ON STRENGTHENING LYAPUNOV TYPE ESTIMATES (THE CASE WHEN THE DISTRIBUTION OF THE SUMMANDS IS CLOSE TO THE NORMAL DISTRIBUTION)

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Introduction

Let $\{X_j\}_{j=1}^n$ be a sequence of random variables (r.v.) such that

$$F_j(x) = \mathbf{P}(X_j < x);\ \mathbf{E}X_j = 0,\ \mathbf{E}X_j^2 = \sigma_j^2,\ B^2 = \sum_{j=1}^n \sigma_j^2,\ \mathbf{E}|X_j|^3 = \mu_j < \infty.$$

Let

$$\zeta_n = \sum_{j=1}^n X_j/B, \qquad \bar{F}(x) = \mathbf{P}(\zeta_n < x),$$

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^x e^{-u^2/2}\, du, \qquad \Delta(x) = \bar{F}(x) - \Phi(x), \qquad \delta = \sup_x |\Delta(x)|.$$

We agree to denote by L and b, with or without subscripts, absolute positive constants.

The problem of obtaining estimates for the quantity δ in which the right-hand sides tend to zero as the distributions of the summands approximate the normal distribution was apparently first considered in [1]. (Earlier in [2], the case was investigated where they become close to stable laws.)

In particular, for a pair of distributions (F, G) [1] and [2] introduced the quantity

$$\tilde{v}^{(r)}(F, G) = \int_{-\infty}^{\infty} |x|^r |(F - G)(dx)|$$

into consideration, termed the r-th absolute pseudomoment, and obtained the estimate (cf. [1])

(0.1) $\delta \leqq L_{01}(\tilde{\Lambda}/B^3)^{1/4},$

where $L_{01} = 0.28845$, $\tilde{\Lambda} = \sum_{j=1}^n \tilde{v}_j$ and $\tilde{v}_j = \tilde{v}^{(3)}(F_j(x), \Phi(x/\sigma_j))$.

In [3], along with the pseudomoment $\tilde{v}^{(r)}$, another characteristic of the distributions was considered which is more natural to the estimation of the closeness of distributions in the uniform metric. The quantity introduced there differs merely by the absence of the factor r from the quantity

$$v^{(r)}(F, G) = r \int_{-\infty}^{\infty} |x|^{r-1}|F(x) - G(x)| \, dx,$$

which in the following we shall call the r-th absolute difference moment of the pair (F, G). In particular, it was noted in [3] that

$$v^{(r)}(F, G) \leqq \tilde{v}^{(r)}(F, G).$$

In the estimation of the closeness of the distributions of two convolutions in the Lévy metric, an inequality was obtained in [3] differing from (0.1) in that the pseudomoments are replaced by difference moments.

In the general case, estimates of type (0.1) do not lead to the estimate of the Berry–Esseen theorem (cf. [4]):

$$(0.2) \qquad\qquad \delta \leqq L_{02}\left(\sum_{j=1}^{n} \mu_j\right)\bigg/ B^3.$$

Nevertheless, as will be shown below, the estimate (0.1) is exact within the limits of the applicable information about the distributions of the r.v. X_j.

We observe first of all that a sequence of distribution functions $G_m(x)$ ($m = 1, \cdots, \infty$) exists such that $\tilde{v}_m = \tilde{v}^{(3)}(G_m, \Phi) \to 0$ as $m \to \infty$ while $\sup_x |G_m(x) - \Phi(x)| \geqq b_1 \tilde{v}_m^{1/4}$. To construct an example, we merely have to set $G_m(x) = \Phi(x)$ for $|x| > 1/m$; $G_m(x) = \Phi(-1/m) + \alpha_1$, where $\alpha_1 = m^2 \int_0^{1/m} x^2 \times d\Phi(x)$, for $-1/m < x < 0$; $G_m(x) = \Phi(-1/m) + \alpha_1 + \alpha_2$, where $\alpha_2 = 2\int_0^{1/m} d\Phi(x) - 2\alpha_1$, for $0 < x < 1/m$. (The existence of such a sequence of functions was pointed out to us by V. V. Sazanov.)

Suppose now $\sigma_2 = \cdots = \sigma_n = 0$ and $\mathbf{P}(X_1 < x) = G_m(x)$. Then the estimate (0.1) is true. Observe that, in that case, $\sigma_1 = 1$. A less trivial example with $\sigma_2, \cdots, \sigma_n$ all non-vanishing is also not hard to construct if the variances $\sigma_2^2, \cdots, \sigma_n^2$ are sufficiently small, the r.v. $X_2, \cdots, X_n$ are normally distributed, and the familiar smoothing theorems are used (see, for example, [4]).

The case described above is of the nature of an extreme case. In other cases, estimates of a different type prove to be exact. Thus, for identically distributed summands, the convolution method was used in [5] when $\sigma_1 = 1$ to obtain the estimate

$$(0.3) \qquad\qquad \Delta(x) \leqq L_{03} \max\{\tilde{v}, \tilde{v}^{1/4}\}/\sqrt{n},$$

where $\tilde{v} = \tilde{v}^{(3)}(F_j, \Phi)$. It is easy to see that (0.3) leads to (0.2).

The convolution method was also used in [6] to obtain estimates for identically distributed multidimensional summands which reduce in the one-dimensional case to the estimate (0.3) with an additional factor of $(1 + |x|^3)^{-1}$.

This paper obtains estimates for the quantity δ in the case of non-identically distributed summands. They imply, in particular, inequalities (0.1)–(0.3). In deriving these estimates, we shall make use of the method of characteristic functions.

Some of the results were reported briefly in [7].

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1. Formulation and Discussion of Results

Apparently, the most graphic consequences of the theorems stated below are the following estimates.

Let $C = \sum_{j=1}^{n} \sigma_j^3$, $v_j = v^{(3)}(F_j(x), \Phi(x/\sigma_j))$ and $\Lambda = \sum_{j=1}^{n} v_j$. Then

$$(1.1) \qquad \delta \leqq L_{11} \max\{\Lambda/B^3\, ; (\Lambda/B^3)^{1/4}(C/B^3)^{3/4}\}.$$

Further, if $B^2/n = 1$, $v = \Lambda/n$ and $\min_j \sigma_j \geqq v^{1/4}$, then

$$(1.1^*) \qquad \delta \leqq 4.2 v^{1/4}/\sqrt{n}.$$

There is no loss of generality in the condition that $B^2 = n$ is finite.

In order to bring the resultant estimates into general form, we need some additional notation.

Let us agree that $\max \sigma_j = \sigma_1 \geqq \sigma_2 \geqq \cdots \geqq \sigma_n = \min \sigma_j$. Let $\sigma(u)$ be a polygonal path defined on $[1, n+1]$ with nodes at the points $(1, \sigma_1), (2, \sigma_2), \cdots, (n, \sigma_n), (n+1, 0)$. Let $B^2(u) = \sigma^2(u)[u] + \sum_{j=[u]+1}^{n} \sigma_j^2$ ($[u]$ is the integral part of u) and $C(u) = \sigma^3(u)[u] + \sum_{j=[u]+1}^{n} \sigma_j^3$. Observe that $B(u)$ and $C(u)$ are continuous functions of u.

Theorem 1. *The following estimate holds:*

$$(1.2) \qquad \delta \leqq L_{12} \min_{u}\{\Lambda/B^3(u) + (\Lambda/B^3)^{1/4}(C(u)/B^3(u))^{3/4}\}.$$

Corollary 1.1. *Since $B(1) = B$ and $C(1) = C$, estimate (1.1) follows from (1.2).*

Denote the first term in the braces on the right-hand side of (1.2) by $A_1(u)$ and the second term by $A_2(u)$. $A_1(u)$ is non-decreasing and $A_2(u)$ non-increasing with increasing u. (The last follows, for example, from the fact that $A_2(u)$ is differentiable for non-integral values of u and its derivative is non-positive.) Further, for $\Lambda \leqq C$, $A_1(1) \leqq A_2(1)$ and the equation $A_1(u) = A_2(u)$ has a solution. This last equation is equivalent to

$$(1.3) \qquad \Lambda B = B(u)C(u).$$

Corollary 1.2. *Suppose $\Lambda \leqq C$ and u_1 is any solution of (1.3). Then*

$$(1.4) \qquad \delta \leqq L_{13}\frac{\Lambda}{B^3(u_1)},$$

and (1.4) is equivalent to (1.2) up to a constant factor.

In many situations, an estimate which is equivalent to (1.2) up to a constant factor can be more convenient and yields the following.

Theorem 2. *The following inequality holds*:

$$(1.5) \qquad \delta \leqq L_{14} \min_u \left\{ \frac{\Lambda}{B^3(u)} + \frac{\sigma(u)}{B} \right\}.$$

The evaluation of L_{14}, which we shall omit, shows that $L_{14} \leqq 2.1$. If $v = \Lambda/n$ and

$$(1.6) \qquad \sigma_n \geqq v^{1/4}(B/\sqrt{n})^{1/4},$$

then $\Lambda/B^3(n) \leqq \sigma(n)/B$, from which one can obtain

Corollary 2.1. *Under condition* (1.6),

$$(1.7) \qquad \delta \leqq 2L_{14}(\Lambda/B^3)^{1/4}n^{-3/8}.$$

If $B^2/n = 1$, (1.7) reduces to the estimate (1.1*) and condition (1.6) to the condition $\sigma_n \geqq v^{1/4}$.

In exactly the same way as (1.2) led to (1.4), (1.5) leads to

Corollary 2.2. *Let $\Lambda \leqq \sigma_1 B^2$ and let u_2 be a solution of the equation*

$$(1.8) \qquad \Lambda B = B^3(u)\sigma(u).$$

Then

$$(1.9) \qquad \delta \leqq L_{15}\Lambda/B^3(u_2),$$

where $L_{15} \leqq 4.2$, and (1.9) is equivalent to (1.5) up to a constant factor.

The equivalence of the estimates (1.2) and (1.5) will be proved later but at this point we state that Theorem 2 is interesting, in particular, in that a direct proof of it is much simpler than the proof of Theorem 1. However, since a direct proof of (1.2) is, in the opinion of the authors, of methodological interest in its own right, we shall prove Theorems 1 and 2 separately.

We shall now make some remarks about the theorems.

REMARK 1. Since $C \leqq B^3$, (1.1) implies (0.1). It is not hard to verify that (1.1) can be reduced in the case of identically distributed summands to (0.3) and since $v_j \leqq \mu_j + 4\sigma_j^3/\sqrt{2\pi}$ and $\sigma_j^3 \leqq \mu_j$, (1.1) implies also (0.2).

REMARK 2. If $\Lambda \geqq C$, it follows from (1.2) that

$$(1.10) \qquad \delta \leqq L_{16}\Lambda/B^3.$$

This same estimate also follows from (1.5) if $\Lambda \geqq \sigma_1 B^2$. These are trivial cases because when $\Lambda \geqq \sigma_1 B^2$ (and when $\Lambda \geq C$), (1.10) follows from (0.2) since $\mu_j \leqq v_j + 4\sigma_j^3/\sqrt{2\pi}$.

REMARK 3. Let $l = [u_2]$ and $\sigma_* = \sigma(u_2)$. Since $\sigma_*/B = \Lambda/B^3(u_2) \leqq \Lambda/(l\sigma_*^2)^{3/2}$, we have $\sigma_* \leqq (\Lambda B)^{1/4}l^{-3/8}$ and, under the condition $\Lambda \leqq \sigma_1 B^2$,

$$\delta \leqq L_{17}\frac{v_*^{1/4}}{\sqrt{l}},$$

where $v_* = \sqrt{l}\Lambda/B^3$ and $L_{17} \leqq 4.2$.

Let us now prove the equivalence of (1.2) and (1.5). We shall confine ourselves to the case $\Lambda \leqq C$ (and, hence, $\Lambda \leqq \sigma_1 B^2$) and prove the equivalence of (1.4) and (1.9). From (1.3) and (1.8) it is not hard to deduce that $u_1 \leqq u_2$ and hence that $B(u_1) \geqq B(u_2)$.

Let us show that

$$(1.11) \qquad B(u_1) \leqq 2B(u_2).$$

If

$$\sum_{j=[u_2]+1}^{n} \sigma_j^2 \geqq \sigma^2(u_1)[u_1] + \sum_{j=[u_1]+1}^{[u_2]} \sigma_j^2,$$

then

$$B^2(u_1) = \sigma^2(u_1)[u_1] + \sum_{j=[u_1]+1}^{[u_2]} \sigma_j^2 + \sum_{j=[u_2]+1}^{n} \sigma_j^2 \leqq 2 \sum_{j=[u_2]+1}^{n} \sigma_j^2 \leqq 2B(u_2).$$

Now let

$$\sum_{j=[u_2]+1}^{n} \sigma_j^2 \leqq \sigma^2(u_1)[u_1] + \sum_{j=[u_1]+1}^{[u_2]} \sigma_j^2.$$

Let us prove that then

$$(1.12) \qquad \sigma(u_2)B^2(u_2) \leqq 2C(u_1).$$

Indeed,

$$\sigma(u_2)B^2(u_2) \leqq \sigma(u_2) \left\{ \sigma^2(u_2)[u_2] + \sigma^2(u_1)[u_1] + \sum_{j=[u_1]+1}^{[u_2]} \sigma_j^2 \right\}$$

$$= \sigma(u_2) \left\{ (\sigma^2(u_2)[u_1] + \sigma^2(u_1))[u_1] + \sigma^2(u_2)([u_2] - [u_1]) \right.$$

$$\left. + \sum_{j=[u_1]+1}^{[u_2]} \sigma_j^2 \right\} \leqq 2C(u_1).$$

Moreover, from (1.3) and (1.8) it follows that

$$(1.13) \qquad \sigma(u_2)B^3(u_2) = C(u_1)B(u_1).$$

From (1.13) and (1.12) results (1.11).

We now pause to consider the possibility of increasing the power $1/4$ occurring in the estimates. In the general case, for example, for $n = 1$, this cannot be done. However, a number of considerations indicate that for sufficiently large n the estimates can be improved in this direction. Thus the following is valid.

Theorem 3. *Suppose the r.v.* X_j, $j = 1, \cdots, n$, *are identically distributed,* $\sigma_1 = 1$ *and* $v = v_j \leqq 1$. *Then*

$$(1.14) \qquad \delta \leqq 6 \frac{v^{1/3}}{\sqrt{n}} + \frac{0.23 \exp\{-0.2n\}}{n}.$$

Corollary 3.1. *For all n such that* $n^{-1/2} \exp(-0.2n) \leqq v^{1/3}$,

(1.15) $$\delta \leqq 6.23 v^{1/3}/\sqrt{n}.$$

2. Subsidiary Results

Let $z = (z_1, \cdots, z_m)$, $\beta = (\beta_1, \cdots, \beta_m)$, $m \geqq 2$, and

$$\varphi(z, \beta) = \prod_{j=1}^{m} (e^{-z_j} + \beta_j), \quad M = \max_{z, \beta} \varphi(z, \beta),$$

where $\max_{z, \beta}(z, \beta)$ is chosen under the conditions

(2.1) $z_j \geqq 0, \quad \beta_j \geqq 0, \qquad\qquad j = 1, \cdots, m,$

(2.2) $$\sum_{j=1}^{m} z_j = D,$$

(2.3) $$\sum_{j=1}^{m} z_j^{3/2} = E,$$

(2.4) $$\sum_{j=1}^{m} \beta_j = K,$$

D and E being such that $U^{(m)}$, the set of all z determined by conditions (2.1)–(2.3), is non-empty.

Lemma 1. *Let*

(2.5) $$\varkappa = (3E/D)^2,$$

(2.6) $$q = [D^3/27E^2].$$

Then

(2.7) $$M \leqq (e^{-\varkappa} + K_1)^q (1 + K_2)^{m-q},$$

where K_1 and K_2 are non-negative numbers satisfying $K_1 q + K_2(m - q) = K$.

PROOF. Point z is an interior point of the set (2.1) if and only if all of its coordinates are strictly positive. Assume that the extremum point z is interior. Using Lagrange's method for finding a conditional extremum, we have

(2.8)
$$-e^{-z_i}(e^{-z_j} + \beta_j) \prod_{k \neq i,j} (e^{-z_k} + \beta_k) + \gamma_1 + \tfrac{3}{2}\gamma_2 z_i^{1/2} = 0,$$
$$-e^{-z_j}(e^{-z_i} + \beta_i) \prod_{k \neq i,j} (e^{-z_k} + \beta_k) + \gamma_1 + \tfrac{3}{2}\gamma_2 z_j^{1/2} = 0,$$

(2.9)
$$(e^{-z_i} + \beta_i) \prod_{k \neq i,j} (e^{-z_k} + \beta_k) + \gamma_3 = 0,$$
$$(e^{-z_j} + \beta_j) \prod_{k \neq i,j} (e^{-z_k} + \beta_k) + \gamma_3 = 0,$$

where γ_1, γ_2, and γ_3 are constants and (i, j) is a selection from the set $(1, \cdots, m)$.

From (2.8) and (2.9) it is not hard to find that

(2.10) $$\gamma_3(e^{-z_i} - e^{-z_j}) = \tfrac{3}{2}\gamma_2(z_j^{1/2} - z_i^{1/2}),$$

(2.11) $$e^{-z_j} - e^{-z_i} = \beta_i - \beta_j.$$

Let $v_i = \sqrt{z_i}$, $\gamma_4 \doteq 3\gamma_2/2\gamma_3$ and $h_i = \exp(-v_i^2)$. Then (2.10) can be rewritten in the form

(2.10*) $$h_i - h_j = \gamma_4(v_i - v_j).$$

All points in the h, v-plane satisfying condition (2.10*) lie on one straight line. But for $v > 0$, a line intersects the curve $h = \exp(-v^2)$ in at most three points. This means that the numbers $z_1, \cdots, z_m$ can be split into at most three groups in each of which all z_i are equal. From (2.11) it follows that when $z_i = z_j$, then so is $\beta_i = \beta_j$.

Now let the point at which an extremum is attained not be interior and for definiteness let $z_1, \cdots, z_l \neq 0$ and $z_j = 0$ for $j = l + 1, \cdots, m$. Having fixed β_j for $j = l + 1, \cdots, m$, we pass to the "l-dimensional case" and consider the function $\varphi_1 = \prod_{j=1}^{l} (\exp(-z_j) + \beta_j)$. It is clear that the stipulation about the splitting of the values of z_j, β_j for $j = 1, \cdots, l$ into three groups remains true. Thus, without loss of generality, we may assume that there exists an integer $p \leq m$ such that $z_j = z_1$ for $j = 1, \cdots, p$ and $pz_1 \geq D/3$. We have

$$M = (e^{-z_1} + \beta_1)^p \prod_{j=p+1}^{l} (e^{-z_j} + \beta_j) \prod_{j=l+1}^{m} (1 + \beta_j)$$

$$\leq (e^{-z_1} + \beta_1)^p \prod_{j=p+1}^{m} (1 + \beta_j).$$

Further, since for fixed sum $\beta_i + \beta_j$, the maximum of $(1 + \beta_i)(1 + \beta_j)$ is attained when $\beta_i = \beta_j$,

$$M \leq (e^{-z_1} + \beta_1)^p(1 + \beta')^{m-p},$$

where $\beta_1 p + \beta'(m - p) = K$.

Since $pz_1 \geq D/3$ and $pz_1^{3/2} \leq E$, we have $z_1 \leq \varkappa = (3E/D)^2$. Further,

$$(e^{-z_1} + \beta_1)^p \leq \max' \prod_{j=1}^{p} (e^{-s_j} + \beta_1),$$

where $\max'$ is taken under the conditions $\sum_{j=1}^{p} s_j = \tilde{D} = pz_1$ and $0 \leq s_j \leq \varkappa$. But $\max'$ can be attained only if (up to a permutation of subscripts) $s_j = \varkappa$ for all $j = 1, \cdots, p_0$, where $p_0 = [\tilde{D}/\varkappa]$, $s_{p_0+1} = \tilde{D} - \varkappa[\tilde{D}/\varkappa]$ and $s_j = 0$ for all $j \geq p_0 + 2$. The latter follows from the fact that if s_i and s_j satisfy $0 < s_i \leq s_j < \varkappa$, then

$$(e^{-s_j} + \beta) < \exp\{-\min(\varkappa, s_i + s_j)\} + \beta)$$

$$\times (\exp\{-s_i - s_j + \min(\varkappa, s_i + s_j)\} + \beta).$$

Further, $p_0 \geq q = [D^3/27E^2]$ and

$$M \leq (e^{-\varkappa} + \beta_1)^q(1 + \beta_1)^{p-q}(1 + \beta')^{m-p} \leq (e^{-\varkappa} + \beta_1)^q(1 + \beta'')^{m-q},$$

where $\beta_1 q + \beta''(m - q) = K$. The lemma is proved.

Lemma 2. *Let $H \geqq B$ and*

$$\chi(t) = \prod_{j=1}^{[u]} (\exp\{-\sigma^2(u)t^2/2\} + v_j t^3/6) \prod_{j=[u]+1}^{n} (\exp\{-\sigma_j^2 t^2/2\} + v_j t^3/6),$$

$$I(s) = \Lambda \int_0^\varepsilon \chi(t)t^2 \, dt + 1/H\varepsilon.$$

Then for any u satisfying $1 \leqq u < n + 1$,

$$\min_{0 < \varepsilon \leqq 1/\sigma(u)} I(\varepsilon) \leqq L_{21}(\Lambda/B^3(u) + \sigma(u)/H).$$

PROOF. Let $0 \leqq t \leqq 1/\sigma(u)$. Then

$$e^{-\sigma^2(u)t^2/2} + v_j t^3/6 \leqq e^{-\sigma^2(u)t^2/2}(1 + \sqrt{e}v_j t^3/6)$$
$$\leqq \exp\{-\sigma^2(u)t^2/2 + \sqrt{e}v_j t^3/6\},$$

and, for $j \geqq [u] + 1$,

$$e^{-\sigma_j^2 t^2/2} + v_j t^3/6 \leqq e^{-\sigma_j^2 t^2/2}(1 + \sqrt{e}v_j t^3/6) \leqq \exp\{-\sigma_j^2 t^2/2 + \sqrt{e}v_j t^3/6\}.$$

From this we obtain

(2.12) $$\chi(t) \leqq \exp\{-B^2(u)t^2/2 + \sqrt{e}\Lambda t^3/6\}.$$

CASE I. Let $\Lambda \geqq \sigma(u)B^2(u)$. Set $\varepsilon = B^2(u)/\Lambda \leqq 1/\sigma(u)$. Then, for $0 \leqq t \leqq \varepsilon$,

$$\chi(t) \leqq \exp\{-0, 2B^2(u)t^2\}.$$

This easily implies that, for $\varepsilon = B^2(u)/\Lambda$,

$$I(\varepsilon) \leqq L_{22}(\Lambda/B^3(u) + \Lambda/HB^2(u)) \leqq 2L_{22}\Lambda/B^3(u).$$

CASE II. Now let $\Lambda < \sigma(u)B^2(u)$. Set $\varepsilon = 1/\sigma(u)$. Then, for $0 \leqq t \leqq \varepsilon$,

$$\chi(t) \leqq \exp\{-(B^2(u)/2 - \sqrt{e}\Lambda/6\sigma(u))t^2\} \leqq \exp\{-0.2B^2(u)t^2\}$$

and

$$I(\varepsilon) \leqq L_{23}\{\Lambda/B^3(u) + \sigma(u)/H\}.$$

The lemma is proved.

In proving Theorem 1, we shall make use of both lemmas proved above. Only Lemma 2 will be needed in the proof of Theorem 2.

3. Proofs of Theorems 1 and 2

Let

$$f_j(t) = \mathbf{E}\exp\{itX_j\}, \quad f(t) = \prod_{j=1}^{n} f_j(t);$$

$$g(t) = \exp\{-t^2/2\}, g_j(t) = g(\sigma_j t); \qquad h(t) = f(t) - g(Bt).$$

We shall use the following variant of the well-known Esseen inequality (cf. [4]):

$$(3.1) \qquad \cdot \delta \leqq (2/\pi) \int_0^\varepsilon |h(t)/t|\, dt + 24/\pi\sqrt{2\pi}B\varepsilon,$$

as well as the elementary inequality

$$(3.2) \qquad |h(t)| \leqq \sum_{j=1}^n |f_j(t) - g_j(t)| \cdot |w_j(t)|,$$

where $w_j(t) = (g_1 \cdots g_{j-1} f_{j+1} \cdots f_n)(t)$.

We shall also assume everywhere that $t \geqq 0$. Integrating by parts, we obtain

$$|f_j(t) - g_j(t)| = \left| \int_{-\infty}^\infty (e^{itx} - 1 - itx + t^2x^2/2)d(F_j(x) - \Phi(x/\sigma_j)) \right|$$

$$(3.3) \qquad = \left| t \int_{-\infty}^\infty (e^{itx} - 1 - itx)(F_j(x) - \Phi(x/\sigma_j))\, dx \right|$$

$$\leqq \frac{1}{2} t^3 \int_{-\infty}^\infty x^2 |F_j(x) - \Phi(x/\sigma_j)| dx = v_j t^3/6.$$

From (3.3) we obtain

$$(3.4) \qquad |f_j(t)| \leqq g_j(t) + v_j t^3/6.$$

Let $1 \leqq u < n + 1$. From (3.4) and the fact that $\sigma_1 \geqq \sigma_2 \geqq \cdots \geqq \sigma_n$, it easily follows that

$$|w_j(t)| \leqq \prod_{j=2}^n (e^{-\sigma_j t^2/2} + v_j t^3/6) \leqq \prod_{j=2}^{[u]} (e^{-\sigma^2(u)t^2/2}$$

$$(3.5) \qquad + v_j t^3/6) \prod_{j=[u]+1}^n (e^{-\sigma_j^2 t^2/2} + v_j t^3/6) = \chi^*(t).$$

From (3.1)–(3.3) and (3.5), we find for any $\varepsilon > 0$ that

$$(3.6) \qquad \delta \leqq L_{31} \left\{ \Lambda \int_0^\varepsilon t^2 \chi^*(t)\, dt + 1/B\varepsilon \right\}.$$

PROOF OF THEOREM 2. For $t \leqq 1/\sigma(u)$, it is clear that

$$(3.7) \qquad \chi^*(t) \leqq \sqrt{e}\chi(t),$$

where $\chi(t)$ is the function occurring in Lemma 2.

The conclusion of Theorem 2 is an easy consequence of (3.6), (3.7) and Lemma 2.

PROOF OF THEOREM 1. Let

$$B_1^2(u) = \sigma^2(u)([u] - 1) + \sum_{j=[u]+1}^{n} \sigma_j^2,$$

$$C_1(u) = \sigma^3(u)([u] - 1) + \sum_{j=[u]+1}^{n} \sigma_j^3,$$

$$\Lambda_1 = \sum_{j=2}^{n} v_j.$$

To estimate the function $\chi^*(t)$, we apply Lemma 1. Let

$$D = B_1^2(u)t^2/2, \quad E = C_1(u)t^3/2^{3/2}, \quad K = \Lambda_1 t^3/6 \leqq \Lambda t^3/6,$$

$$q\Lambda'(t) + (n - q)\Lambda''(t) = \Lambda_1$$

(see Section 2),

$$q = [B_1^6(u)/27C_1^2(u)], \quad a = 3C_1(u)/B_1^2(u).$$

Then using (2.7), we obtain

(3.8) $$\chi^*(t) \leqq (e^{-a^2 t^2/2} + \Lambda'(t)t^3/6)^q (1 + \Lambda''(t)t^3/6)^{n-q}.$$

The quantities Λ' and Λ'' depend on t but a and q do not.

Consider the case

(3.9) $$C_1(u) \leqq B_1^3(u)/3^{3/2},$$

which implies the condition $q \geqq 1$.

Now let v be such that $a \geqq v > 0$,

$$\chi_1^*(t) = (e^{-v^2 t^2/2} + \Lambda'(t)t^3/6)^q (1 + \Lambda''(t)t^3/6)^{n-q}.$$

From (3.6) and (3.8) we obtain

(3.10) $$\delta \leqq L_{31} \left\{ \Lambda \int_0^\varepsilon t^2 \chi^*(t) \, dt + 1/B\varepsilon \right\} = L_{31} I_1(\varepsilon),$$

It is easy to see that the expression we still have to estimate differs from $I(\varepsilon)$ (see Section 2) in notation only and in the fact that Λ' and Λ'' depend on t. (The condition $H \geqq B$ is satisfied since $B^2 \geqq a^2 q$.) Repeating exactly the computations in the proof of Lemma 2 and satisfying ourselves that the dependence of Λ' and Λ'' on t requires no changes in the estimates, we easily obtain

(3.11) $$\min_\varepsilon I_1(\varepsilon) \leqq L_{32}(\Lambda/v^3 q^{3/2} + v/B) = L_{32}(Q_1(v) + Q_2(v)).$$

If $Q_1(a) \geqq Q_2(a)$, we take $v = a$. If $Q_1(a) < Q_2(a)$, then the quantity v^* chosen from the condition $Q_1(v) = Q_2(v)$ (and equal to $(\Lambda B)^{1/4} q^{3/8}$) is less than a and we take $v = v^*$. From the above, it follows that

(3.12) $$\min_\varepsilon I_1(\varepsilon) \leqq 2L_{32}(Q_1(a) + Q_2(v^*)).$$

Further, under condition (3.9), $q \geqq 1$ and hence $q \geqq B_1^6(u)/54C_1^2(u)$. From this and (3.12) it is easy to see that

$$(3.13) \qquad \delta \leqq L_{33}\{\Lambda/B_1^3(u) + (\Lambda/B^3)^{1/4}(C_1(u)/B_1^3(u))^{3/4}\}$$

when (3.9) holds. Now using the fact that $|w_j(t)| \leqq 1$, we can easily obtain from (3.1)–(3.3) the estimate

$$(3.14) \qquad \delta \leqq L_{34}(\Lambda/B^3)^{1/4}.$$

Since estimate (3.13) follows from (3.14) when the opposite condition to (3.9) holds, we conclude that (3.13) is also valid in the general case.

It remains for us to show that (3.13) implies (1.2). Since $C_1(u) \leqq C(u)$, this will be true if

$$(3.15) \qquad B_1^2(u) \geqq B^2(u)/2.$$

For $u \geqq 2$, (3.15) is trivial. For $[u] = 1$ and $\sigma^2(u) \leqq B^2(u)/2$, (3.15) follows from the fact that $B_1^2(u) = \sum_{j=[u]+1}^n \sigma_j^2 \geqq B^2(u)/2$. The case $[u] = 1$ and $\sigma^2(u) \geqq B^2(u)/2$ is considered separately. From (3.14) we find

$$\delta \leqq L_{35}(\Lambda/B^3)^{1/4}(\sigma^3(u)/B^3(u))^{3/4} \leqq L_{35}(\Lambda/B^3)^{1/4}(C(u)/B^3(u))^{3/4}.$$

The theorem is proved.

4. Proof of Theorem 3

If the summands are identically distributed and $\sigma_1 = 1$, then (3.2) and (3.3) lead to

$$(4.1) \qquad |h(t)| \leqq v|t|^3 \sum_{j=1}^n |w_j(t)|/6,$$

where $w_j(t) = (g^{j-1} \cdot f_1^{n-j})(t)$.

Let $\varepsilon = 1/v^{1/3}$. For $0 \leqq t \leqq 1$, we have

$$(4.2) \quad |f_1(t)| \leqq \exp\{-t^2/2\}(1 + \sqrt{e}\, vt^3/6) \leqq \exp\{-t^2/2 + \sqrt{e}\, vt^3/6\}.$$

For $1 < t \leqq \varepsilon$,

$$|f_1(t)| \leqq e^{-1/2} + vt^3/6 \leqq \exp\{-\tfrac{1}{2} + \sqrt{e}\, vt^3/6\}.$$

Thus, for $0 \leqq t \leqq 1$,

$$(4.3) \quad \begin{aligned} |w_j(t)| &\leqq \exp\{-(j-1)t^2/2 - (n-j)t^2/2 + (n-j)\sqrt{e}\, vt^3/6\} \\ &\leqq \sqrt{e}\exp\{-nt^2(1 - \sqrt{e}\, v^{2/3}/3)/2\} \leqq \sqrt{e}\exp\{-0.2nt^2\}, \end{aligned}$$

and, for $1 < t \leqq \varepsilon$,

$$(4.4) \quad |w_j(t)| \leqq \exp\{-(j-1)t^2/2 - (n-j)/2 + (n-j)\sqrt{e}\, vt^3/6\}.$$

Let

$$I_j = \int_0^\varepsilon t^2 |\omega_j(t)|\, dt = \int_0^1 + \int_1^\varepsilon = I_{1j} + I_{2j}.$$

Using (4.3), we obtain

(4.5) $$I_{1j} \leqq 2\sqrt{2\pi e}\, n^{-3/2}.$$

Further, for $j \geqq 2$, we find from (4.4) that

(4.6) $$I_{2j} \leqq \exp\{-0.2(n - j)\} \int_1^\infty t^2 \exp\{-(j - 1)t^2/2\}\, dt \leqq 15n^{-3/2}.$$

We estimate I_{21} obtaining

(4.7) $$I_{21} \leqq \sqrt{e}\exp\{-n/2\}\int_0^\varepsilon t^2 \exp\{n\sqrt{e}\, t^3 v/6\}\, dt \leqq 2\exp\{-0.2n\}/vn.$$

From (3.1), (4.1), (4.5)–(4.7), we conclude that the assertion of the theorem holds.

5. Supplement

In this section, we shall state without proof some estimates which contain a large amount of information about the distributions of the r.v. X_j. These estimates generalize (1.9) and in certain situations are essentially more exact than (1.9) (and (1.2)). For simplicity, we put $B^2/n = 1$. Let

$$v = n^{-1}\sum_{j=1}^n v_j, \qquad v^{(1)}(u) = n^{-1}\left\{\sum_{j=1}^{[u]} v_j + (u - [u])v_{[u]+1}\right\},$$

$$v^{(2)}(u) = n^{-1}\left\{([u] + 1 - u)v_{[u]+1} + \sum_{j=[u]+2}^n v_j\right\}, \qquad q(u) = u/n,$$

$$\varkappa^2(u) = n^{-1}\left\{([u] + 1 - u)\sigma_{[u]+1}^2 + \sum_{j=[u]+2}^n \sigma_j^2\right\}, \qquad \varepsilon^2(u) = \sigma^2(u)q(u) + \varkappa^2(u).$$

Introduce into consideration the functions

$$V_1(u) = v^{(1)}(u)/(\varepsilon^2(u) - \sqrt{e}\, v/3\sigma(u))^{3/2} + v^{(2)}(u),$$

$$V_2(u) = v^{(1)}(u)/\sigma^3(u)q^{3/2}(u) + v^{(2)}(u),$$

$$\overline{V}(u) = \min\{V_1(u), V_2(u)\},$$

$$V_3(u) = v^{(1)}(u)/\varepsilon^3(u) + v^{(2)}(u).$$

Theorem 4. *Suppose*

(5.1) $$0 < v < \sigma_1/2$$

and $\bar{u}_*$ is any solution of the equation $\overline{V}(u) = \sigma(u)$. Then

$$\delta \leqq 5.6\overline{V}(\bar{u}_*)/\sqrt{n}.$$

Theorem 5. *If a number u_{2*} exists such that $1 \leqq u_{2*} < n + 1$ and $V_2(u_{2*}) = \sigma(u_{2*})$, then*

$$\delta \leqq 5.6V_2(u_{2*})/\sqrt{n}.$$

Theorem 6. *Suppose condition* (5.1) *holds and* u_{3*} *is a solution of*

$$V_3(u) = \sigma(u).$$

If u_{3*} *is such that there exists a number* $a < 1$ *for which*

$$v/\sigma(u_{3*}) \leqq 3a\varepsilon^2(u_{3*})/\sqrt{e},$$

then

$$\delta \leqq \frac{4.2}{(1-a)^{3/2}} \cdot \frac{V_3(u_{3*})}{\sqrt{n}}.$$

REMARK 5. The last condition of Theorem 6 is satisfied if, for example, $v^{(2)}(u_{3*}) \leqq bv^{(1)}(u_{3*})$, where $b < 0.1$.

REFERENCES

[1] V. M. ZOLOTAREV, *On the closeness of the distributions of two sums of independent random variables*, Theory Prob. Applications, 10 (1965), pp. 472–479.

[2] V. M. ZOLOTAREV, *An analogue of the Edgeworth–Cramér asymptotic expansion for the case of approximation by stable laws*, Proceedings of the 6th All-Union Conference on Probability and Statistics, Vil'nyus, 1962, pp. 49–51. (In Russian.)

[3] V. M. ZOLOTAREV, *Some probability inequalities related to the Lévy metric*, Dokl. Akad. Nauk SSSR, 190, 5 (1970), pp. 1019–1021. (In Russian.)

[4] C. G. ESSEEN, *Fourier analysis of distribution functions: a mathematical study of the Laplace–Gaussian law*, Acta Math., 77, 1 (1945), pp. 3–125.

[5] V. PAULAUSKAS, *On a strengthening of Lyapunov's theorem*, Litovskii Mat. Sb., 9, 2 (1969), pp. 323–328. (In Russian.)

[6] V. V. SAZANOV, *On a bound for the rate of convergence in the multi-dimensional central limit theorem*, Proc. 6th Berkeley Symp. Math. Statist. and Prob., Vol. II, 1972, pp. 563–582.

[7] S. V. NAGAEV and V. I. ROTAR', *On Lyapunov type estimates for the case where the distributions of the summands are close to the normal law*, Dokl. Akad. Nauk SSSR, 199, 4 (1971), pp. 778–779. (In Russian.)