THEORY OF PROBABILITY AND ITS APPLICATIONS

Volume XVII | 1972 | Number 4

# ON NECESSARY AND SUFFICIENT CONDITIONS FOR THE STRONG LAW OF LARGE NUMBERS

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## 0. Formulation and Discussion of Results

Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of independent random variables with respective distribution functions $F_1(x), F_2(x), \cdots, F_n(x), \cdots$. We shall say that the sequence obeys the strong law of large numbers (S.L.L.N.) if

$$\mathbf{P}\left\{\frac{1}{n}\sum_{i=1}^{n} \xi_i \to 0\right\} = 1.$$

When considering conditions for the S.L.L.N. to be applicable, we may assume without loss of generality that the random variables $\xi_n$ are symmetrically distributed (see, for example, [1], Section 1).

Let

$$I_r = \{n: 2^r + 1 \leqq n \leqq 2^{r+1}\} \quad \text{and} \quad \chi_r = 2^{-r} \sum_{n \in I_r} \xi_n.$$

Yu. V. Prokhorov [2] showed that the S.L.L.N. is satisfied if and only if

$$\sum_{r=0}^{\infty} \mathbf{P}(\chi_r \geqq \varepsilon) < \infty. \tag{0.1}$$

Thus, the problem of determining necessary and sufficient conditions for the S.L.L.N. reduces to obtaining upper and lower bounds for the probability that a sum of independent random variables exceeds a prescribed level. As a result, the necessary conditions will coincide with the sufficient ones if there exist positive sequences $\varepsilon_m$ and $\eta_m, \varepsilon_m \leqq \eta_m, m = 1, \cdots, \infty$, such that $\eta_m \to 0$ and the lower bound for $\varepsilon = \varepsilon_m$ is an upper bound to within a factor constant relative to $r$ for $\varepsilon = \eta_m$.

The question naturally arises: In terms of what characteristics of the individual summands must these estimates be formulated if the aim is to obtain sufficient conditions which are at the same time necessary?

In [3], Yu. V. Prokhorov constructed two sequences of independent random variables $\xi_n'$ and $\xi_n''$ such that $\mathbf{D}\xi_n' = \mathbf{D}\xi_n''$ and one obeys the S.L.L.N. while the other does not.

Hence, it follows that necessary and sufficient conditions for the S.L.L.N. cannot be expressed in terms of variances alone. From general considerations, it is very likely that necessary and sufficient conditions for the S.L.L.N. cannot be formulated with the help of a finite number $s$ of moments, since in most cases lower bounds in terms of moments are excessive.

Such a conjecture was expressed earlier in [3] by Yu. V. Prokhorov.

The following example shows that this is actually so.

Let $L_s(x)$ be the Laguerre polynomial $e^x d^{s+1}(x^{s+1} e^{-x})/dx^{s+1}$. Let $a_s$ denote the largest root of $L_s(x)$. Let $p_s(x) = 1/2a_s$ for $|x| \leqq a_s$, and $p_s(x) = 0$ for $|x| > a_s$. Set $q_s = p_s + b_s e^{-x} L_s(x)$ for $x \geqq 0$, where

$$b_s = \frac{1}{2a_s} \min_{0 \leqq x \leqq a_s} |e^{-x} L_s(x)|^{-1},$$

and $q_s(x) = q_s(-x)$ for $x < 0$. Clearly, $q_s(x) \geqq 0$ and

$$\int_{-\infty}^{\infty} q_s(x)\, dx = \int_{-\infty}^{\infty} p_s(x)\, dx = 1.$$

Define two sequences of independent random variables $\xi'_n$ and $\xi''_n$ as follows. Let $\xi'_1$ and $\xi''_1$ have distributions with respective densities $p_s(x)$ and $q_s(x)$ and let $\xi'_n = \xi''_n = 0$ for $2^r < n < 2^{r+1}, r > 0$, while $\xi'_{2^r}$ and $\xi''_{2^r}$ are distributed, respectively, like $c_r \xi'_1$ and $c_r \xi''_1$, where $c_r = 2^r/\sqrt{\log r}$. The necessary and sufficient condition (0.1) becomes in this case

$$\sum_{r=1}^{\infty} \mathbf{P}(\xi_1 > \varepsilon\sqrt{\log r}) < \infty, \qquad \xi_1 = \xi'_1, \xi''_1, \varepsilon > 0. \tag{0.2}$$

For sufficiently large $r$, clearly

$$\mathbf{P}(\xi''_1 > \varepsilon\sqrt{\log r}) > b_s e^{-\varepsilon\sqrt{\log r}} L_s(a_s + 1).$$

Hence, the sequence $\xi''_n$ does not obey the S.L.L.N. At the same time, it is clear that the sequence $\xi'_n$ does. On the other hand, $\mathbf{E}\xi'^k_n = \mathbf{E}\xi''^k_n$, $k \leqq s$, since

$$\int_0^{\infty} x^k L_s(x)\, e^{-x}\, dx = 0, \qquad k \leqq s.$$

The following assertion answers the question; in what terms can necessary and sufficient conditions for the S.L.L.N. be expressed?

Let

$$f_n(h, \varepsilon) = \int_{-n\varepsilon}^{n\varepsilon} e^{hx}\, dF_n(x).$$

Define $h_r(\varepsilon)$ to be the solution of the equation

$$\Psi_r(h, \varepsilon) \equiv \sum_{n \in I_r} f'_n(h, \varepsilon)/f_n(h, \varepsilon) = \varepsilon n_r, \quad \left(f'_n(h, \varepsilon) = \frac{d}{dh} f_n(h, \varepsilon)\right),$$

where $n_r = 2^{r+1}$, for the case where $\sup_h \Psi_r(h, \varepsilon) \geqq \varepsilon n_r$ (the solution is unique by virtue of the monotonicity of $\Psi_r(h, \varepsilon)$). Otherwise, set $h_r(\varepsilon) = \infty$.

**Theorem.** *The S.L.L.N. holds if and only if, for* $\varepsilon > 0$,

(I) $$\sum_{n=1}^{\infty} \mathbf{P}(\xi_n > n\varepsilon) < \infty,$$

(II) $$\sum_{r=1}^{\infty} e^{-\varepsilon h_r(\varepsilon) n_r} < \infty.$$

We shall postpone proving the theorem until Section 1 and in the meantime deduce as corollaries the two validity criteria for the S.L.L.N. due to Yu. V. Prokhorov [3].

**Corollary 1.** *If* $\xi_n < \varphi(n)$ *and* $\varphi(n) = O(n/\log\log n)$, *then the S.L.L.N. is valid if and only if*

$$\sum_{r=1}^{\infty} \exp\left\{-\frac{\varepsilon}{H_r}\right\} < \infty,$$

*where* $H_r = n_r^{-2} \sum_{n \in I_r} \mathbf{D}\xi_n$.

PROOF. Without loss of generality, we may assume that $\varphi(n_r) = n_r/\log r$.

NECESSITY. Let $h_r(\varepsilon) > \bar{h}_r = \varphi^{-1}(n_r)$. Clearly, $f_n(\bar{h}_r, \varepsilon) < e$ for $n \leqq n_r$. If, in addition, $n\varepsilon > \varphi(n_r)$, then $f'_n(\bar{h}_r, \varepsilon) > \sigma_n^2 \bar{h}_r/e$, where $\sigma_n^2 = \mathbf{D}\xi_n$. Therefore,

$$\bar{h}_r \sum_{n \in I_r} \sigma_n^2/e^2 < \Psi_r(\bar{h}_r, \varepsilon) \leqq \varepsilon n_r.$$

Thus, for sufficiently large $r$,

(0.3) $$-e^2\varepsilon/H_r < -\log r.$$

Now let $h_r(\varepsilon) \leqq \bar{h}_r$. Clearly, $f_n(h_r(\varepsilon), \varepsilon) < e$ and $f'_n(h_r(\varepsilon), \varepsilon) > \sigma_n^2 h_r(\varepsilon)/e$ for $n \leqq n_r$ such that $n\varepsilon > \varphi(n_r)$. Hence, in exactly the same way as (0.3), we can deduce that

(0.4) $$-e^2\varepsilon/H_r < -n_r h_r(\varepsilon).$$

From (0.3) and (0.4) follows the convergence of $\sum_{r=1}^{\infty} \exp\{-\varepsilon/H_r\}$ for $\varepsilon > 0$.

SUFFICIENCY. If $\varepsilon h_r(\varepsilon) < 2\bar{h}_r$, then

$$f'_n(h_r(\varepsilon), \varepsilon) < e^{2/\varepsilon}\sigma_n^2 h_r(\varepsilon).$$

If, in addition, $n\varepsilon > \varphi(n_r)$, then $f_n(h_r(\varepsilon), \varepsilon) \geqq 1$. Using these two estimates, we find analogously to (0.4) that

$$-e^{-2/\varepsilon}\varepsilon/H_r > -n_r h_r(\varepsilon),$$

if $r$ is sufficiently large.

Thus, for $\varepsilon > 0$,

$$\sum_{r=r(\varepsilon)}^{\infty} e^{-\varepsilon n_r h_r(\varepsilon)} < \sum_{r=r(\varepsilon)}^{\infty} (r^{-2} + \exp\{-e^{-2/\varepsilon}\varepsilon^2/H_r\}) < \infty,$$

i.e., condition (II) of the theorem is satisfied.

Condition (I) is evidently also satisfied.

**Corollary 2.** *If* $\mathbf{P}(\xi_n = \pm a_n) = p_n/2$, $\mathbf{P}(\xi_n = 0) = 1 - p_n$, $a_n = o(n)$ *and there are* $c_1 > 0$ *and* $c_2 > 0$ *such that*

$$c_1 \leqq \min_{n\in I_r} a_n/\max_{n\in I_r} a_n, \qquad c_2 \leqq \min_{n\in I_r} p_n/\max_{n\in I_r} p_n,$$

*then the S.L.L.N. is applicable to the sequence* $\xi_n$ *if and only if*

$$\sum_{r=1}^{\infty} \exp\left\{-\frac{\varepsilon n_r}{a_{n_r}} \operatorname{arc\,sinh} \frac{\varepsilon}{a_{n_r}p_{n_r}}\right\} < \infty, \qquad \varepsilon > 0.$$

Proof. Necessity. Without loss of generality, we may assume that $n\varepsilon > a_n$ for all $n$. Suppose $\varepsilon/c_1 a_{n_r} < 1/4$. If, moreover, $\min_{n\in I_r} f_n(h_r(\varepsilon), \varepsilon) > 2$, then

$$e^{h_r(\varepsilon)a_n}p_n > f_n(h_r(\varepsilon), \varepsilon) - 1 > f_n(h_r(\varepsilon), \varepsilon)/2, \qquad n \in I_r.$$

Hence,

$$\begin{aligned}\varepsilon n_r &= \sum_{n\in I_r} \frac{2a_n p_n \sinh h_r(\varepsilon)a_n}{f_n(h_r(\varepsilon), \varepsilon)} > \frac{1}{2}\sum_{n\in I_r} a_n(1 - e^{-2h_r(\varepsilon)a_n}) \\ &\geqq \tfrac{1}{2} n_r c_1 a_{n_r}(1 - e^{-2c_1 h_r(\varepsilon)a_{n_r}}).\end{aligned}$$

This implies that $\exp\{2c_1 h_r(\varepsilon)a_{n_r}\} < 2$ and hence $f_n(h_r(\varepsilon), \varepsilon) < 2^{1/2c_1^2}$. But if $\min_{n\in I_r} f_n(h_r(\varepsilon), \varepsilon) \leqq 2$, then $\max_{n\in I_r} f_n(h_r(\varepsilon), \varepsilon) < 2^{1/c_1} + 1$. Thus, for $\varepsilon/c_1 a_{n_r} > 1/4$,

$$f_n(h_r(\varepsilon), \varepsilon) < \Lambda_1, \qquad n \in I_r,$$

where $\Lambda_1 = \max[2^{1/2c_1^2}, 2^{1/c_1} + 1]$. Therefore,

$$\varepsilon n_r = \sum_{n\in I_r} \frac{2a_n p_n \sinh h_r(\varepsilon)a_n}{f_n(h_r(\varepsilon), \varepsilon)} > 2\Lambda_1^{-1} n_r c_1 c_2 a_{n_r} p_{n_r} \sinh h_r(\varepsilon)c_1 a_{n_r}.$$

Hence, we find

$$h_r(\varepsilon) < \frac{1}{c_1 a_{n_r}} \operatorname{arc\,sinh} \frac{\Lambda_2 \varepsilon}{a_{n_r}p_{n_r}},$$

where $\Lambda_2 = \Lambda_1/2c_1c_2$ providing that $\varepsilon/c_1 a_{n_r} < 1/4$.

But if $\varepsilon/c_1 a_{n_r} \geqq 1/4$, then

$$\frac{\varepsilon}{a_{n_r}} \operatorname{arc\,sinh} \frac{\varepsilon}{a_{n_r}p_{n_r}} \geqq \frac{c_1}{4} \operatorname{arc\,sinh} \frac{c_1}{4}.$$

The last two estimates and hypothesis (II) of the theorem imply the necessity of the conditions of Corollary 2.

Sufficiency. Observe that $f_n(h, \varepsilon) \geqq 1$ for $h > 0$. Therefore,

$$\Psi_r(h, \varepsilon) \leqq \sum_{n\in I_r} f_n'(h, \varepsilon) \leqq 2n_r a_{n_r} p_{n_r} c_1^{-1} c_2^{-1} \sinh a_{n_r} h/c_1 c_2, \qquad h > 0.$$

Hence we obtain

$$h_r(\varepsilon) \geqq \frac{c_1 c_2}{a_{n_r}} \operatorname{arc\,sinh} \frac{c_1 c_2 \varepsilon}{2a_{n_r}p_{n_r}}.$$

From the last estimate it follows that condition (II) of the theorem is satisfied.

Let $\bar{h}_r(\varepsilon)$ be the root of the equation

$$\sum_{n\in I_r} \int_{-n}^{n} x\, e^{hx}\, dF_n(x) = n_r\varepsilon.$$

In [1], it is proved that the condition

$$\sum_{r=1}^{\infty} \exp\{-n_r\varepsilon\bar{h}_r(\varepsilon)\} < \infty, \qquad \varepsilon > 0 \tag{0.5}$$

is sufficient for the S.L.L.N. if $|\xi_n| < n$ for all $n$. Let us now show that condition (0.5) easily leads to conditions (I) and (II) of the theorem.

Indeed, $h_r(\varepsilon) \geqq \bar{h}_r(\varepsilon')$, $\varepsilon' = \varepsilon \min_{n\in I_r} \mathbf{P}(|\xi_n| < n\varepsilon)$. Hence condition (II) results if

$$\lim_{n\to\infty} \mathbf{P}(|\xi_n| < n\varepsilon) > 0, \qquad \varepsilon > 0.$$

Further, for $2n_r\varepsilon < n$,

$$\sum_{n\in I_r} \mathbf{P}(\xi_n > 2n_r\varepsilon) < e^{-2n_r\bar{h}_r(\varepsilon)\varepsilon} \int_{2n_r\varepsilon}^{n} e^{\bar{h}_r(\varepsilon)x}\, dF_n(x).$$

Without loss of generality, we may assume that $2\varepsilon n_r\bar{h}_r(\varepsilon) > 1$. Therefore,

$$\sum_{n\in I_r} \int_{2n_r\varepsilon}^{n} e^{\bar{h}_r(\varepsilon)x}\, dF_n(x) < 4\bar{h}_r(\varepsilon) \sum_{n\in I_r} \int_{2n_r\varepsilon}^{n} (e^{\bar{h}_r(\varepsilon)x} - e^{-\bar{h}_r(\varepsilon)x})x\, dF_n(x) < 4\varepsilon n_r\bar{h}_r(\varepsilon).$$

The last two estimates imply conditions (I) and (II). We can now derive conditions (I) and (II) from Kolmogorov's sufficient condition

$$\sum_{n=1}^{\infty} \frac{\mathbf{D}\xi_n}{n^2} < \infty.$$

It is not hard to see that

$$\int_{-n_r\varepsilon}^{n_r\varepsilon} e^{hx}x\, dF_n(x) = \int_{0}^{n_r\varepsilon} (e^{hx} - e^{-hx})x\, dF_n(x)$$

$$< h\, e^{hn_r\varepsilon} \int_{0}^{n_r\varepsilon} x^2\, dF_n(x) < e^{2hn_r\varepsilon} n_r^{-1}\varepsilon^{-1} \int_{0}^{n_r\varepsilon} x^2\, dF_n(x).$$

On the other hand,

$$\sum_{n\in I_r} \int_{-n_r\varepsilon}^{n_r\varepsilon} e^{h_r(\varepsilon)x}x\, dF_n(x) > n_r\varepsilon \min_{n\in I_r} \int_{-n_{r-1}\varepsilon}^{n_{r-1}\varepsilon} e^{h_r(\varepsilon)x}\, dF_n(x).$$

Hence,

$$e^{-2h_r(\varepsilon)n_r\varepsilon} < \sum_{n\in I_r} \mathbf{D}\xi_n/\varepsilon^2 n_r^2(1 - n_{r-1}^{-2}\varepsilon^{-2} \max_{n\in I_r} \mathbf{D}\xi_n).$$

This clearly implies condition (II). The validity of condition (I) is a consequence of Chebyshev's inequality.

## 1. Proof of the Theorem

NECESSITY. Condition (I) is known to be necessary for the S.L.L.N. (see, for example, [4], p. 60, Theorem 2.7.2).

Let $Q_r(h, \delta) = \prod_{n \in I_r} f_n(h, \delta)$. Let

$$F_n(x, \delta) = \begin{cases} F_n(x), |x| \leqq n\delta, \\ F_n(n\delta), |x| > n\delta, \end{cases}$$

and $G_r(x, \delta) = F_{n_{r-1}+1} * F_{n_{r-1}+2} * \cdots * F_{n_r}(x, \delta)$. Denote by $h_r(\delta, \varepsilon)$ the root of the equation

$$\frac{d}{dh} \log Q_r(h, \delta) = n_r \varepsilon,$$

if it exists. Otherwise, set $h_r(\delta, \varepsilon) = \infty$. It is not hard to see that

$$G_r(\infty, \delta) - G_r(\eta n_r, \delta) = Q_r(h, \delta) \int_{n_r \eta}^{\infty} e^{-hx} \, d\bar{G}_r(x, h, \delta), \tag{1.1}$$

for $\eta > 0$, where

$$\bar{G}_r(x, h, \delta) = \int_{-\infty}^{x} e^{hy} \, dG_r(y, \delta)/Q_r(h, \delta).$$

Let $\zeta_r(h, \delta)$ be a random variable with distribution function $\bar{G}_r(x, h, \delta)$. It is not hard to see that

$$\zeta_r(h, \delta) = \sum_{n \in I_r} \xi_n(h, \delta),$$

where the $\xi_n(h, \delta)$ are mutually independent and

$$\mathbf{P}(\xi_n(h, \delta) < x) = \int_{-\infty}^{x} e^{hy} \, dF_n(y, \delta)/f_n(h, \delta).$$

In consequence of (0.1), $\chi_r \to 0$ in probability as $r \to \infty$. Condition (I) implies that

$$\lim_{r \to \infty} \mathbf{P}\left( \max_{n \in I_r} \left| \frac{\xi_n}{n_r} \right| > \delta \right) = 0,$$

for $\delta > 0$. Now applying the criterion for degenerate convergence, we obtain

$$\lim_{r \to \infty} \frac{1}{n_r^2} \sum_{n \in I_r} \int_{|x| < n_r \delta} x^2 \, dF_n(x) = 0$$

(see, for example, [5], p. 317). Hence,

$$\lim_{r \to \infty} n_r^{-2} \sum_{n \in I_r} \int_{-n\delta}^{c/h} e^{hx} x^2 \, dF_n(x)/f_n(h, \delta) = 0,$$

for $c > 0$ uniformly in $h > 0$. Further, for $\eta > 0$, there is a $c > 0$ such that

$$\sum_{n \in I_r} \int_{c/h_r(\delta,\varepsilon)}^{n\delta} e^{h_r(\delta,\varepsilon)x} \, dF_n(x)/f_n(h_r(\delta, \varepsilon), \delta) < (1 - e^{-2c})^{-1}$$
$$\times \sum_{n \in I_r} \int_{c/h_r(\delta,\varepsilon)}^{n\delta} (e^{h_r(\delta,\varepsilon)x} - e^{-h_r(\delta,\varepsilon)x}) x \, dF_n(x)/f_n(h_r(\delta, \varepsilon), \delta) < (1 + \eta)\varepsilon n_r.$$

Hence, for $\eta$, there is an $r_0$ such that, for $r > r_0$,

$$\mathbf{D}\zeta_r(h_r(\delta,\varepsilon),\delta) \leqq \sum_{n\in I_r} \mathbf{E}\xi_n^2(h_r(\delta,\varepsilon),\delta)$$

$$\leqq \sum_{n\in I_r}\left[\int_{-n\delta}^{c/h_r(\delta,\varepsilon)} e^{hx}x^2\,dF_n(x) + \delta n_r \int_{c/h_r(\delta,\varepsilon)}^{n\delta} e^{hx}x\,dF_n(x)\right]$$

$$\Big/ f_n(h_r(\delta,\varepsilon),\delta) < (1+\eta)\,\delta\varepsilon n_r^2.$$

Thus, without loss of generality we may assume that, for $\delta < \varepsilon$,

$$\mathbf{D}\zeta_r(h_r(\delta,\varepsilon),\delta) < \frac{\delta+\varepsilon}{2}\varepsilon n_r^2.$$

Therefore,

$$\mathbf{P}\{|\zeta_r(h_r(\delta,\varepsilon),\delta) - \varepsilon n_r| > n_r(\varepsilon-\eta)\} \leqq \frac{(\delta+\varepsilon)\varepsilon}{2(\varepsilon-\eta)^2},$$

providing $h_r(\delta,\varepsilon) < \infty$ and $\delta < \varepsilon$. Hence we obtain

$$\int_{n_r\eta}^{\infty} e^{-h_r(\delta,\varepsilon)x}\,d\bar{G}_r(x,h,\delta) > \frac{\varepsilon(\varepsilon-\delta) - 2\varepsilon\eta + 2\eta^2}{2(\varepsilon-\eta)^2}\, e^{-n_r h_r(\delta,\varepsilon)(2\varepsilon-\eta)},$$

(1.2)

$$2(\varepsilon-\eta)^2 > (\delta+\varepsilon)\varepsilon.$$

From (1.1) and (1.2) it follows that, for $2(\varepsilon-\eta)^2 > (\delta+\varepsilon)\varepsilon$,

$$\frac{2(\varepsilon-\eta)^2}{\varepsilon(\varepsilon-\delta) - 2\varepsilon\eta + 2\eta^2}(G_r(\infty,\delta) - G_r(n_r\eta,\delta))/(G_r(\infty,\delta) - G_r(-\infty,\delta))$$

(1.3)

$$> e^{-n_r(2\varepsilon-\eta)h_r(\delta,\varepsilon)},$$

since $Q_r(h,\delta) \geqq G_r(\infty,\delta) - G_r(-\infty,\delta)$. On the other hand,

$$0 \leqq G_r(x,\infty) - G_r(x,\delta) \leqq \sum_{n\in I_r} \mathbf{P}(|\xi_n| > \delta n). \tag{1.4}$$

From (0.1), (1.3), (1.4) and condition (I), we conclude that

$$\sum_{r=1}^{\infty} e^{-n_r(\varepsilon+2\delta)h_r(\delta,\varepsilon)} < \infty, \qquad \frac{\varepsilon}{2,3} \leqq \delta < \varepsilon. \tag{1.5}$$

Observe that (1.5) is even more valid if all or a part of the quantities $h_r(\delta,\varepsilon)$ are infinite.

Suppose that

$$\sum_{n\in I_r}\int_{n\delta}^{n\varepsilon} x\,e^{h_r(\varepsilon)x}\,dF_n(x) > \frac{\varepsilon n_r}{3}, \qquad \delta < \varepsilon.$$

Then

$$n_r e^{\varepsilon n_r h_r(\varepsilon)} \sum_{n\in I_r}(1 - F_n(n\delta)) > \frac{\varepsilon n_r}{3}$$

and hence

$$e^{-n_r\varepsilon h_r(\varepsilon)} < \frac{3}{\varepsilon}\sum_{n\in I_r}(1 - F_n(n\delta)). \tag{1.6}$$

Now let

$$\sum_{n\in I_r}\int_{n\delta}^{n\varepsilon} x\, e^{h_r(\varepsilon)x}\, dF_n(x) < \frac{\varepsilon n_r}{3}, \qquad \delta < \varepsilon.$$

Then, for sufficiently large $r$,

$$\sum_{n\in I_r}\frac{\int_{n\delta\leqq|x|\leqq n\varepsilon} x\, e^{h_r(\varepsilon)x}\, dF_n(x)}{f_n(h_r(\varepsilon),\varepsilon)} \leqq \tfrac{1}{2}\varepsilon n_r,$$

since $f_n(h_r(\varepsilon),\varepsilon) \geqq 2/3$ if $n$ is sufficiently large. This implies that

$$\sum_{n\in I_r} f_n'(h_r(\varepsilon),\delta)/f_n(h_r(\varepsilon),\delta) \geqq \sum_{n\in I_r} f_n'(h_r(\varepsilon),\delta)/f_n(h_r(\varepsilon),\varepsilon) \geqq \frac{\varepsilon n_r}{2}$$

and hence,

$$h_r(\varepsilon) \geqq h_r\left(\delta,\frac{\varepsilon}{2}\right), \tag{1.7}$$

if $r$ is sufficiently large.

From (1.6) and (1.7), we conclude that

$$e^{-n_r\varepsilon h_r(\varepsilon)} < \max\left[\frac{3}{\varepsilon}\sum_{n\in I_r}\left(1 - F_n\left(\frac{n\varepsilon}{4}\right)\right), e^{-n_r\varepsilon h_r(\varepsilon/4,\varepsilon/2)}\right], \qquad r > r(\varepsilon).$$

Thus, by virtue of (I) and (1.5),

$$\sum_{r=r(\varepsilon)}^{\infty} e^{-n_r\varepsilon h_r(\varepsilon)} < \frac{3}{\varepsilon}\sum_{n=2^{r(\varepsilon)}}^{\infty}\left(1 - F_n\left(\frac{n\varepsilon}{4}\right)\right) + \sum_{n=2^{r(\varepsilon)}}^{\infty} e^{-n_r\varepsilon h_r(\varepsilon/4,\varepsilon/2)} < \infty,$$

as required.

Sufficiency. On account of (1.1),

$$G_r(\infty,\delta) - G_r(n_r\eta,\delta) \leqq e^{-hn_r\eta} Q_r(h,\delta). \tag{1.8}$$

Observe that

$$\frac{d^2}{dh^2}\log Q_r(h,\delta) \geqq 0,$$

i.e., $\log Q_r(h,\delta)$ is convex down with respect to $h$. Therefore,

$$\int_0^x \frac{d}{dh}\log Q_r(h,\delta)\, dh \leqq x n_r\varepsilon, \qquad 0 < x \leqq h_r(\delta,\varepsilon).$$

Hence, we obtain

$$\log Q_r(h,\delta) - \log Q_r(0,\delta) \leqq \varepsilon n_r h_r(\delta,\varepsilon), \qquad 0 \leqq h \leqq h_r(\delta,\varepsilon),$$

and therefore

$$Q_r(h(\delta,\varepsilon),\delta) \leqq e^{\varepsilon n_r h_r(\delta,\varepsilon)}, \tag{1.9}$$

since $Q_r(0,\delta) \leqq 1$.

From (1.8) and (1.9), we see that, for $h_r(\delta, \varepsilon) < \infty$,

$$G_r(\infty, \delta) - G_r(n_r\eta, \delta) \leqq e^{(\varepsilon - \eta)n_r h_r(\delta,\varepsilon)}.$$

Now letting $\delta = \varepsilon$ and $\eta = 2\varepsilon$, we have, for $h_r(\varepsilon) < \infty$,

$$G_r(\infty, \varepsilon) - G_r(2n_r\varepsilon, \varepsilon) \leqq e^{-\varepsilon n_r h_r(\varepsilon)}. \tag{1.10}$$

But if $h_r(\varepsilon) = \infty$, then

$$\sum_{n \in I_r} M_n \leqq \varepsilon n_r, \tag{1.11}$$

where $M_n = \text{ess sup } \xi_n$.

From (1.10), (1.11), (1.14) and conditions (I) and (II), it follows that

$$\sum_{r=1}^{\infty} \mathbf{P}(\chi_r \geqq \varepsilon) < \infty,$$

as required.

*Received by the editors*
*April* 12, 1971

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