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LIMIT THEOREMS FOR A CRITICAL GALTON-WATSON BRANCHING PROCESS WITH MIGRATION

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1. Introduction. Notation. Formulation of Basic Results

A Galton–Watson branching process with immigration and emigration is studied in this paper. More precisely, a population is considered in which each particle reproduces according to the scheme of a Galton–Watson process, and moreover at each time $n, n=0, 1, \cdots$, either k particles immigrate into the population with probability p_k, $k=0, 1, \cdots$, or r of the particles existing at the instant n emigrate from the population with probability q_r, $r=1, \cdots, m$, where m is an arbitrary fixed positive integer,

$$\sum_{k=0}^{\infty} p_k + \sum_{r=1}^{m} q_r = 1.$$

The particles reproduce independently of one another and independently of their own origin.

We shall now turn to the formal description of the process under study. Let independent integer-valued random variables

$$\xi_i^{(n)}, \zeta_n, i=1, 2, \cdots, \qquad n=0, 1, \cdots,$$

be assigned, where the ζ_n, $n=0, 1, \cdots$, are identically distributed and

$$\mathbf{P}\{\zeta_n = k\} = p_k, \qquad k=0, 1, \cdots,$$

$$\mathbf{P}\{\zeta_n = -r\} = q_r, \qquad r=1, \cdots, m,$$

in turn, the $\xi_i^{(n)}$, $i=1, 2, \cdots, n=0, 1, \cdots$, are identically distributed with generating function

$$f(s) = \mathbf{E}s^{\xi^{(n)}}, \qquad |s| \leqq 1.$$

We shall define the process $\{Z_n, n=0, 1, \cdots\}$ in the following manner:

$$Z_0 = 0, \qquad Z_{n+1} = \begin{cases} \xi_1^{(n)} + \cdots + \xi_{Z_n+\zeta_n}^{(n)}, & Z_n + \zeta_n > 0, \\ 0, & Z_n + \zeta_n \leqq 0. \end{cases}$$

It is not difficult to see that $\{Z_n, n=0, 1, \cdots\}$ is a homogeneous Markov process.

The random variables Z_n, $\xi_i^{(n)}$ and ζ_n, $n=0, 1, \cdots, i=1, 2, \cdots$, are interpreted, respectively, as the number of particles in the population at the time n; the number of particles generated by the ith of the particles existing at the instant n, at the $(n+1)$st time instant; and the number of particles migrating at the nth time instant.

Throughout in what follows we assume that

$$f'(1-)=1, \tag{1}$$

$$f(0)>0, \tag{2}$$

$$B=\frac{f''(1-)}{2}<\infty, \tag{3}$$

$$q_m>0, \tag{4}$$

$$\sum_{k=1}^{\infty} kp_k - \sum_{k=1}^{m} kq_k = 0, \tag{5}$$

$$\sum_{k=2}^{\infty} k^2 p_k < \infty. \tag{6}$$

It obviously follows from (1), (2) that

$$B=\frac{f''(1-)}{2}>0. \tag{7}$$

Among the Galton–Watson processes considered earlier with various forms of immigration, the critical Galton–Watson process with immigration depending on the state, which was studied in [1]–[3], is the closest to the one cited above.

For the critical Galton–Watson process with immigration defined in [4], limit theorems were obtained in [5]–[10]. Galton–Watson processes with emigration were studied in [11], [12].

We also point out that the process $\{Z_n, n=0, 1, \cdots\}$ can be represented in the form of a ϕ-branching process considered in [13]–[15]. However, the results of this paper do not follow from known results for ϕ-branching processes.

Let $\mathfrak{N}$ be the state set of the Markov chain $\{Z_k, k=0, 1, \cdots\}$, $p_{rj}(n)$ the transition probability from state r to state j over n steps $r, j \in \mathfrak{N}$, $n=0, 1, \cdots$ ($p_{rr}(0)=1, r \in \mathfrak{N}$, and ${}_0p_{0r}(n)$ the probability that the process $\{Z_k, k=0, 1, \cdots\}$, starting from 0, reaches state r over n steps without coming to 0 ($n=1, 2, \cdots, r \in \mathfrak{N}$),

$$F(x)=\sum_{k=[x]+1}^{\infty} {}_0p_{00}(k), \qquad x \geqq 0.$$

We set

$$W_{0r}(s)=\sum_{n=0}^{\infty} p_{0r}(n)s^n, \qquad W_0(s)=W_{00}(s),$$

where $0 \leqq s<1$, $r \in \mathfrak{N}$. We denote by $f_n(s)$ the nth iteration of the function $f(s)$,

$|s| \leqq 1$, $n = 1, 2, \cdots$. Let us introduce the generating functions

$$g(s) = \sum_{k=0}^{\infty} p_k s^k, \qquad \Psi_n(s) = \mathbf{E} s^{Z_n} = \sum_{j \in \mathfrak{N}} p_{0j}(n) s^j, \qquad G_r(s) = \sum_{k=1}^{\infty} {}_0p_{0r}(k) s^k,$$

where $0 \leqq s \leqq 1$, $r \in \mathfrak{N}$, $n = 0, 1, \cdots$.

The basic purpose of this paper is the proof of the following theorems.

Theorem 1. *As* $n \to \infty$,

$$F(n) \sim \frac{1}{A_0 n},$$

where A_0 *is a positive constant.*

Theorem 2. *As* $n \to \infty$,

$$p_{0r}(n) \sim \frac{A_r}{\log n},$$

where the A_r *are positive constants,* $r \in \mathfrak{N}$.

Theorem 3. *As* $n \to \infty$,

$$\mathbf{E} Z_n \sim B \frac{n}{\log n}, \tag{8}$$

$$\operatorname{Var} Z_n \sim 2B^2 \frac{n^2}{\log n}. \tag{9}$$

Theorem 4. *For* $x \in [0, 1]$,

$$\lim_{n \to \infty} \mathbf{P} \left\{ \frac{\log Z_n}{\log n} < x \right\} = x.$$

REMARK. Theorem 4 is an analogue of the limit theorem obtained by Foster for a Galton–Watson process with immigration depending on the state (see [1]).

We also point out that the representation (17) obtained in Section 2 for the generating function $\Psi_n(s)$, $0 \leqq s \leqq 1$, $n = 0, 1, \cdots$ serves as the basis for the proof of Theorem 4. In the case $m = 1$ a similar representation was given in [1], [2] for a Galton–Watson process with immigration depending on the state.

2. Study of the Generating Functions Connected with the Process $\{Z_n, n = 0, 1, \cdots\}$

The following lemma obviously holds.

Lemma 1. *Let* $\Phi(y) = \sum_{n=0}^{\infty} \beta_n y^n$, $0 \leqq y < 1$, *and* $\beta_n = o(1/n)$, $n \to \infty$. *Then, as* $y \to 1-$,

$$\Phi(y) = o\left(\log \frac{1}{1-y}\right), \quad \Phi'(y) = o\left(\frac{1}{1-y}\right), \quad \Phi''(y) = o\left(\frac{1}{(1-y)^2}\right).$$

From (4), (5) it follows that there exists a $k \geqq 1$ such that $p_k > 0$. Therefore $\mathfrak{N} \neq \{0\}$.

Lemma 2. *The set $\mathfrak{N}$ consists of a single class of information states.*

PROOF. Since $Z_0=0$, $\mathfrak{N}$ coincides with the state set attainable from 0. Hence, for any $r\in\mathfrak{N}$, there exists a $k_r\geqq 0$ such that

$$p_{0r}(k_r)>0.$$

Moreover, in view of (2), (4),

$$p_{j0}(1)\geqq q_m\{f(0)\}^{\max(0,j-m)}>0, \qquad j\in\mathfrak{N}. \tag{10}$$

Therefore

$$p_{jr}(k_r+1)\geqq p_{j0}(1)p_{0r}(k_r)>0, \qquad j, r\in\mathfrak{N}. \tag{11}$$

Corollary. *For $r\in\mathfrak{N}$,*

$$0<G_r(1)<\infty. \tag{12}$$

PROOF. According to Lemma 2 the state 0 is attainable from any state $r\in\mathfrak{N}$. Hence (see [16]),

$$G_r(1)<\infty, \qquad r\in\mathfrak{N}.$$

In turn, from (11) for $j=0$ it evidently follows that

$$G_r(1)>0, \qquad r\in\mathfrak{N}.$$

Lemma 3. *The generating functions $\Psi_n(s)$, $0\leqq s\leqq 1$, are connected by the recurence relation*

$$\Psi_{n+1}(s)=\Psi_n(f(s))g(f(s))+\sum_{k=1}^{m} q_k\left[\Psi_n(f(s))-\sum_{r=0}^{k-1} p_{0r}(n)f^r(s)\right]f^{-k}(s)$$

$$+\sum_{k=1}^{m} q_k\sum_{r=0}^{k-1} p_{0r}(n), \qquad n=0,1,\cdots, \quad \Psi_0(s)=1. \tag{13}$$

PROOF. Let $\mathfrak{M}$ be the set of $r\in\{-m,\cdots,-1,0,1,\cdots\}$ such that $\mathbf{P}\{\zeta_0=r\}>0$. Taking into account the independence of the $\xi_i^{(k)}$, ζ_k, $i=1,2,\cdots$, $k=0,1,\cdots$, we have, for any $r\in\mathfrak{M}$, $n=0,1,\cdots$,

$$\mathbf{E}(s^{Z_{n+1}}|\zeta_n=r)=\begin{cases}\mathbf{E}s^{\xi_1^{(n)}+\cdots+\xi_{Z_n+r}^{(n)}}, & r>0,\\ \sum_{k=-r+1}^{\infty} p_{0k}(n)\mathbf{E}s^{\xi_1^{(n)}+\cdots+\xi_{k+r}^{(n)}}+\sum_{k=0}^{-r} p_{0k}(n), & r=-m,\cdots,0.\end{cases}$$

Hence, for $r\in\mathfrak{M}$, $n=0,1,\cdots$,

$$\mathbf{E}(s^{Z_{n+1}}|\zeta_n=r)$$

$$=\begin{cases}\Psi_n(f(s))f^r(s), & r\geqq 0,\\ \left[\Psi_n(f(s))-\sum_{k=0}^{-r-1} p_{0k}(n)f^k(s)\right]f^r(s)+\sum_{k=0}^{-r-1} p_{0k}(n), & -m\leqq r<0.\end{cases} \tag{14}$$

Since, for $0\leqq s\leqq 1$, $n=0,1,\cdots$,

$$\Psi_{n+1}(s)=\mathbf{E}s^{Z_{n+1}}=\sum_{r\in\mathfrak{M}}\mathbf{E}(s^{Z_{n+1}}|\zeta_n=r)\mathbf{P}\{\zeta_n=r\},$$

it is not difficult to see that (13) follows from (14). The lemma is proved.

We set

$$g_1(s)=g(s)+\sum_{k=1}^{m} q_k s^{-k}, \qquad \varkappa_r(s)=\sum_{k=r+1}^{m} q_k(s^{r-k}-1),$$

where $0<s\leqq 1$, $r=0,\cdots,m-1$. Obviously, for $0<s\leqq 1$,

$$0<g_1(s)<\infty, \quad 0\leqq\varkappa_r(s)<\infty, \quad r=0,\cdots,m-1. \tag{15}$$

Moreover, by virtue of (1) we have (see [17], Chap. I, § 8)

$$f_n(s)\geqq f(s)\geqq f(0), \quad 0\leqq s\leqq 1, \quad n=1,2,\cdots$$

Hence, taking (2), (15) into account, we obtain

$$0<g_1(f_n(s))<\infty, \qquad 0\leqq\varkappa_r(f_n(s))<\infty, \tag{16}$$

where $0\leqq s\leqq 1$, $r=0,\cdots,m-1$ and $n=1,2,\cdots$.

Lemma 4. *For* $0\leqq s\leqq 1$, $n=1,2,\cdots$,

$$\Psi_n(s)=c_n(s)-\sum_{r=0}^{m-1}\sum_{k=1}^{n} h_k^{(r)}(s)p_{0r}(n-k), \tag{17}$$

where $c_n(s)=g_1(f_1(s))g_1(f_2(s))\cdots g_1(f_n(s))$,

$$h_n^{(r)}(s)=\varkappa_r(f_n(s))c_{n-1}(s), \quad n>1, \qquad h_1^{(r)}(s)=\varkappa_r(f(s)), \quad r=0,\cdots,m-1.$$

PROOF. We represent (13) in the form

$$\begin{aligned}\Psi_{n+1}(s)&=\Psi_n(f(s))g_1(f(s))-\sum_{k=0}^{m-1}\varkappa_k(f(s))p_{0k}(n), \quad n=0,1,\cdots,\\ \Psi_0(s)&=1, \qquad 0\leqq s\leqq 1.\end{aligned} \tag{18}$$

Hence (17) follows in an obvious way.

Lemma 5. *For* $0\leqq s<1$, $n=1,2,\cdots$,

$$c_n(s)=c(s)\left(1-\frac{L}{(1-s)^{-1}+nB}\right)+\lambda_n(s), \tag{19}$$

$$h_n^{(r)}(s)=\frac{c(s)}{(1-s)^{-1}+nB}\sum_{k=r+1}^{m} q_k(k-r)+\lambda_n^{(r)}(s), \tag{20}$$

where

$$0<c(s)=\prod_{k=1}^{\infty} g_1(f_k(s))<\infty, \quad L=\frac{g_1''(1-)}{2}, \quad B=\frac{f''(1-)}{2} \tag{21}$$

and as $n\to\infty$

$$\lambda_n(s)=o\left(\frac{1}{n}\right), \qquad \lambda_n^{(r)}(s)=o\left(\frac{1}{n}\right) \tag{22}$$

uniformly in $0\leqq s<1$, $r=0,\cdots,m-1$.

PROOF. Since conditions (1)–(3) are fulfilled, we have for $0\leqq s<1$, $n=1,2,\cdots$ (see [6], [18])

$$f_n(s)=1+\frac{1+\delta_n(s)}{(1-s)^{-1}+nB}, \quad \text{where } \lim_{n\to\infty}\sup_{0\leqq s<1}\delta_n(s)=0. \tag{23}$$

Further, since in view of (5), (6), $g_1'(1-)=0$, $L=g_1''(1-)/2<\infty$, we have

$$g_1(s)=1+L(s-1)^2+o((s-1)^2), \qquad s\to 1-. \tag{24}$$

From (23) it follows that, as $n\to\infty$,

$$f_n^{-k}(s)=1+\frac{k}{(1-s)^{-1}+nB}+o\left(\frac{1}{n}\right) \tag{25}$$

uniformly in $0\leqq s<1$, $k=1,\cdots,m$. Taking (23)–(25) into account, we conclude that uniformly in $0\leqq s<1$

$$g_1(f_n(s))=1+\frac{L}{((1-s)^{-1}+nB)^2}+o\left(\frac{1}{n^2}\right), \tag{26}$$

$$\varkappa_r(f_n(s))=\frac{1}{(1-s)^{-1}+nB}\sum_{k=r+1}^{m} q_k(k-r)+o\left(\frac{1}{n}\right), \tag{27}$$

where $r=0,\cdots,m-1$, $n\to\infty$. Obviously (21) follows from (16), (26). Using (16), (26), we have

$$\begin{aligned} c_n(s) &= \prod_{k=1}^{n} g_1(f_k(s)) = c(s)\prod_{k=n+1}^{\infty}\frac{1}{g_1(f_k(s))} \\ &= c(s)\exp\left\{-\sum_{k=n+1}^{\infty}\log\left(1+\frac{L}{((1-s)^{-1}+kB)^2}+\varepsilon_k(s)\right)\right\} \\ &= c(s)\exp\left\{-\sum_{k=n+1}^{\infty}\frac{L}{((1-s)^{-1}+kB)^2}+\tilde{\varepsilon}_n(s)\right\} \\ &= c(s)\left(1-\frac{L}{(1-s)^{-1}+nB}\right)+\lambda_n(s), \qquad 0\leqq s<1, \end{aligned}$$

where as $n\to\infty$ uniformly in $0\leqq s<1$

$$\varepsilon_n(s)=o\left(\frac{1}{n^2}\right),\quad \tilde{\varepsilon}_n^{(s)}=o\left(\frac{1}{n}\right),\quad \lambda_n(s)=o\left(\frac{1}{n}\right).$$

In turn, (20) follows from (19), (27). The lemma is proved.

We set $c_0=1$, $c_n=c_n(0)$, $h_n^{(r)}=h_n^{(r)}(0)$, $r=0,\cdots,m-1$, $n=1,2,\cdots$. Let us introduce the generating functions

$$U(y)=\sum_{n=0}^{\infty} c_n y^n,\quad H_r(y)=\sum_{n=1}^{\infty} h_n^{(r)} y^n, \qquad r=0,\cdots,m-1,\quad 0\leqq y<1.$$

Using lemma 5, it is not difficult to show that the following lemma holds.

Lemma 6. *For* $0\leqq y<1$,

$$U(y)=\frac{c}{1-y}+\frac{cL}{B}\log(1-y)+\alpha(y), \tag{28}$$

$$H_r(y)=\frac{c}{B}\log\frac{1}{1-y}\sum_{k=r+1}^{m} q_k(k-r)+\alpha_r(y), \tag{29}$$

where $0<c<\infty$, $\alpha(y)=\sum_{n=0}^{\infty}\lambda_n y^n$, $\alpha_r(y)=\sum_{n=0}^{\infty}\lambda_n^{(r)}y^n$, *and as* $n\to\infty$

$$\lambda_n=o\left(\frac{1}{n}\right), \tag{30}$$

$$\lambda_n^{(r)}=o\left(\frac{1}{n}\right), \qquad r=0,\cdots,m-1. \tag{31}$$

We set

$$H(y)=1+H_0(y)+\sum_{r=1}^{m-1}G_r(y)H_r(y), \qquad 0\leqq y<1.$$

We agree to say that $\sum_{r=1}^{m-1}\cdot=0$, if $m=1$.

Lemma 7. *For* $0\leqq y<1$,

$$W_0(y)H(y)=U(y). \tag{32}$$

PROOF. After multiplying both sides of (17) by y^n, $0<y<1$, and summing over n for $s=0$, we obtain

$$W_0(y)(H_0(y)+1)+\sum_{r=1}^{m-1}W_{0r}(y)H_r(y)=U(y). \tag{33}$$

Further, for $0\neq r\in\mathfrak{R}$ we have (see [16])

$$p_{0r}(n)=\sum_{k=0}^{n-1}p_{00}(k)\,{}_0p_{0r}(n-k), \qquad n=1,2,\cdots \tag{34}$$

Hence, $W_{0r}(y)=W_0(y)G_r(y)$, $0\leqq y<1$, $0\neq r\in\mathfrak{R}$. The assertion of the lemma now follows from (33).

Lemma 8. *For* $\frac{1}{2}\leqq y<1$,

$$G_r'(y)\leqq b_rG_0'(y), \tag{35}$$

$$G_r''(y)\leqq b_rG_0''(y), \tag{36}$$

where $0<b_r<\infty$, $r\in\mathfrak{R}$.

PROOF. Obviously, ${}_0p_{0r}(n)\,{}_0p_{r0}(1)\leqq{}_0p_{00}(n+1)$, $0\neq r\in\mathfrak{R}$, $n=1,2,\cdots$. Hence, taking (10) into account, we obtain

$${}_0p_{0r}(n)\leqq\frac{{}_0p_{00}(n+1)}{{}_0p_{r0}(1)}=\frac{{}_0p_{00}(n+1)}{p_{r0}(1)}, \qquad 0\neq r\in\mathfrak{R}, \quad n=1,2,\cdots \tag{37}$$

Hence (35), (36) easily follow. The lemma is proved.

We set

$$A_0=B\left(\sum_{k=1}^{m}kq_k+\sum_{r=1}^{m-1}G_r(1)\sum_{k=r+1}^{m}(k-r)q_k\right)^{-1}. \tag{38}$$

By virtue of (3), (4), (7), (12),

$$0<A_0<\infty. \tag{39}$$

Lemma 9. *As* $y \to 1-$,

(40) $$W_0(y) \sim \frac{A_0}{(1-y)\log((1-y)^{-1})},$$

(41) $$G_0'(y) \sim \frac{1}{A_0} \log \frac{1}{1-y}.$$

PROOF. Using (28)–(31) and Lemma 1, it is not difficult to see that (40) follows from (32). Further (see [16]),

(42) $$G_0(y) = 1 - \frac{1}{W_0(y)}, \qquad 0 \leqq y < 1.$$

Since $U(y) > 0$, $0 \leqq y < 1$, we have, in view of (32), (42),

(43) $$G_0'(y) = -\frac{H'(y)}{U(y)} + \frac{H(y)U'(y)}{U^2(y)}, \qquad 0 \leqq y < 1.$$

Taking (12), (28)–(31), (35) and Lemma 1 into account, we deduce that, as $y \to 1-$,

(44) $$\frac{H'(y)}{U(y)} = o(G_0'(y)),$$

(45) $$\frac{H(y)U'(y)}{U^2(y)} \sim \frac{1}{A_0} \log \frac{1}{1-y}.$$

From (43)–(45) follows (41).

Lemma 10. *As* $y \to 1-$,

$$G_0''(y) \sim \frac{1}{A_0(1-y)}.$$

PROOF. By differentiating (43) with respect to y, we obtain

(46) $$G_0''(y) = -\frac{H''(y)}{U(y)} + 2H'(y)\frac{U'(y)}{U^2(y)} - \left(\frac{1}{U(y)}\right)'' H(y), \qquad 0 \leqq y < 1.$$

Using (12), 28)–(31) and Lemma 1, it is not difficult to show that

(47) $$\left(\frac{1}{U(y)}\right)'' H(y) = O\left(\log^2 \frac{1}{1-y}\right), \qquad y \to 1-.$$

Moreover, taking (12), (28)–(31) and Lemma 1 into account, we have

(48) $$\begin{aligned} &\frac{2U'(y)}{U^2(y)}\left[H_0'(y) + \sum_{r=1}^{m-1} G_r(y)H_r'(y)\right] \\ &\quad - \frac{1}{U(y)}\left[H_0''(y) + \sum_{r=1}^{m-1} G_r(y)H_r''(y)\right] = \frac{1+o(1)}{A_0(1-y)}, \qquad y \to 1-. \end{aligned}$$

In view of (28)–(31), (35), (36), (41) and Lemma 1, we conclude that, as $y \to 1-$,

(49) $$\frac{U'(y)}{U^2(y)} \sum_{r=1}^{m-1} G_r'(y)H_r(y) = O\left(\log^2 \frac{1}{1-y}\right),$$

$$\frac{1}{U(y)}\sum_{r=1}^{m-1} G_r'(y)H_r'(y)=O\left(\log\frac{1}{1-y}\right), \tag{50}$$

$$\frac{1}{U(y)}\sum_{r=1}^{m-1} G_r''(y)H_r(y)=o(G_0''(y)). \tag{51}$$

The assertion of the lemma follows from (46)–(51).

3. Proof of Theorem 1

In view of a Tauberian theorem (see [19], p. ch. XIII, § 5, ff.), we have from (41)

$$\sum_{k=1}^{n} k_0p_{00}(k)\sim\frac{1}{A_0}\log n, \qquad n\to\infty. \tag{52}$$

It is not difficult to see that

$$\frac{1-G_0(y)}{1-y}=\sum_{n=0}^{\infty} F(n)y^n, \qquad 0\leqq y<1. \tag{53}$$

Moreover, according to (40) and (42)

$$\frac{1-G_0(y)}{1-y}=\frac{1}{(1-y)W_0(y)}\sim\frac{1}{A_0}\log\frac{1}{1-y}, \qquad y\to 1-. \tag{54}$$

Using (53), (54) and the Tauberian theorem, we obtain

$$\sum_{k=0}^{n} F(k)\sim\frac{1}{A_0}\log n, \qquad n\to\infty. \tag{55}$$

But, for $n=1, 2, \cdots$,

$$\sum_{k=0}^{n} F(k)=\sum_{k=1}^{n} {}_0p_{00}(k)k+(n+1)F(n). \tag{56}$$

From (52), (55) and (56) it follows that

$$F(n)=o\left(\frac{\log n}{n}\right), \qquad n\to\infty. \tag{57}$$

Taking (52), Lemma 10 and the Tauberian theorem into account, we have

$$\sum_{k=1}^{n} k^2{}_0p_{00}(k)\sim\frac{1}{A_0}n, \qquad n\to\infty.$$

Hence it follows (see [19], ch. VIII, § 9) that $\lim_{n\to\infty} F(n)n=\mu$ exists, where $0<\mu\leqq\infty$. We shall show that $\mu<\infty$. In fact, if $\mu=\infty$, then $F(x)$ is slowly varying as $x\to\infty$. Therefore in view of a known representation for a slowly-varying function (see [19], p. 282), we obtain

$$\lim_{n\to\infty}\frac{F(n)n}{\log n}=\infty,$$

which contradicts (57). Thus $F(n)\sim\mu/n$, $n\to\infty$, where $0<\mu<\infty$. Taking (55) into account, we conclude that $\mu=1/A_0$.

4. Proof of Theorem 2

By using Theorem 1 and Erickson's result (see [20], p. 266), we obtain

$$p_{00}(n) \sim \left(\sum_{k=0}^{n} F(k) \right)^{-1}, \qquad n \to \infty.$$

Therefore, in view of (55),

$$p_{00}(n) \sim \frac{A_0}{\log n}, \qquad n \to \infty. \tag{58}$$

According to (34) we have

$$\begin{aligned} p_{0r}(n) &= \sum_{k=1}^{n} {}_0p_{0r}(k) p_{00}(n-k) \\ &= \sum_{k-1}^{[n/2]} {}_0p_{0r}(k) p_{00}(n-k) + \sum_{k=[n/2]+1}^{n} {}_0p_{0r}(k) p_{00}(n-k), \qquad n > 1, r \neq 0. \end{aligned} \tag{59}$$

By virtue of (37) and Theorem 1,

$$\sum_{k=[n/2]+1}^{n} {}_0p_{0r}(k) p_{00}(n-k) = O\left(\sum_{k=[n/2]}^{n} {}_0p_{0r}(k) \right) = O\left(\frac{1}{n}\right), \qquad n \to \infty. \tag{60}$$

As a consequence of (58),

$$\lim_{n\to\infty} \frac{p_{00}(n-k)}{p_{00}(n)} = 1.$$

uniformly in $0 \leqq k \leqq n/2$. Taking (12) into account, we obtain

$$\sum_{k=1}^{[n/2]} {}_0p_{0r}(k) p_{00}(n-k) = p_{00}(n) G_r(1)(1+o(1)), \qquad n \to \infty. \tag{61}$$

From (58)–(61) it follows that $p_{0r}(n) \sim A_r/\log n$, $n \to \infty$, where

$$A_r = G_r(1) A_0, \qquad r \in \mathfrak{R}. \tag{62}$$

In view of (12), (39) we obtain $0 < A_r < \infty$, $r \in \mathfrak{R}$.

5. Proof of Theorem 3

By differentiating (18) with respect to s and taking (1), (5) into account, we obtain $\Psi'_0(1-) = 0$,

$$\Psi'_{n+1}(1-) = \Psi'_n(1-) + \sum_{r=0}^{m-1} p_{0r}(n) \sum_{k=r+1}^{m} (k-r) q_k, \qquad n = 0, 1, \cdots$$

Hence, for $n = 1, 2, \cdots$,

$$\mathbf{E} Z_n = \Psi'_n(1-) = \sum_{r=0}^{m-1} \left(\sum_{u=1}^{m-r} u q_{r+u} \right) \sum_{k=0}^{n-1} p_{0r}(k). \tag{63}$$

By virtue of (38), (62) and Theorem 2,

$$\sum_{r=0}^{m-1} p_{0r}(n) \sum_{k=r+1}^{m} (k-r) q_k \sim \frac{B}{\log n}, \qquad n \to \infty. \tag{64}$$

Obviously, (8) follows from (63), (64).

Differentiating relation (18) twice and taking (1), (3), (5) and (6) into account, we have $\Psi_0''(1-)=0$, while for $n \geqq 0$

$$\Psi_{n+1}''(1-)=\Psi_n''(1-)+\Psi_n'(1-)f''(1-)+g_1''(1-)-\sum_{r=0}^{m-1} p_{0r}(n)$$
$$\times \sum_{k=r+1}^{m}(k-r)(k-r+1)q_k+f''(1-)\sum_{r=0}^{m-1} p_{0r}(n)\sum_{k=r+1}^{m}(k-r)q_k.$$

Hence, we deduce from (63), (64) and Theorem 2 that

$$\Psi_n''(1-)\sim 2B^2\frac{n^2}{\log n}, \qquad n\to\infty. \tag{65}$$

Since

$$\operatorname{Var} Z_n=\Psi_n''(1-)+\mathbf{E}Z_n-(\mathbf{E}Z_n)^2,$$

(9) follows from (8), (65).

6. Proof of Theorem 4

Let us consider the Laplace transform

$$\mathbf{E}\exp(-Z_n t/n^x)=\Psi_n(\exp(-t/n^x)), \qquad t\in[0,\infty),\ n=1,\cdots$$

for arbitrary $x\in[0,1]$. Setting $s=\exp(-t/n^x)$ in (17) and using (19)–(22) and (64) we obtain, for $t>0$, $x\in[0,1]$, and as $n\to\infty$,

$$\Psi_n(\exp(-t/n^x))=c(\exp(-t/n^x))$$
$$-c(e^{-t/n^x})\sum_{k=1}^{n-1}\left(\left[\frac{1}{1-e^{-t/n^x}}+kB\right]^{-1}+\gamma_k\right)\left(\frac{B}{\log(n-k)}+d_{n-k}\right)+o(1), \tag{66}$$

where $\gamma_k=o(1/k)$, $d_k=o(1/\log k)$, $k\to\infty$. Since

$$\lim_{s\to 1-} g_1(f_k(s))=g_1(1-)=1, \qquad k=1,2,\cdots,$$

we have, by taking (26) into account, for $x\in(0,1]$, $t\geqq 0$,

$$\lim_{n\to\infty} c_n(\exp(-t/n^x))=1. \tag{67}$$

With the aid of a formula for the partial sum of a harmonic series (see [21], p. 270), it is not difficult to show that

$$\lim_{n\to\infty}\sum_{k=1}^{[n/2]}\left[\frac{1}{(1-e^{-t/n^x})B}+k\right]^{-1}\frac{1}{\log(n-k)}=1-x, \qquad x\in[0,1],\ t>0. \tag{68}$$

Moreover,

$$\lim_{n\to\infty}\sum_{k=[n/2]+1}^{n-1}\left[\frac{1}{(1-e^{-t/n^x})B}+k\right]^{-1}\frac{1}{\log(n-k)}=0, \qquad x\in[0,1],\ t>0. \tag{69}$$

As a consequence of (68), (69), for $x\in[0,1]$, $t>0$,

$$\lim_{n\to\infty}\sum_{k=1}^{n-1}\left(\left[\frac{1}{1-e^{-t/n^x}}+kB\right]^{-1}+\gamma_k\right)\left(\frac{B}{\log(n-k)}+d_{n-k}\right)=1-x. \tag{70}$$

From (66), (67) and (70) it follows that

$$\lim_{n\to\infty} \Psi_n\,(\exp(-t/n^x)) = x, \qquad x \in [0, 1], \quad t \in (0, \infty).$$

Hence, according to a continuity theorem (see [19], p. 481),

$$\lim_{n\to\infty} \mathbf{P}\left\{\frac{Z_n}{n^x} < \beta\right\} = x$$

for all $\beta > 0$, $x \in [0, 1]$. The assertion of the theorem obviously follows from this.

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