\exp\big((1+\gamma)nz\big)g_n(s)}{\mathbf{P}(N = n)}, \tag{3.22}$$

where $s = e^{-z}$. Let $z = t(1-A_0)^{3/2}/B_0 n^{1/2}$. In this case, taking into account (1.3) and (2.36) we get $z = tQ_n$ and $W = (2t/(n|1-A|)^{1/2})(1+tQ_n) \to 0$ as $\sigma_1 \to 0$, i.e., the conditions of Lemma 20 are satisfied. Moreover, we have $\gamma z = tQ_n$. Using (2.69) we deduce that

$$a\exp\big((1+\gamma)z\big) = A_0\Big\{1 + \Big(\beta - \frac{1}{2}(1+\gamma)^2\Big)z^2(1+tQ_n)\Big\},$$

where $\beta = B_0^2/2(1-A_0)^3$. On account of (1.3) and (2.36) we have $(1+\gamma)^2 = \beta Q_n$. Let us also point out that $\beta z^2 = t^2/2n$. Hence, we have

$$a \exp\big((1+\gamma)z\big) = A_0\big(1 + (t^2/2n)(1+tQ_n)\big).$$

Combining the last representation with (2.68) we conclude that

$$\exp\big((1+\gamma)nz\big)\, g_n(s) = \Big(2(1-A_0)^2 A_0^n B_0\Big) \exp(t^2/2)(1+t^3 Q_n). \tag{3.23}$$

Relations (3.22), (3.23), (3.12) and the continuity theorem give us (3.16).

Putting $A = 1$ in Theorem 5 we obtain the following corollary.

COROLLARY 6. *Let* $A = 1$. *Then,*

$$\lim_{n\to\infty} \mathbf{P}(S_N/m_N < x \mid N = n) = G(x), \qquad x \geqq 0,$$

where

$$\int_0^\infty e^{-tx}\, dG(x) = \frac{3t}{\sinh^2\big((3t)^{1/2}\big)}, \qquad 0 \leqq t \leqq \infty.$$

Remark 5. Using [19] it is possible to find the inverse Laplace transform for $t/\sinh^2(t^{1/2})$. It has the following form:

$$g(x) = \frac{4}{\pi^{1/2}} \frac{d}{dx}\left\{\frac{1}{x^{3/2}} \sum_{k=1}^{\infty} k^2 \exp\left(-\frac{k^2}{x}\right)\right\} = \frac{2}{\pi^{1/2}} \frac{d}{dx}\left\{x^{1/2} \frac{d}{dx}\left\{v_3\left(0, \frac{1}{x}\right)\right\}\right\},$$

where $v_3(\alpha, x) = 1 + 2\sum_{k=1}^{\infty} \exp(-xk^2)\cos(2k\alpha)$ is a theta function.

THEOREM 6. *Let* $A = \text{const} \neq 1$. *Then, for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbf{P}\big(|S_N/m_N - 1| > \varepsilon \mid N = n\big) = 0. \tag{3.24}$$

Moreover, if $L < \infty$ *in the case* $A < 1$, *then*

$$\lim_{n\to\infty} \mathbf{P}\Big(\big(S_N - (1+\gamma)N\big)/TN^{1/2} < x \,|\, N = n\Big) = \Phi(x), \tag{3.25}$$

where $\Phi(x)$ *is the standard normal distribution, and*

$$T^2 = \Big(\gamma(1+A_0) + \gamma^2(2A_0 - 1) + \lambda^2 L_0/A_0(1-A_0)\Big)\,/(1-A_0).$$

Proof. Let $t > 0$, $s = \exp(-t/m_n)$. In view of (3.5), $m_n \to \infty$ as $n \to \infty$, and hence, we have

$$1 - s = \frac{t}{m_n}\,(1 + tQ_n^0) = \frac{t}{(1+\gamma)n}(1+tQ_n^0).$$

The conditions of Lemma 22 are satisfied. Combining (3.17), (2.76), and (3.3) we come to the conclusion that $\varphi_n(t) = \exp\{-(1+\gamma)n(1-s)\}\,(1+Q_n^0)$, and hence,

$$\lim_{n\to\infty} \varphi_n(t) = e^{-t}.$$

Applying the continuity theorem we obtain (3.24).

To prove (3.25), we start by considering the representation (3.22), where $s = e^{-z}$. Let $z = t/Tn^{1/2}$. By virtue of (2.79) we can write

$$a \exp\big((1+\gamma)z\big) = A_0\left(1 + \left(\beta + \frac{1}{2}\gamma(1-\gamma)\right)z^2(1+tQ_n^0)\right), \tag{3.26}$$

where β is defined in Lemma 23. It is not difficult to see that $\beta + \gamma(1-\gamma)/2 = T^2/2$. Hence, $(\beta + \gamma(1-\gamma)/2)z^2 = t^2/2n$. Now applying (2.78) and (3.26) we obtain

$$\exp\left((1+\gamma)nz\right)g_n(s) = R(1-A_0)A_0^{n-1}\exp(t^2/2)(1+t^3Q_n^0). \tag{3.27}$$

Relation (3.25) follows from (3.22), (3.27), (3.3) and the continuity theorem.

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