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ON THE RATE OF CONVERGENCE TO NORMAL LAW IN HILBERT SPACE

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(*Translated by Amal Ellis*)

1. Introduction

Let $X_1, X_2, \cdots$ be independent identically distributed random variables (r.v.'s) in a real separable Hilbert space H, $\mathbf{E}X_1=0$, $\beta_3=\mathbf{E}|X_1|^3<\infty$. Here and below, the symbol $|x|$, $x\in H$, denotes the norm in H.

Set $\sigma^2=\mathbf{E}|X_1|^2$. Let Λ denote the covariance operator of the r.v. X_1. Let $\sigma_1^2\geqq\sigma_2^2\geqq\cdots\geqq\sigma_k^2\geqq\cdots$ denote the eigenvalues of Λ.

Let $Z_1, Z_2, \cdots$ take values in H, be independent and have Gaussian distribution with covariance operator Λ, where $\mathbf{E}Z_1=0$.

The goal of the present paper is estimation of the distance

$$\Delta_n(a)=\sup_r\left|\mathbf{P}\left(\sum_1^n X_i/\sqrt{n}\in S(a,r)\right)-\mathbf{P}\left(\sum_1^n Z_i/\sqrt{n}\in S(a,r)\right)\right|,$$

where $S(a, r)$ is the sphere $\{x: |x+a|\leqq r, x\in H\}$.

This problem has been taken up by many authors (see [1]-[18]). The estimates improved steadily; however, for a long time it was unclear whether the estimate $\Delta_n(a)=O(1/\sqrt{n})$ was valid—the best one in the sense of dependence on n. Such an estimate was first derived in 1978 (see [12]) under the condition $H=l_2$ (this of course is no loss in generality), $a=0$, while the coordinates of the random vector X_1 are mutually independent. More precisely, the estimate derived in [12] has the form

$$\Delta_n(0)<c\left(\beta_3\Big/\left(\prod_1^4\sigma_j\right)^{3/4}+\sum_1^4\mathbf{E}|\xi_j|^3/\sigma_j^3\right)\Big/\sqrt{n},$$

where ξ_j is the jth coordinate of X_1, $\sigma_j^2=\mathbf{E}|\xi_j|^2$, and c is an absolute constant.

In [13], [16] this estimate is generalized to the case when the first 7 coordinates are independent of the others and $a\neq 0$.

As for the general case, in 1979 Götze [14] obtained the estimate $\Delta_n(0)<c(\beta_6/\sigma_{31}^6)^3n^{-1/2}$, where $\beta_6=\mathbf{E}|X_1|^6$.

Finally, in 1981, Yurinskii [17], combining the methods of [13] and [14], obtained the estimate

$$(1)\qquad \Delta_n(a)<c(\Lambda)\beta_3(1+|a|^3/\sigma^3)n^{-1/2},$$

where $c(\Lambda)$ is a constant depending only on Λ.

A somewhat less accurate estimate $O(n^{-1/2}\log n)$ was announced in [18].

The drawback of estimate (1) is the fact that the explicit dependence of $c(\Lambda)$ on Λ is not indicated in it.

In the present paper we study the dependence of $\Delta_n(a)$ on the covariance operator of the r.v. X_1 truncated on a certain level L.

Let the operator $\Lambda_0(L)$ be defined by the relation

$$(\Lambda_0(L)x, x)=\mathbf{E}\{(X_1-\mathbf{E}\{X_1\|X_1|\leqq L\}, x)^2\|X_1|\leqq L\}.$$

Set $\Lambda(L)=p_L\Lambda_0(L)$, where $p_L=\mathbf{P}(|X_1|\leqq L)$. Let $\sigma_1^2(L)\geqq\sigma_2^2(L)\cdots\geqq\sigma_k^2(L)\geqq\cdots$ be the eigenvalues of the operator $\Lambda(L)$. We shall now formulate the main result.

Theorem. *There exists an absolute constant c such that, for* $0<L<\infty$,

$$\Delta_n(a)\leqq c\left(\beta_3\Big/\left(\prod_1^7\sigma_j(L)\right)^{6/7}+L/\sigma^2\sigma_1\sigma_2\right)(\sigma^3+|a|^3)n^{-1/2}.$$

The proof is based on the method of estimating the characteristic function of the r.v. $\sum_1^n X_j$ suggested in [14].

Now let us agree on some notation to be used throughout the paper. We shall denote one-dimensional r.v.'s by ξ, η, ζ, $\cdots$, other r.v.'s by X, Y, Z, W, $\cdots$. If X is a r.v., then X' denotes an independent copy of X, while X^s denotes the symmetrization of X, i.e., $X^s=X-X'$. The symbol η_0 denotes the r.v. with density $p(u)=c_0((60/u)\sin(u/60))^{60}$, where c_0 is a normalizing factor. It is important for us that $\mathbf{E}\,e^{it\eta_0}=0$ for $|t|\geqq 1$.

Let $\chi(t)$ denote the indicator of the set $\{t\colon |t|\leqq 1\}$. For any r.v. X, set $X(L)=\chi(|X|/L)X$. The symbol c stands for various absolute constants. We shall also use Vinogradov's symbol $\ll$ ($A\ll B$ if there exists an absolute constant c such that $|A|\leqq cB$). If a constant depends on some parameter α, then we shall use the notation $c(\alpha)$. Whenever the symbol O is used, the corresponding constant is absolute.

2. Ancillary Results

First of all we shall prove a special variant of a smoothing lemma.

Lemma 1. *For* $i=1,2$, *let* X, Y_i, η *be mutually independent and* $Z_i(a)=X+Y_i+a$, $V_i(a,v)=|Z_i(a)|^2+(\eta-v)|X|$, $F_i(r,a)=\mathbf{P}(|Z_i(a)|^2<r)$, $G_i(r,a,v)=\mathbf{P}(V_i(a,v)<r)$, $\Delta=\sup(|F_1(r,a)-F_2(r,a)|/(\beta+|a|^3))$,

$$\Delta(v)=\sup_{r,a}(|G_1(r,a,v)-G_2(r,a,v)|/(\beta+|a|^3)),$$

$$\delta=\sup_{r,a}(|\mathbf{P}(|Y_1+a|^2<r)-\mathbf{P}(|Y_2+a|^2<r)|/(\beta+|a|^3)),$$

where $\beta=\mathbf{E}|X|^3$. *Then* $(\forall\varepsilon>0)$

$$\Delta<\alpha^{-1}\max[\Delta(\varepsilon),\Delta(-\varepsilon)]+8\delta\alpha^{-1}(1-\alpha)+2q\varepsilon\beta^{-1}\mathbf{E}|X|,\ \textit{where}\ \alpha=\mathbf{P}(|\eta|\leqq\varepsilon),$$

$$q=\sup_{r,a,h>0}h^{-1}\mathbf{P}(r\leqq|Y_2+a|^2\leqq r+h).$$

Lemma 1 is not entirely traditional in the sense that the operator carrying out the smoothing is not convolution.

PROOF. It is not hard to see that $\mathbf{P}(|Z_1(a)|^2+(\eta-\varepsilon)|X|<r,\ |\eta|\leqq\varepsilon)\geqq \mathbf{P}(|Z_1(a)|^2<r)\mathbf{P}(|\eta|\leqq\varepsilon)$. On the other hand, $\mathbf{P}(|Z_2(a)|^2+(\eta-\varepsilon)|X|<r, |\eta|\leqq\varepsilon)\leqq \mathbf{P}(|Z_2(a)|^2<r+2\varepsilon|X|)\mathbf{P}(|\eta|\leqq\varepsilon)\leqq\mathbf{P}(|Z_2(a)|^2<r)\mathbf{P}(|\eta|\leqq\varepsilon)+2q\varepsilon\mathbf{E}|X|\mathbf{P}(|\eta|\leqq\varepsilon)$. Thus,

$$\begin{aligned}(\mathbf{P}(|Z_1(a)|^2<r)-\mathbf{P}(|Z_2(a)|^2<r))\mathbf{P}(|\eta|\leqq\varepsilon)\\ \leqq\mathbf{P}(V_1(a,\varepsilon)<r,|\eta|\leqq\varepsilon)-\mathbf{P}(V_2(a,\varepsilon)<r,|\eta|\leqq\varepsilon)\\ +2q\varepsilon\mathbf{E}|X|\mathbf{P}(|\eta|\leqq\varepsilon).\end{aligned}\tag{2}$$

Further,

$$\mathbf{P}(V_i(a,\varepsilon)<r,|\eta|>\varepsilon|X)=\int_{|u|>\varepsilon}\mathbf{P}(|Z_i(a)|^2<r+(\varepsilon-u)|X||X)Q(du),\tag{3}$$

where Q is the distribution of η. By definition of δ,

$$\begin{aligned}|\mathbf{P}(|Z_1(a)|^2<r+(\varepsilon-u)|X||X)-\mathbf{P}(|Z_2(a)|^2<r+(\varepsilon-u)|X||X)|\\ \leqq\delta(\beta+|a+X|^3)\leqq4\delta(\beta+|a|^3+|X|^3).\end{aligned}\tag{4}$$

From (3) and (4) it follows that

$$|\mathbf{P}(V_1(a,\varepsilon)<r,|\eta|>\varepsilon)-\mathbf{P}(V_2(a,\varepsilon)<r,|\eta|>\varepsilon)|\leqq4\delta(2\beta+|a|^3)\mathbf{P}(|\eta|>\varepsilon).\tag{5}$$

Combining (2) and (5), we arrive at the inequality $\alpha(F_1(r,a)-F_2(r,a))\leqq G_1(r,a,\varepsilon)-G_2(r,a,\varepsilon)+4\delta(1-\alpha)(2\beta+|a|^3)+2q\varepsilon\alpha\mathbf{E}|X|$.
Hence we have:

$$\begin{aligned}\sup_{r,a}((F_1(r,a)-F_2(r,a))/(\beta+|a|^3))\\ \leqq\alpha^{-1}\Delta(\varepsilon)+8\delta\alpha^{-1}(1-\alpha)+2q\varepsilon\mathbf{E}|X|/\beta.\end{aligned}$$

Analogous arguments show that

$$\begin{aligned}\sup_{r,a}((F_2(r,a)-F_1(r,a))/(\beta+|a|^3))\\ \leqq\alpha^{-1}\Delta(-\varepsilon)+8\delta\alpha^{-1}(1-\alpha)+2q\varepsilon\mathbf{E}|X|/\beta.\end{aligned}$$

The assertion of the lemma follows from the last two inequalities.

Lemma 2. *Let X, Y, V, η be mutually independent, $Z=X+Y$, $\mathbf{E}e^{it\eta}=0$, $\forall|t|\geqq\omega$,* Im $t=0$. *Then* $(\forall\varepsilon>0)$

$$\begin{aligned}&\mathbf{E}\exp\{it(|V+Z|^2+(\eta+\varepsilon)|X|)\}\\ &=(1+it\mathbf{E}|V|^2)\,\mathbf{E}\exp\{it(|Z|^2+(\eta+\varepsilon)|X|)\}\\ &\quad+2it\,\mathbf{E}(Z,V)\exp\{it(|Z|^2+(\eta+\varepsilon)|X|)\}\\ &\quad-2t^2\,\mathbf{E}\,(Z,V)^2\exp\{it(|Z|^2+(\eta+\varepsilon)|X|\}+R(t),\end{aligned}$$

where

$$\begin{aligned}R(t)\ll\mathbf{E}|V|^3[\mathbf{E}^{1/2}|f_{X^s}(2t)|\chi(t,X,X',\omega)(|t|^{3/2}+\mathbf{E}^{1/2}|Y|^2t^2)\\ +t^2\,\mathbf{E}^{1/2}|X||X'||f_{X^s}(2t)|\chi(t,X,X',\omega)\end{aligned}$$

$$+|t|^3 \sum_{j+k=3} \mathbf{E}^{1/2}|Y|^{2j}\, \mathbf{E}^{1/2}|X|^k|X'|^k|f_{X^s}(2t)|\chi(t, X, X', \omega)],$$

$$\chi(t, X, X', \omega) = \chi(t \max[|X|, |X'|]/\omega),$$

$$f_x(t) = \mathbf{E}\exp\{it(Y, x)\}.$$

PROOF. For brevity, we shall use the notation $s = it$. We start from the representation

$$\begin{aligned} &\mathbf{E}\exp\{s(|Z+V|^2 + (\eta+\varepsilon)|X|)\} \\ &\quad = \mathbf{E}\exp\{s(\Omega + |V|^2)\} \\ &\quad = \mathbf{E}\exp\{s\Omega\} + s\mathbf{E}|V|^2\exp\{s\Omega\} + \mathbf{E}r(s, V)\exp\{s\Omega\} \\ &\quad = A_1 + A_2 + A_3, \end{aligned} \tag{6}$$

where $\Omega = |Z|^2 + 2(Z, V) + (\eta+\varepsilon)|X|$, $r(s, V) = \exp\{s|V|^2\} - 1 - s|V|^2$.

It is not hard to see that

$$r(s, V) = O(|s|^{3/2}|V|^3). \tag{7}$$

Obviously,

$$A_3 = \mathbf{E}r(s, V)\,\mathbf{E}\{\exp\{s\Omega\}|V\}. \tag{8}$$

Further, $(\forall x \in H)$

$$\begin{aligned} \mathbf{E}_1(x) &\equiv \mathbf{E}\exp\{s(|Z|^2 + 2(Z, x) + (\eta+\varepsilon)|X|)\} \\ &= \mathbf{E}\exp\{s(|Y|^2 + 2(Y, x))\}\,\mathbf{E}\{\exp\{s(|X|^2 + 2(X, x+Y) + (\eta+\varepsilon)|X|)\}|Y\}. \end{aligned}$$

Using Cauchy's inequality, we have:

$$\begin{aligned} |\mathbf{E}_1(x)| &\leqq \mathbf{E}^{1/2}|\mathbf{E}\{\exp\{s(|X|^2 + 2(X, x+Y) + (\eta+\varepsilon)|X|)/Y\}|^2 \\ &= \mathbf{E}^{1/2}\mathbf{E}\{\exp\{s(|X|^2 - |X'|^2 + 2(X - X', x+Y) \\ &\qquad + (\eta+\varepsilon)|X| - (\eta'+\varepsilon)|X'|)\}|Y\}. \end{aligned}$$

Changing the order of integration, we get

$$\begin{aligned} |\mathbf{E}_1(x)| \leqq \mathbf{E}^{1/2}|&\mathbf{E}\{\exp\{s\eta|X|\}|X\}\,\mathbf{E}\exp\{-s\eta'|X'|\}|X'\} \\ &\cdot \mathbf{E}\{\exp\{2s(Y+x, X-X')\}|X, X'\}|. \end{aligned}$$

Since $\mathbf{E}\exp\{s|y|\eta\} = 0$, $\mathbf{E}\exp\{s|y|\eta'\} = 0$, $\forall|s| > \omega/|y|$, $y \in H$, this means that

$$\begin{aligned} |\mathbf{E}_1(x)| &\leqq \mathbf{E}^{1/2}|f_{X^s}(2t)|\chi(t|X|/\omega)\chi(t|X'|/\omega) \\ &= \mathbf{E}^{1/2}|f_{X^s}(2t)|\chi(t, X, X', \omega). \end{aligned} \tag{9}$$

From (7)-(9) it follows that

$$A_3 \ll |t|^{3/2}\,\mathbf{E}|V|^3\,\mathbf{E}^{1/2}|f_{X^s}(2t)|\chi(t, X, X', \omega). \tag{10}$$

Let us now turn to the estimation of A_1. It is not hard to see that

$$\begin{aligned} A_1 &= \mathbf{E}e^{s\Omega_1} + 2s\mathbf{E}(Z, V)e^{s\Omega_1} + 2s^2\mathbf{E}(Z, V)^2 e^{s\Omega_1} \\ &\quad + \mathbf{E}(Z, V)^3 R_1(s, V, Z)e^{s\Omega_1} \equiv \sum_1^4 A_{1,j}, \end{aligned} \tag{11}$$

where

$$\Omega_1=|Z|^2+(\eta+\varepsilon)|X|,$$
$$R_1(s, V, Z)=(e^{2s(Z,V)}-2(Z, V)s-2s^2(Z, V)^2)/(Z, V)^3.$$

Writing $R_1(s, V, Z)$ in integral form, we have

$$R_1(s, V, Z)=-4i\int_0^t \exp\{2iu(Z, V)\}(t-u)^2\,du.$$

Consequently,

$$\begin{aligned} A_{1,4}&=-4i\int_0^t \mathbf{E}(Z, V)^3\exp\{s\Omega_1+2iu(Z, V)\}(t-u)^2\,du \\ &=-4i\sum_0^3 C_3^k\int_0^t \mathbf{E}(X, V)^k(Y, V)^{3-k}\exp\{s\Omega_1+2iu(Z, V)\}(t-u)^2\,du. \end{aligned}\tag{12}$$

Now let us estimate

$$\mathbf{E}_{kj}(x)\equiv\mathbf{E}(X, x)^k(Y, x)^j\exp\{s\Omega_1+iu(Z, x)\},\ x\in H.$$

Obviously,

$$\begin{aligned}\mathbf{E}_{kj}(x)=&\,\mathbf{E}\{(Y, x)^j\exp\{s|Y|^2+iu(Y, x)\} \\ &\cdot\mathbf{E}\exp\{s(|X|^2+2(X, Y)+(\eta+\varepsilon)|X|)+iu(X, x)\}(X, x)^k|Y\}.\end{aligned}$$

Whence

$$|\mathbf{E}_{kj}(x)|\leqq\mathbf{E}^{1/2}|(Y, x)|^{2j}\mathbf{E}^{1/2}|\mathbf{E}\{(X, x)^k\exp\{s\Omega_2+iu(X, x)\}|Y\}|^2,\tag{13}$$

where $\Omega_2=|X|^2+2(X, Y)+(\eta+\varepsilon)|X|$.

Using the same arguments as in the derivation of (9), we arrive at the estimate

$$\begin{aligned} &\mathbf{E}|\mathbf{E}\{(X, x)^k\exp\{s\Omega_2+iu(X, x)\}|Y\}|^2 \\ &\quad=\mathbf{E}(X, x)^k(X', x)^k\exp\{s(|X|^2-|X'|^2 \\ &\qquad+2(X-X', Y)+(\eta+\varepsilon)|X|-(\eta'+\varepsilon)|X'|)+iu(X-X', x)\} \\ &\quad\leqq|x|^{2k}\mathbf{E}|X|^k|X'|^k|f_{X^s}(2t)|\chi(t, X, X', \omega)\equiv A(k)|x|^{2k}. \end{aligned}\tag{14}$$

From (13) and (14) it follows that

$$|\mathbf{E}_{kj}(x)|\leqq|x|^{k+j}\mathbf{E}^{1/2}|Y|^{2j}A^{1/2}(k).\tag{15}$$

Combining (12) and (15), we obtain

$$A_{1,4}\ll|t|^3\mathbf{E}|V|^3\sum_{j+k=3}A^{1/2}(k)\mathbf{E}^{1/2}|Y|^{2j}.\tag{16}$$

From (11) and (16) it follows that

$$A_1=\sum_1^3 A_{1,j}+O\left(|t|^3\mathbf{E}|V|^3\sum_{j+k=3}\mathbf{E}^{1/2}|Y|^{2j}A^{1/2}(k)\right).\tag{17}$$

Now let us get an estimate for A_2. Evidently,

$$A_2 = s\mathbf{E}\exp\{s(|Z|^2+(\eta+\varepsilon)|X|)\}\mathbf{E}|V|^2 \\ +2si\mathbf{E}\exp\{s\Omega_1\}(Z, V)|V|^2R_2(s, V, Z) \\ \equiv A_{2,1}+A_{2,2}, \tag{18}$$

where

$$R_2(s, V, Z) = \int_0^t \exp\{2iu(Z, V)\}\,du.$$

The same arguments as in the derivation of (16) lead to

$$A_{2,2} \ll t^2\mathbf{E}|V|^3 \sum_{j+k=1} \mathbf{E}^{1/2}|Y|^{2j}A^{1/2}(k). \tag{19}$$

From (6), (10), (11), (17)-(19) follows the assertion of the lemma.

Lemma 3. *Let X, Y, V, η satisfy the conditions of Lemma* 2. *Then* $(\forall\varepsilon>0, m>0)$

$$|\mathbf{E}(Z, V)^m \exp\{it(|Z|^2+(\eta+\varepsilon)|X|)\}| \\ \leqq c(m)\mathbf{E}|V|^m \sum_{j+k=m} \mathbf{E}^{1/2}|Y|^{2j}\mathbf{E}^{1/2}|X|^k|X'|^k|f_{X^s}(2t)|\chi(t, X, X', \omega).$$

PROOF. Introduce the notation

$$\mathbf{E}_m = \mathbf{E}(Z, V)^m \exp\{it(|Z|^2+(\eta+\varepsilon)|X|)\},$$
$$\mathbf{E}_{kj}(x, X, Y) = \mathbf{E}(x, X)^k(x, Y)^j \exp\{it(|Z|^2+(\eta+\varepsilon)|X|)\}.$$

Setting $s = it$, we have the following chain of inequalities:

$$\begin{aligned}
|\mathbf{E}_{kj}(x, X, Y)| &= |\mathbf{E}(Y, x)^j \exp\{s|Y|^2\}\mathbf{E}\{(X, x)^k \\
&\quad\cdot \exp\{s(|X|^2+2(X, Y)+(\eta+\varepsilon)|X|)\}|Y\}| \\
&\leqq \mathbf{E}^{1/2}(Y, x)^{2j}\mathbf{E}^{1/2}|\mathbf{E}\{(X, x)^k \exp\{s(|X|^2+2(X, Y) \\
&\quad+(\eta+\varepsilon)|X|)\}|Y\}|^2 \\
&\leqq |x|^j\mathbf{E}^{1/2}|Y|^{2j}\mathbf{E}^{1/2}\mathbf{E}\{(X, x)^k(X', x)^k \\
&\quad\cdot \exp\{s(|X|^2-|X'|^2+2(X^s, Y) \\
&\quad+\varepsilon(|X|-|X'|)+\eta|X|-\eta'|X'|)\}|Y\} \\
&= |x|^j\mathbf{E}^{1/2}|Y|^{2j}\mathbf{E}^{1/2}(X, x)^k(X', x)^k \\
&\quad\cdot \exp\{s(|X|^2-|X'|^2+\varepsilon(|X|-|X'|)\} \\
&\quad\cdot \mathbf{E}\{\exp\{s(2(X^s, Y)+\eta|X|-\eta'|X'|)\}|X, X'\} \\
&\leqq |x|^{j+k}\mathbf{E}^{1/2}|Y|^{2j}\mathbf{E}^{1/2}|X|^k|X'|^k|\mathbf{E}\{\exp\{s\eta|X|\}|X\} \\
&\quad\cdot \mathbf{E}\{\exp\{s\eta'|X'|\}|X'\}\mathbf{E}\exp\{2s(X^s, Y)\}|X, X'\}| \\
&\leqq |x|^{j+k}\mathbf{E}^{1/2}|X|^k|X'|^k|f_{X^s}(2t)|\chi(t, X, X', \omega)\mathbf{E}^{1/2}|Y|^{2j}.
\end{aligned}$$

It remains to apply the identity

$$\mathbf{E}_m = \sum_{k+j=m} C_m^k \mathbf{E}\mathbf{E}_{kj}(V, X, Y).$$

Lemma 3, just as the preceding Lemma 2, is a variant of Lemma (3.37) in [14].

Lemma 4. *Let* X_i, Y, $i=1,2$, *be independent and*

$$\tilde{F}_i(r, a) = \mathbf{P}(|X_i + a| < r),$$
$$\tilde{G}_i(r, a) = \mathbf{P}(|X_i + Y + a| < r),$$
$$\tilde{\Delta}(a) = \sup_r |\tilde{F}_1(r, a) - \tilde{F}_2(r, a)|,$$
$$\tilde{\delta} = \sup_r |\tilde{G}_1(r, a) - \tilde{G}_2(r, a)|.$$

Then $(\forall \varepsilon > 0, \alpha > \frac{1}{2})$

$$\tilde{\Delta}(a) < (\tilde{\delta} + 2\alpha\tilde{q}\varepsilon)/(2\alpha - 1),$$

where

$$\alpha = \mathbf{P}(|Y| \leqq \varepsilon),\ \tilde{q} = \sup_{r,h>0} h^{-1}(\tilde{F}_2(r+h, a) - \tilde{F}_2(r, a)).$$

PROOF. By the independence of X_1 and Y,

$$\begin{aligned}&\mathbf{P}(|X_1 + Y + a| < r + \varepsilon, |Y| \leqq \varepsilon)\\ &\quad \geqq \mathbf{P}(|X_1 + a| < r, |Y| \leqq \varepsilon) = \mathbf{P}(|X_1 + a| < r)\mathbf{P}(|Y| \leqq \varepsilon).\end{aligned}$$

Analogously,

$$\begin{aligned}&\mathbf{P}(|X_2 + Y + a| < r + \varepsilon, |Y| \leqq \varepsilon)\\ &\quad \leqq \mathbf{P}(|X_2 + a| < r + 2\varepsilon, |Y| \leqq \varepsilon)\\ &\quad = \mathbf{P}(|X_2 + a| < r + 2\varepsilon)\mathbf{P}(|Y| \leqq \varepsilon)\\ &\quad \leqq (\mathbf{P}(|X_2 + a| < r) + 2\tilde{q}\varepsilon)\mathbf{P}(|Y| \leqq \varepsilon).\end{aligned}$$

Thus,

$$\begin{aligned}(\tilde{F}_1(r, a) - \tilde{F}_2(r, a))\alpha \leqq\ & \tilde{G}_1(r+\varepsilon, a) - \tilde{G}_2(r+\varepsilon, a)\\ &+ 2\tilde{q}\varepsilon\alpha + \mathbf{P}(|X_2 + Y + a| < r + \varepsilon, |Y| > \varepsilon)\\ &- \mathbf{P}(|X_1 + Y + a| < r + \varepsilon, |Y| > \varepsilon).\end{aligned}$$

On the other hand,

$$\begin{aligned}&|\mathbf{P}(|X_1 + Y + a| < r, |Y| > \varepsilon) - \mathbf{P}(|X_2 + Y + a| < r, |Y| > \varepsilon)|\\ &\quad = \left| \int_{|y|>\varepsilon} (\tilde{F}_1(r, a+y) - \tilde{F}_2(r, a+y))Q(dy) \right| < \tilde{\Delta}(a)(1-\alpha),\end{aligned}$$

where Q is the distribution of Y. Consequently,

$$\sup_r (\tilde{F}_1(r, a) - \tilde{F}_2(r, a)) \leqq \tilde{\delta}/\alpha + 2\tilde{q}\varepsilon + \tilde{\Delta}(a)(1-\alpha)/\alpha.$$

Similarly,

$$\sup_r (\tilde{F}_2(r, a) - \tilde{F}_1(r, a)) \leqq \tilde{\delta}/\alpha + 2\tilde{q}\varepsilon + \tilde{\Delta}(a)(1-\alpha)/\alpha.$$

The assertion of the lemma follows from the last two estimates.

Lemma 4 carries over the corresponding assertions for R^k to Hilbert space (e.g., see [6], [20]).

Lemma 5. *Let ξ_1 and ξ_2 be independent and normal, respectively, $N(a_1, \sigma_1)$ and $N(a_2, \sigma_2)$, let $p(r)$ be the distribution density of $\xi_1^2 + \xi_2^2$.*

Then $(\forall a_1, a_2, \sigma_1, \sigma_2)$

$$p(r) \ll \min [(1/\sigma_1 + 1/\sigma_2)/\sqrt{r}, 1/\sigma_1\sigma_2]. \tag{20}$$

PROOF. It is not hard to see that

$$2\pi\sigma_1\sigma_2 p(r) \leqq \int_0^r (r-\rho)^{-1/2}\rho^{-1/2} \exp\{-((r-\rho)^{1/2} - |a_1|)^2/2\sigma_1^2 - (\rho^{1/2} - |a_2|)^2/2\sigma_2^2\}\, d\rho = \int_0^{r/2} + \int_{r/2}^r.$$

Further,

$$\int_0^{r/2} < (2/r)^{1/2} \int_0^{r/2} \rho^{-1/2} \exp\{-(\rho^{1/2} - |a_2|)^2/2\sigma_2^2\}\, d\rho \ll \sigma_2/r^{1/2}.$$

Similarly,

$$\int_{r/2}^r \ll \sigma_1/r^{1/2}.$$

Consequently,

$$p(r) \ll (1/\sigma_1 + 1/\sigma_2)/r^{1/2}.$$

On the other hand,

$$\int_0^r < \int_0^r (r-\rho)^{-1/2}\rho^{-1/2}\, d\rho \leqq c.$$

This means that $p(r) \ll 1/\sigma_1\sigma_2$.

The lemma is completely proved.

Lemma 6. *Let $X = \{\xi_i\}_1^\infty$, $X \in l_2$, where the ξ_i are mutually independent and normal $N(0, \sigma_i)$, $p_k(r, a)$ is the distribution density of $|X + a|^k$, $a \in l_2$, $k = 1, 2$.*

Then

$$p_1(r, a) \ll (B + |a|)/\sigma_1\sigma_2, \quad p_2(r, a) \ll \min [(B + |a|)/\sqrt{r}, 1]/\sigma_1\sigma_2,$$

$$\text{where} \quad B^2 = \sum_1^\infty \sigma_i^2.$$

PROOF. Let $a=\{a_i\}_1^\infty$. Let us represent $|X+a|^2$ in the form

$$|X+a|^2=(\xi_1+a_1)^2+(\xi_2+a_2)^2+|Y|^2, \qquad \text{where } Y=\{\xi_i+a_i\}_3^\infty.$$

Obviously,

$$p_2(r,a)=\int_0^r p(r-\rho)\,d\mathbf{P}(|Y|^2<\rho),$$

where $p(r)$ is the density of $(\xi_1+a_1)^2+(\xi_2+a_2)^2$. Using the estimate (see (20)) $p(r)\ll 1/\sigma_1\sigma_2$, we get from here that $p_2(r,a)\ll 1/\sigma_1\sigma_2$.

Further,

$$\int_0^{r/2} p(r-\rho)\,d\mathbf{P}(|Y|^2<\rho)\ll(1/\sigma_1+1/\sigma_2)/\sqrt{r},$$

since, due to Lemma 5, $p(r)<(1/\sigma_1+1/\sigma_2)/\sqrt{r}$.

On the other hand, by the same Lemma 5,

$$\int_{r/2}^r \ll \sigma_1^{-1}\sigma_2^{-1}\mathbf{P}(|Y|^2>r/2).$$

Obviously,

$$\mathbf{P}(|Y|^2>r/2)\ll(B+|a|)r^{-1/2}.$$

Therefore,

$$p_2(r,a)\ll\min\,[(B+|a|)/\sqrt{r},1]/\sigma_1\sigma_2.$$

Finally,

$$p_1(r,a)=2rp_2(r^2,a)\ll(B+|a|)/\sigma_1\sigma_2.$$

Lemma 7. *Let the r.v.'s $Y_1, Y_2,\cdots,Y_m$ take values in R^k and be identically distributed, $\theta=\{\theta_j\}_1^k$,*

$$|\mathbf{E}\exp\{i(Y_1,\theta)\}|\leqq\exp\left\{-\sum_1^k\sigma_j^2\theta_j^2\right\},$$

if

$$\max_j|\theta_j|\leqq\gamma,\quad S=\sum_1^m Y_j+Z,$$

where Z does not depend on Y_j, $j=1,\cdots,m$,

$$\mathbf{E}\exp\{i(Z,\theta)\}=\prod_1^k\varphi(\theta_j),\ \textit{where}\ \varphi_j(\theta_j)=0\ \textit{for}\ |\theta_j|>\gamma.$$

Then there exists a distribution density $p(u_1,u_2,\cdots,u_k)$ of the r.v. S and $p(u_1,u_2,\cdots,u_k)\leqq c(k)/m^{k/2}\Pi_1^k\sigma_j$.

PROOF. By the inversion formula,

$$(21)\qquad p(u_1,u_2,\cdots,u_k)=(2\pi i)^{-k}\int_{R^k}\exp\{-i(u,\theta)\}f^m(\theta)\varphi(\theta)\,d\theta,$$

where

$$f(\theta)=\mathbf{E}\exp\{i(Y_1,\theta)\},\qquad \varphi(\theta)=\prod_1^k\varphi_j(\theta_j),\qquad d\theta=\prod_1^k d\theta_j,\qquad \theta=\{\theta_j\}_1^k,$$

$$u=\{u_j\}_1^k.$$

From the conditions of the lemma it follows that

$$\int_{R^k}|f^m(\theta)\varphi(\theta)|\,d\theta\leqq\int_{-\infty}^{\infty}\exp\left\{-m\sum_1^k\sigma_j^2\theta_j^2\right\}d\theta\leqq c(k)/m^{k/2}\prod_1^k\sigma_j. \tag{22}$$

Combining (21) and (22), we get the assertion of the lemma.

3. Proof of the Main Theorem

Let us choose in H an orthonormal basis $\{e_j\}_1^\infty$ so that in it the covariance operator $\Lambda(L)$ has diagonal form. Below, when referring to the coordinates of a vector $x\in H$, we shall always mean the coordinates with respect to this basis.

Introduce the notation $\bar{X}_j=X_j(\sigma\sqrt{n})$. Let P_k be the distribution of $n^{-1/2}\sum_1^k\bar{X}_j$, Φ_k be the distribution of $n^{-1/2}\sum_1^k Z_j$. For even n,

$$P_n-\Phi_n=(P_{n/2}-\Phi_{n/2})*P_{n/2}+(P_{n/2}-\Phi_{n/2})*\Phi_{n/2}. \tag{23}$$

Now let us introduce a smoothing r.v. W not depending on $\{X_j\}_1^\infty$ and $\{Z_j\}_1^\infty$. Let $W=\sum_1^7\eta_je_j$, where the η_j are mutually independent and coincide in distribution with $4\eta_0L/\sqrt{n}$, P_W is the distribution of the r.v. W.

Set

$$\rho_n=\sup_{r,a}|(P_{n/2}-\Phi_{n/2})*P_{n/2}*P_W(S(a,r))|/(\sigma^3+|a|^3).$$

Obviously, P_n*P_W is the distribution of $n^{-1/2}\sum_1^n\bar{X}_j+W$, while $\Phi_{n/2}*P_{n/2}*P_W$ is the distribution of

$$n^{-1/2}\left(\sum_1^{n/2}\bar{X}_j+\sum_{n/2+1}^{n}Z_j\right)+W$$

under the condition that $\{X_j\}_1^\infty$ does not depend on $\{Z_j\}_1^\infty$.

Let us estimate ρ_n by means of Lemma 1, setting

$$X=n^{-1/2}\sum_{n/2+1}^{n}\bar{X}_j+W,$$

$$Y_1=n^{-1/2}\sum_1^{n/2}\bar{X}_j,\qquad Y_2=n^{-1/2}\sum_1^{n/2}Z_j,\qquad \eta=6\eta_0L/\sqrt{n}.$$

First let us estimate $\Delta(\varepsilon)$. Set

$$Q_j(r,a,\varepsilon)=\mathbf{P}\left(\left|\left(\sum_1^j Z_i+\sum_{j+1}^n\bar{X}_i\right)n^{-1/2}+W+a\right|^2+(\eta-\varepsilon)|X|<r\right),$$

$$j=0,\cdots,n/2.$$

Obviously,

$$Q_0(r, a, \varepsilon) = G_1(r, a, \varepsilon),$$
$$Q_{n/2}(r, a, \varepsilon) = G_2(r, a, \varepsilon).$$

Therefore,

$$G_1(r, a, \varepsilon) - G_2(r, a, \varepsilon) = \sum_{0}^{n/2-1} (Q_j(r, a, \varepsilon) - Q_{j+1}(r, a, \varepsilon)). \tag{24}$$

By the inversion formula,

$$Q_j(r, a, \varepsilon) - Q_{j+1}(r, a, \varepsilon) = (1/2\pi i) \int_{-\infty}^{\infty} (q_j(t) - q_{j+1}(t))(e^{-itr}/it)\, dt, \tag{25}$$

where

$$q_j(t) = \int_{-\infty}^{\infty} e^{itr} Q_j(dr, a, \varepsilon).$$

To estimate $q_j(t) - q_{j+1}(t)$ we apply Lemma 2, setting

$$X = n^{-1/2} \sum_{n/2+1}^{n} \bar{X}_i + W,$$
$$Y = n^{-1/2}\left(\sum_{1}^{j} Z_i + \sum_{j+2}^{n/2} \bar{X}_i\right) + a,$$
$$V = n^{-1/2}\bar{X}_{j+1} \text{ or } n^{-1/2} Z_{j+1}.$$

As a result we obtain:

$$\begin{aligned} q_j(t) - q_{j+1}(t) &= (\mathbf{E}|\bar{X}_{j+1}|^2 - \mathbf{E}|Z_{j+1}|^2)\mathbf{E}f(t, X, Y)\, itn^{-1} \\ &\quad + 2\mathbf{E}f(t, X, Y)((\bar{X}_{j+1}, Z) - (Z_{j+1}, Z))\, itn^{-1/2} \\ &\quad + 2\mathbf{E}f(t, X, Y)((Z_{j+1}, Z)^2 - (\bar{X}_{j+1}, Z)^2)t^2 n^{-1} + R_j(t) - R_{j+1}(t) \\ &\equiv A(t) + R_j(t) - R_{j+1}(t), \end{aligned} \tag{26}$$

where

$$f(t, X, Y) = \exp\{it(|Z|^2 + (\eta - \varepsilon)|X|)\},$$

and $R_j(t)$ and $R_{j+1}(t)$ are the remainder terms in the expansion for $q_j(t)$ and $q_{j+1}(t)$.

By Lemma 3,

$$\begin{aligned} |\mathbf{E}(\bar{X}_{j+1}, Z) f(t, X, Y)| &= |\mathbf{E}(X_{j+1} - \bar{X}_{j+1}, Z) \\ f(t, X, Y)| &\ll \mathbf{E}|X_{j+1} - \bar{X}_{j+1}| \sum_{j+k=1} \mathbf{E}^{1/2}|Y|^{2j} \\ &\quad \cdot \mathbf{E}^{1/2}|X|^k |X'|^k |f_{X^s}(2t)| \chi(t, X, X', \omega), \\ |\mathbf{E}f(t, X, Y)| &\ll \mathbf{E}^{1/2}|f_{X^s}(2t)|\chi(t, X, X', \omega). \end{aligned} \tag{27}$$

Further,

$$\mathbf{E}(\bar{X}_{j+1}, Z)^2 f(t, X, Y) = \mathbf{E}(X_{j+1}, Z)^2 f(t, X, Y) - \mathbf{E}(X_{j+1} - \bar{X}_{j+1}, Z)^2 f(t, X, Y).$$

It is not hard to see that

$$\begin{aligned} \mathbf{E}(X_{j+1}, Z)^2 f(t, X, Y) &= \mathbf{E} f(t, X, Y) \mathbf{E}\{(X_{j+1}, Z)^2 | X, Y\} \\ &= \mathbf{E} f(t, X, Y) \mathbf{E}\{(Z_{j+1}, Z)^2 | X, Y\} \\ &= \mathbf{E}(Z_{j+1}, Z)^2 f(t, X, Y). \end{aligned}$$

On the other hand, by Lemma 3,

$$\begin{aligned} &\mathbf{E}(X_{j+1} - \bar{X}_{j+1}, Z)^2 f(t, X, Y) \\ &\quad \ll \mathbf{E}|X_{j+1} - \bar{X}_{j+1}|^2 \sum_{j+k=2} \mathbf{E}^{1/2} |Y|^{2j} \mathbf{E}^{1/2} |X|^k \\ &\quad \cdot |X'|^k |f_{X^s}(2t)| \chi(t, X, X', \omega). \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbf{E}((\bar{X}_{j+1}, Z)^2 - (Z_{j+1}, Z)^2) f(t, X, Y) \\ &\quad = \mathbf{E}((X_{j+1}, Z)^2 - (Z_{j+1}, Z)^2 - (X_{j+1} - \bar{X}_{j+1}, Z)^2) f(t, X, Y) \\ &\quad \ll \sum_{j+k=2} \mathbf{E}^{1/2} |Y|^{2j} \mathbf{E}^{1/2} |f_{X^s}(2t)| \, |X|^k |X'|^k \chi(t, X, X', \omega). \end{aligned} \tag{28}$$

Obviously,

$$\begin{aligned} &\mathbf{E}|X_j|^2 - \mathbf{E}|\bar{X}_j|^2 = \mathbf{E}\{|X_j|^2; |X_j| > \sigma\sqrt{n}\} \leqq \beta_3 / \sigma\sqrt{n}, \\ &\mathbf{E}|X_j - \bar{X}_j| \leqq \beta_3 / \sigma^2 n, \quad \mathbf{E}|X_j - \bar{X}_j|^2 \leqq \beta_3 / \sigma\sqrt{n}. \end{aligned} \tag{29}$$

Finally,

$$\mathbf{E}(Z_{j+1}, Z) f(t, X, Y) = \mathbf{E} f(t, X, Y) \mathbf{E}\{(Z_{j+1}, Z) | X, Y\} = 0. \tag{30}$$

From (27)-(30) it follows that

$$\begin{aligned} A(t) \ll \beta_3 n^{-3/2} (&\sigma^{-1} |t| \mathbf{E}^{1/2} |f_{X^s}(2t)| \chi(t, X, X', \omega) \\ &+ \sigma^{-3} \sum_{1 \leqq j+k \leqq 2} \sigma^{j+k} |t|^{j+k} \mathbf{E}^{1/2} |Y|^{2j} \mathbf{E}^{1/2} |X|^k |X'|^k |f_{X^s}(2t)| \chi(t, X, X', \omega)). \end{aligned} \tag{31}$$

Using the inequality for the moments of sums of independent r.v.'s with values in a Banach space (e.g., see [19]), we have

$$\mathbf{E}|X|^t \leqq c(t) (\mathbf{E}|\bar{X}_1|^t n^{1-t/2} + \mathbf{E}^{t/2} |\bar{X}_1|^2 + \mathbf{E}^t |X - W| + \mathbf{E}|W|^t), \qquad t \geqq 1.$$

Further,

$$\mathbf{E}|\bar{X}_1|^t \leqq \sigma^{t-2} n^{t/2-1} \mathbf{E}|X_1|^2 = \sigma^t n^{t/2-1}, \qquad t \geqq 2.$$

It is not hard to see that

$$\mathbf{E}|X - W| \leqq (\mathbf{E}|X_1|^2/2)^{1/2} + \beta_3 / 2\sigma^2 \sqrt{n} \leqq 2\sigma,$$

since without loss of generality $\beta_3/\sigma^3\sqrt{n} \leqq 2$. Finally,

$$\mathbf{E}|W|^t \ll (L/\sqrt{n})^t \leqq \sigma^t, \qquad 0 < t \leqq 58,$$

since without loss of generality $L\sigma/\sigma_1\sigma_2\sqrt{n} \leqq 1$.

As a result we have:

$$\mathbf{E}|X|^t \leqq c(t)\sigma^t, \qquad 1 \leqq t \leqq 58. \tag{32}$$

Similarly,

$$\mathbf{E}|Y|^t \leqq c(t)(\sigma^t + |a|^t). \tag{33}$$

Let us turn to estimation of $f_x(t)$. In the case at hand,

$$\begin{aligned} f_x(t) = (\mathbf{E} \exp\{it(\bar{X}_1, x)/n^{1/2}\})^{n/2-j-1} \\ \cdot (\mathbf{E} \exp\{it(Z_1, x)/n^{1/2}\})^j \exp\{it(a, x)\}. \end{aligned} \tag{34}$$

From the identity

$$\begin{aligned} \mathbf{E} \exp\{it(\bar{X}_1, x)\} = \mathbf{E}\{\exp\{it(\bar{X}_1, x)\}; |\bar{X}_1| \leqq L\} \\ + \mathbf{E}\{\exp\{it(\bar{X}_1, x)\}; |\bar{X}_1| > L\} \end{aligned}$$

it follows that

$$|\mathbf{E} \exp\{it(\bar{X}_1, x)\}| \leqq 1 + (|\varphi(t)| - p),$$

where

$$\varphi(t) = \mathbf{E}\{\exp\{it(\bar{X}_1, x)\}; |\bar{X}_1| \leqq L\}, \qquad p = \mathbf{P}(|\bar{X}_1| \leqq L).$$

Obviously,

$$\begin{aligned} &p - |\varphi| = (p^2 - |\varphi|^2)/(p + |\varphi|) \geqq (p^2 - |\varphi|^2)/2p, \\ &|p^2 - |\varphi(t)|^2 - 2^{-1}t^2\mathbf{E}\{(\bar{X}_1^s, x)^2; |\bar{X}_1| \leqq L, |\bar{X}_1'| \leqq L\}| \\ &\quad \leqq 6^{-1}|t|^3\mathbf{E}\{|(\bar{X}_1^s, x)|^3; |\bar{X}_1| \leqq L, |\bar{X}_1'| \leqq L\} \\ &\quad \leqq 3^{-1}|t|^3|x|L\mathbf{E}\{(\bar{X}_1^s, x)^2; |\bar{X}_1| \leqq L, |\bar{X}_1'| \leqq L\}. \end{aligned}$$

It is not hard to see that

$$\begin{aligned} \mathbf{E}\{(\bar{X}_1^s, x)^2; |\bar{X}_1| \leqq L, |\bar{X}_1'| \leqq L\} &= 2(p\mathbf{E}(X_1(L), x)^2 - (\mathbf{E}X_1(L), x)^2) \\ &\geqq 2p(\Lambda(L)x, x). \end{aligned}$$

As a result we obtain that $(\forall |t| \leqq \frac{3}{4}|x|L)$

$$\begin{aligned} |\mathbf{E} \exp\{it(\bar{X}_1, x)\}| &\leqq 1 - t^2(\Lambda(L)x, x)/4 \\ &\leqq \exp\{-t^2(\Lambda(L), x, x)/4\}. \end{aligned} \tag{35}$$

Further,

$$\begin{aligned} \mathbf{E} \exp\{it(Z_1, x)\} &= \exp\{-t^2\mathbf{E}(Z_1, x)^2/2\} \\ &\leqq \exp\{-t^2\mathbf{E}(X_1(L), x)^2/2\} \\ &< \exp\{-t^2(\Lambda(L)x, x)/4\}. \end{aligned} \tag{36}$$

From (34)-(36) it follows that $(\forall |t| \leqq 3\sqrt{n}/4|x|L,\ n \geqq 4)$

(37) $$|f_x(t)| \leqq \exp\{-(\Lambda(L)x, x)^2 t^2/16\}.$$

Let us estimate

$$\mathbf{E}_k \equiv \mathbf{E}|X|^k|X'|^k|f_{X^s}(2t)|\chi(t, X, X', \omega),$$

setting $\omega = 3\sqrt{n}/16L$.

By (37),

$$\mathbf{E}_k \leqq \mathbf{E}|X|^k|X'|^k \exp\{-t^2(\Lambda(L)X^s, X^s)^2/4\}.$$

Using Hölder's inequality now we have

(38) $$\mathbf{E}_k \leqq \mathbf{E}^{1/p}|X|^{kp}|X'|^{kp}\mathbf{E}^{1/q}\exp\{-4^{-1}qt^2(\Lambda(L)X^s, X^s)\}, \qquad 1/p+1/q=1.$$

Further,

(39) $$\mathbf{E}\exp\{-4^{-1}qt^2(\Lambda(L)X^s, X^s)\} \leqq \mathbf{E}\exp\{-4^{-1}qt^2\sum_1^7 \sigma_j^2(L)\zeta_j^2\},$$

where $\zeta_j = (X^s, e_j)$.

Let $\xi_j = (\bar{X}_1, e_j)$. Obviously,

$$\mathbf{E}\exp\left\{i\sum_1^7 \xi_j\theta_j\right\} = \mathbf{E}\exp\{i(\bar{X}_1, \theta)\}, \text{ where } \theta = \sum_1^7 \theta_j e_j.$$

Hence by (35),

$$\left|\mathbf{E}\exp\left\{i\sum_1^7 \xi_j\theta_j\right\}\right| < \exp\left\{-\sum_1^7 \sigma_j^2(L)\theta_j^2/4\right\}$$

for $(\sum_1^7 \theta_j^2)^{1/2} < \frac{3}{4}L$ and consequently for $\max_{1\leqq j\leqq 7}|\theta_j| \leqq \frac{1}{4}L$.

Applying now Lemma 7, we find that the density $p(u_1, u_2, \cdots, u_7)$ of the distribution of the first seven coordinates of X satisfies the inequality

$$p(u_1, u_2, \cdots, u_7) \ll \left(\prod_1^7 \sigma_j(L)\right)^{-1}.$$

Consequently,

(40) $$p_s(u_1, u_2, \cdots, u_7) \ll 1\Big/\prod_1^7 \sigma_j(L),$$

where $p_s(u_1, u_2, \cdots, u_7)$ is the distribution density of the vector $\sum_1^7 \xi_j e_j$.

From (39) and (40) it follows that

(41) $$\mathbf{E}\exp\{-4^{-1}qt^2(\Lambda(L)X^s, X^s)\} \ll 1/q^{7/2}|t|^7\prod_1^7 \sigma_j^2(L).$$

Combining (32), (38) and (41), we get

(42) $$\mathbf{E}_k \ll \sigma^{2k}\left(q^{7/2}|t|^7\prod_1^7 \sigma_j^2(L)\right)^{-1/q}, \qquad q > 19/18.$$

Estimates (31), (33) and (42) give us

$$|A(t)| \ll \beta_3 n^{-3/2} \frac{|t|/\sigma + \sum_{1 \leqq j+k \leqq 2} \sigma^{j+2k-3} |t|^{j+k} (\sigma^j + |a|^j)}{(q^{7/4} |t|^{7/2} \prod_1^7 \sigma_j(L))^{1/q} + 1}.$$

Similarly, using the estimate for the remainder in Lemma 2, we obtain

$$R_j(t) \ll \beta_3 n^{-3/2} \bigg(|t|^{3/2} + (\sigma + |a|) t^2 + |t|^3 \sum_{i+k=3} (\sigma^i + |a|^i) \sigma^k \bigg) \Big/ \bigg(\Big(q^{7/4} |t|^{7/2} \prod_1^7 \sigma_i(L) \Big)^{1/q} + 1 \bigg).$$

From the last two estimates, choosing $q < 8/7$, we obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} |A(t)/t| \, dt \ll \beta_3 n^{-3/2} (\sigma^3 + |a|^3) \Big(\prod_1^7 \sigma_j(L) \Big)^{-6/7}, \\ &\int_{-\infty}^{\infty} |R_j(t)/t| \, dt \ll \beta_3 n^{-3/2} (\sigma^3 + |a|^3) \Big(\prod_1^7 \sigma_j(L) \Big)^{-6/7}. \end{aligned} \tag{43}$$

From (25), (26) and (43) it follows that

$$|Q_j(r, a, \varepsilon) - Q_{j+1}(r, a, \varepsilon)| \ll \beta_3 n^{-3/2} (\sigma^3 + |a|^3) \Big(\prod_1^7 \sigma_j(L) \Big)^{-6/7}.$$

Whence, by (24),

$$\Delta(\varepsilon) \ll \beta_3 \sigma^3 / \mathbf{E}|X|^3 \Big(\prod_1^7 \sigma_j(L) \Big)^{6/7} \sqrt{n}. \tag{44}$$

Let us estimate $\sigma^3/\mathbf{E}|X|^3$. To this end we note that $\mathbf{E}|X|^3 \geqq (\mathbf{E}|X|^2)^{3/2}$. Further, $\mathbf{E}|X|^2 > \mathbf{E}|\sum_{n/2+1}^n \bar{X}_i|^2/n$, since $\mathbf{E}W = 0$.

On the other hand,

$$\begin{aligned} \mathbf{E}\Big|\sum_{n/2+1}^n \bar{X}_i\Big|^2 &\geqq \mathbf{E}\Big|\sum_{n/2+1}^n (\bar{X}_i - \mathbf{E}\bar{X}_i)\Big|^2 + 4^{-1} n^2 |\mathbf{E}\bar{X}_1|^2 \\ &= 2^{-1} n \mathbf{E}|\bar{X}_1 - \mathbf{E}\bar{X}_1|^2 + 4^{-1} n^2 |\mathbf{E}\bar{X}_1|^2 \\ &= 2^{-1} n \mathbf{E}|\bar{X}_1|^2 + (n^2/4 - n/2) |\mathbf{E}\bar{X}_1|^2. \end{aligned}$$

It is easy to see that

$$\mathbf{E}|\bar{X}_1|^2 \geqq \sigma^2 - \beta_3/\sigma\sqrt{n}.$$

Without loss of generality, we may assume that $\beta_3/\sigma^3\sqrt{n} < 1/2$. Thus, $\mathbf{E}|\sum_{n/2+1}^n \bar{X}_i|^2 > n\sigma^2/4$ and consequently

$$\sigma^3/\mathbf{E}|X|^3 < 8. \tag{45}$$

From Lemmas 1, 6 and inequalities (44), (45) it follows that

$$\begin{aligned} &\sup_{r,a} (|(P_{n/2} - \Phi_{n/2}) * P_{n/2} * P_W(S(a, r))|/(\mathbf{E}|X|^3 + |a|^3)) \\ &\leqq c\alpha^{-1} \bigg(\beta_3 \Big/ \Big(\prod_1^7 \sigma_j(L) \Big)^{6/7} \sqrt{n} + \varepsilon \mathbf{E}|X|/\mathbf{E}|X|^3 \sigma_1 \sigma_2 \bigg) \\ &\quad + 8\alpha^{-1}(1-\alpha) \sup_{r,a} |P_{n/2}(S(a, r)) - \Phi_{n/2}(S(a, r))|/(\mathbf{E}|X|^3 + |a|^3). \end{aligned} \tag{46}$$

Due to (32) and (45),

$$1/c(3) \leqq (\sigma^3 + |a|^3)/(\mathbf{E}|X|^3 + |a|^3) \leqq 8.$$

Hence (46) turns into the inequality

$$\rho_n \leqq c\alpha^{-1}\left(\beta_3 \Big/ \left(\prod_1^7 \sigma_j(L)\right)^{6/7} \sqrt{n} + \varepsilon/\sigma_1\sigma_2\sigma^2\right) + 64c(3)\delta_n \alpha^{-1}(1-\alpha),$$

where

$$\delta_n = \sup_{r,a} \left(|P_{n/2}(S(a, r)) - \Phi_{n/2}(S(a, r))|/(\sigma^3 + |a|^3)\right).$$

Now let us choose ε so that $64c(3)\alpha^{-1}(1-\alpha) = \frac{1}{16}$. It is not hard to see that for such a choice

$$\varepsilon \ll L/\sqrt{n}. \tag{47}$$

As a result we get

$$\rho_n \leqq c\Gamma/\sqrt{n} + \delta_n/16, \tag{48}$$

where

$$\Gamma = \beta_3 \Big/ \left(\prod_1^7 \sigma_j(L)\right)^{6/7} + L/\sigma_1\sigma_2\sigma^2.$$

It is not hard to show that

$$\left| P_{n/2}(S(a, r)) - P\left(\sqrt{2/n} \sum_1^{n/2} X_j \in S(\alpha\sqrt{2}, r\sqrt{2})\right)\right| < \sqrt{2}\beta_3/\sigma^3\sqrt{n}.$$

Consequently,

$$\delta_n 2^{3/2}\Delta_{n/2} + \sqrt{2}\beta_3/\sigma^6\sqrt{n}, \tag{49}$$

where $\Delta_n = \sup_a (\Delta_n(a)/(\sigma^3 + |a|^3))$.

By (48) and (49),

$$\rho_n \leqq c\Gamma/\sqrt{n} + \Delta_{n/2}/4. \tag{50}$$

Similarly,

$$\rho'_n \equiv \sup_{r,a} \left(|(P_{n/2} - \Phi_{n/2}) * \Phi_{n/2} * P_W(S(a, r))|/(\sigma^3 + |a|^3)\right) \leqq c\Gamma/\sqrt{n} + \Delta_{n/2}/4. \tag{51}$$

By Lemmas 4 and 6,

$$\sup_r |P_n(S(a, r)) - \Phi_n(S(a, r))| \ll \sup_r |(P_n - \Phi_n) * P_W(S(a, r))| + L(\sigma + |a|)/\sigma_1\sigma_2\sqrt{n}.$$

This means in view of (23) that

$$\Delta'_n \ll \rho_n + \rho'_n + L/\sigma^2\sigma_1\sigma_2\sqrt{n}, \tag{52}$$

where

$$\Delta_n' = \sup_{r,a} (|P_n(S(a, r)) - \Phi_n(S(a, r))|)/(\sigma^3 + |a|^3)).$$

On the other hand,

$$\sup_r |P_n(S(a, r)) - \Phi_n(S(a, r))| > \Delta_n(a) - \beta_3/\sigma^3\sqrt{n}.$$

Therefore,

$$\Delta_n' \geqq \Delta_n - \beta_3/\sigma^6\sqrt{n}. \tag{53}$$

From (50)-(53) it follows that $\Delta_n < c\Gamma/\sqrt{n} + \Delta_{n/2}/2$. Hence, for $n = 2^k$,

$$\Delta_n \ll \Gamma/\sqrt{n}. \tag{54}$$

Let $2^k < n < 2^{k+1}$. Denoting by Ψ_j the distribution of the r.v. $\sum_1^j X_m/\sqrt{n}$, we have

$$\begin{aligned}\Psi_n(S(a, r)) - \Phi_n(S(a, r)) &= (\Psi_{2^k} - \Phi_{2^k}) * \Psi_{n-2^k}(S(a, r)) \\ &+ (\Psi_{n-2^k} - \Phi_{n-2^k}) * \Phi_{2^k}(S(a, r)) = K_1 + K_2.\end{aligned} \tag{55}$$

It is not hard to see that

$$\left|\int_H (\Psi_{2^k} - \Phi_{2^k})(S(a, r) - x)\Psi_{n-2^k}(dx)\right| \ll \Delta_{2^k} \int_H (\sigma^3 + |x|^3 + |a|^3)\Psi_{n-2^k}(dx).$$

Using the moment inequality from [19], we obtain

$$\int_H |x|^3\Psi_{n-2^k}(dx) = n^{-3/2}\mathbf{E}\left|\sum_1^{n-2^k} X_j\right|^3 \ll \sigma^3 + \beta_3/\sqrt{n}.$$

Since without loss of generality $\beta_3/\sqrt{n} < \sigma^3$, we conclude in view of the two preceding inequalities that

$$K_1 \ll \Delta_{2^k}(\sigma^3 + |a|^3). \tag{56}$$

It remains to estimate K_2. We shall avail ourselves of the representation

$$\begin{aligned}&\mathbf{E}\exp\left\{it\left|n^{-1/2}\left(\sum_1^{2^k} Z_m + \sum_{2^k+1}^{m} X_m\right) + a\right|^2\right\} - \mathbf{E}\exp\left\{it\left|n^{-1/2}\sum_1^n Z_m + a\right|^2\right\} \\ &= \sum_{2^k}^{n-1}\left(\mathbf{E}\exp\left\{it\left|n^{-1/2}\left(\sum_1^j Z_m + \sum_{j+1}^n X_m\right) + a\right|^2\right\}\right. \\ &\left. -\mathbf{E}\exp\left\{it\left|n^{-1/2}\left(\sum_1^{j+1} Z_m + \sum_{j+2}^n X_m\right) + a\right|^2\right\}\right) \equiv \sum_{2^k}^{n-1} \gamma_j(t).\end{aligned}$$

Now we apply Lemma 2, setting

$$X = n^{-1/2}\sum_1^{2^{k-1}} Z_m, \qquad Y = n^{-1/2}\left(\sum_{2^{k-1}+1}^{j} Z_m + \sum_{j+2}^n X_m\right) + a,$$

$$V = X_{j+1}/\sqrt{n}, \text{ or } Z_{j+1}'/\sqrt{n}, \qquad \eta = 0$$

(the latter implies that $\omega = \infty$, i.e., $\chi(t, \omega, X, X') \equiv 1$).

Due to the normality of Z_m we have

$$|f_x(t)| \leqq \exp\{-t^2(\Lambda x, x)/8\} \leqq \exp\{-t^2(\Lambda(L)x, x)/8\} \leqq \exp\left\{-t^2 \sum_1^7 x_i^2\sigma_i^2(L)/8\right\},$$

where $x_i = (x, e_i)$.

On the other hand, $|X|$ has moments of all orders and $\mathrm{E}|X|^t \leqq c(t)\sigma^t$. Now repeating the arguments which led us to (43), we see that

$$\int_{-\infty}^{\infty} |\gamma_j(t)/t|\, dt \ll \beta_3(\sigma^3 + |a|^3) \Big/ \left(\prod_1^7 \sigma_j(L)\right)^{6/7} n^{3/2}.$$

Hence, by the inversion formula,

$$K_2 \ll \beta_3(\sigma^3 + |a|^3) \Big/ \left(\prod_1^7 \sigma_j(L)\right)^{6/7} n^{1/2}. \tag{57}$$

The assertion of the theorem follows from (54)-(57).

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