

PROBABILITY INEQUALITIES FOR A CRITICAL GALTON–WATSON PROCESS*

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(Translated by the authors)

Abstract. The upper bounds for the large deviation probabilities of a critical Galton–Watson process are derived under various conditions on the offspring distribution.

Key words. Galton–Watson process, martingale, Doob inequality, Cramèr’s condition, Chebyshev inequality

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1. Introduction and statement of the main results. Let Z_n be the critical Galton–Watson process, and let M_n be its maximum up to time n , i.e., $M_n = \max_{k \leq n} Z_k$. In what follows, unless otherwise noted, it is assumed that $Z_0 = 1$. By ξ we denote the random variable with the distribution coinciding with that of offspring. Put $\mathbf{P}\{\xi = k\} = p_k$. We assume that the process Z_n is critical, i.e., $f'(1) = 1$, where $f(s) = \sum_{k=0}^{\infty} p_k s^k$. To exclude the deterministic case $f(s) = s$ we assume also that $p_0 > 0$. Denote $B = f''(1) = \mathbf{E}\xi(\xi - 1)$, $C = f'''(1)$, $B_r = \mathbf{E}\xi^r$, $r > 1$. For every $N > 0$ put $\bar{B} = \mathbf{E}\{\xi(\xi - 1); \xi \leq N\}$, $\bar{\beta}_r = \mathbf{E}\{\xi^{r-1}(\xi - 1); \xi \leq N\}/2$.

The main purpose of this work is to obtain the upper bounds for $\mathbf{P}\{M_n \geq k\}$ and $\mathbf{P}\{Z_n \geq k\}$ under various conditions on the distribution of ξ .

It should be noted that there are only a few papers devoted to the probability inequalities for branching processes. In all these papers it is assumed that Cramèr’s condition holds (the convergence radius R of $f(s)$ is strictly bigger than one). To all appearances, for the first time the upper bounds for $\mathbf{P}\{Z_n \geq k\}$ were the subject of investigation in [1], where the following inequality was obtained:

$$(1) \quad \mathbf{P}\{Z_n \geq k\} \leq (1 + y_0) \left(1 + \frac{1}{1/y_0 + f''(1 + y_0) n/2} \right)^{-k}$$

for every $0 < y_0 < R - 1$. It is also shown in this paper that the following inequality follows from (1):

$$\limsup_{n \rightarrow \infty} \mathbf{P}\{Z_n \geq k\} \leq e^{-u},$$

if setting $k = [Bnu/2]$. On the other hand, according to the limit theorem for the critical Galton–Watson process (see, for example, [2]),

$$(2) \quad \limsup_{n \rightarrow \infty} \frac{Bn}{2} \mathbf{P}\{Z_n \geq k\} = e^{-u}.$$

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Subsequently Makarov [3] proved that there exists n_0 such that for all $n \geq n_0$ the upper bound

$$(3) \quad \mathbf{P}\{Z_n \geq k\} \leq \frac{c_0}{n} \left(1 - \frac{2}{2 + B(n + \log_{(N)} n)}\right)^k$$

is valid, where c_0 is some constant and $\log_{(N)} n$ is the N th iteration of $\log n$.

From the asymptotical point of view, the last inequality is more preferable than (1) because for $k = \lfloor Bnu/2 \rfloor$ it implies that

$$\limsup_{n \rightarrow \infty} \frac{Bn}{2} \mathbf{P}\{Z_n \geq k\} \leq \frac{c_0 B}{2} e^{-u}.$$

But without the prior estimation of the parameters n_0 and c_0 inequality (3) does not allow us to find the numerical bounds of the tail probabilities of Z_n .

Concerning the maximum of the critical Galton–Watson process, the main efforts were directed at studying the tail behavior of $M_\infty = \sup_k Z_k$ (see [4], [5]) and deriving the asymptotic formulas for the expectation $\mathbf{E}M_n$ (see [6] and references therein). The probability inequalities for M_n were studied in the dissertation of Karpenko [7] who, in particular, proved the inequality

$$(4) \quad \mathbf{P}\{M_n \geq k\} \leq \frac{\mathbf{P}\{Z_n \geq \nu k\}}{\min_{i < n} \mathbf{P}\{Z_i \geq \nu k \mid Z_0 = k\}}, \quad \nu \leq 1,$$

which connects tail probabilities of the random variables M_n and Z_n . It is easily seen that

$$\mathbf{D}\{Z_i \mid Z_0 = k\} = k\mathbf{D}\{Z_i \mid Z_0 = 1\} = kiB.$$

Hence, by the Chebyshev inequality,

$$\mathbf{P}\left\{Z_i < \frac{k}{2} \mid Z_0 = k\right\} = \mathbf{P}\left\{Z_i - \mathbf{E}Z_i < -\frac{k}{2} \mid Z_0 = k\right\} \leq 4 \frac{Bi}{k},$$

and consequently,

$$\min_{i < n} \mathbf{P}\left\{Z_i \geq \frac{k}{2} \mid Z_0 = k\right\} \geq 1 - \frac{4B(n-1)}{k}.$$

From this bound, letting $\nu = \frac{1}{2}$ in (4), we conclude that for every $k \geq 8Bn$,

$$\mathbf{P}\{M_n \geq k\} < 2\mathbf{P}\left\{Z_n \geq \frac{k}{2}\right\}.$$

Therefore, we can derive probability inequalities for the maximum from the inequalities for the random variable Z_n .

In the present paper we will use another approach which consists of the application in classical bounds for maxima of sub- and supermartingales.

This approach allows us to get probability inequalities directly for M_n , avoiding (4). Of course, the same bounds will hold also for Z_n .

THEOREM 1. *If $R > 1$, $0 < y_0 < R - 1$, $B_0 = f''(1 + y_0)$, then the inequality*

$$(5) \quad \mathbf{P}\{M_n \geq k\} \leq y_0 \left[\left(1 + \frac{1}{1/y_0 + B_0 n/2}\right)^k - 1 \right]^{-1}$$

holds.

Letting $y_0 \rightarrow 0$ in the bound (5), we arrive at the inequality

$$(6) \quad \mathbf{P}\{M_n \geq k\} \leq \frac{1}{k}$$

which can also be derived from the well-known Doob inequality for the maximum of submartingale (see [8])

$$(7) \quad \mathbf{P}\left\{\max_{0 \leq k \leq n} X_k > x\right\} \leq \frac{\mathbf{E}X_n}{x}.$$

Naturally we question the relation between the right-hand sides in (1) and (5) which we denote, for brevity, by $g_1(y_0)$ and $g_2(y_0)$, respectively. It is easy to check that if

$$\left(1 + \frac{1}{1/y_0 + B_0 n/2}\right)^k > 2,$$

then

$$\frac{y_0}{1 + y_0} g_1(y_0) < g_2(y_0) < 2y_0 \left(1 + \frac{1}{1/y_0 + B_0 n/2}\right)^{-k} = 2 \frac{y_0}{1 + y_0} g_1(y_0).$$

If

$$\left(1 + \frac{1}{1/y_0 + B_0 n/2}\right)^k \leq 2,$$

then

$$g_2(y_0) \geq 2 \frac{y_0}{1 + y_0} g_1(y_0).$$

Let y_* be the value of y which minimizes $g_2(y)$, i.e., $g_2(y_*) = \min g_2(y)$. It does not seem possible to find a simple expression for y_* . However, we can localize y_* more or less precisely. To demonstrate this we consider the binary critical Galton–Watson process. The approximation for y_* which we derive below will be used in Corollary 2. It is easy to verify that for the binary process,

$$g_2(y) = y \left[\left(1 + \frac{y}{1 + ay}\right)^k - 1 \right]^{-1} \equiv y\psi(y),$$

where $a = n/2$. Obviously, $\log g_2(y) = \log y + \log \psi(y)$. Simple calculations show that

$$-(\log \psi(y))' > \frac{k}{(1 + ay)(1 + (a + 1)y)} > \frac{k}{(1 + (a + 1)y)^2}.$$

Therefore,

$$(\log g_2(y))' < \frac{1}{y} - \frac{k}{(1 + (a + 1)y)^2} = \frac{P(y)}{y(1 + (a + 1)y)^2},$$

where $P(y) = (1 + (a + 1)y)^2 - ky$. The quadratic polynomial $P(y)$ has different real roots $y_- < y_+$ if and only if $k > 4(a + 1)$. Under this condition, $(\log g_2(y))' < 0$ for $y_- \leq y \leq y_+$. Consequently,

$$(8) \quad \min_{y_- \leq y \leq y_+} g_2(y) = g_2(y_+).$$

Note that

$$y_{\pm} = \frac{k - 2(a+1) \pm \sqrt{k^2 - 4k(a+1)}}{2(a+1)^2}.$$

Hence, $y_+ > (a+1)^{-1}$. If $k/n \rightarrow \infty$, then

$$(9) \quad y_+ = \frac{k}{(a+1)^2} - \frac{2}{a+1} + O(k^{-1}).$$

It is easily seen that $P(2/k) < 0$ if $k > 2(a+1)/(\sqrt{2}-1)$. Therefore, for $k > 2(a+1)/(\sqrt{2}-1)$,

$$(10) \quad y_- < \frac{2}{k}.$$

Assume that $y < 2/k$. Then

$$\left(1 + \frac{y}{1+ay}\right)^k - 1 < ky \left(1 + \frac{y}{1+ay}\right)^{k-1} < ky \left(1 + \frac{2}{k}\right)^k < kye^2.$$

Hence,

$$(11) \quad \min_{0 < y \leq 2/k} g_2(y) > \frac{1}{ke^2}.$$

It follows from (9)–(11) that

$$(12) \quad \min_{0 < y \leq y_+} g_2(y) = \min\left(\min_{0 < y \leq 2/k} g_2(y), g_2(y_+)\right).$$

Obviously,

$$\frac{y}{1+ay} = \frac{1}{a} \frac{1}{1+(ay)^{-1}} = \frac{1}{a} - \frac{1}{a^2y} + \left(\frac{1}{a^3y^2}\right)$$

as $ny \rightarrow \infty$. According to (9),

$$y_+ = \frac{k}{(a+1)^2} \left(1 - 2\frac{a+1}{k} + O\left(\frac{n^2}{k^2}\right)\right)$$

as $k/n \rightarrow \infty$. Hence, as $k/n \rightarrow \infty$, we have

$$\frac{1}{a^2y_+} = k^{-1}(a^{-1}+1)^2 \left(1 + O\left(\frac{n}{k}\right)\right) = k^{-1} \left(1 + O\left(\frac{1}{nk} + \frac{n}{k^2}\right)\right).$$

As a result we get

$$(13) \quad \frac{y_+}{1+ay_+} = \frac{1}{a} - \frac{1}{k} + O\left(\frac{1}{nk} + \frac{n}{k^2}\right).$$

Consequently,

$$(14) \quad \log\left(1 + \frac{y_+}{1+ay_+}\right) > \frac{\log 2}{n}$$

if the ratio k/n is large enough. By (9) and (14) there exists the constant $L > 0$ such that

$$g_2(y_+) < \frac{8k}{n^2} \exp\left\{-\frac{k \log 2}{n}\right\} < \frac{1}{ke^2}$$

if $k/n > L$. Comparing the latter inequality with (11) and (12), we see that

$$g_2(y_+) = \min_{0 < y \leq y_+} g_2(y)$$

for $k/n > L$, i.e., $y_* \geq y_+$. Hence, using (9), we conclude that $y_* \rightarrow \infty$ if $k/n^2 \rightarrow \infty$.

Putting $y_0 = y_+$ in (5) and applying (9) and (13), we get

$$(15) \quad \mathbf{P}\{M_n \geq k\} \leq \frac{2}{n} \exp\left\{-\frac{2k}{n} + \log\left(\frac{2ek}{n}\right)\right\} \left(1 + O\left(\frac{n}{k} + \frac{k}{n^2}\right)\right)$$

if $k/n \rightarrow \infty$ but $k/n^2 \rightarrow 0$.

Now we return to the general situation and state two corollaries from Theorem 1.

COROLLARY 1. Assume that $1 < \rho < R$ and that n satisfies the condition

$$(16) \quad n \geq \frac{2}{(\rho - 1)\Omega_\rho},$$

where $\Omega_\rho = f''(\rho)$. Then for $k > n\Omega_\rho + 1$,

$$(17) \quad \mathbf{P}\{M_n \geq k\} \leq \frac{4}{n\Omega_\rho} \exp\left\{-\frac{k}{n\Omega_\rho + 1}\right\}.$$

One may consider inequality (17) as an analogue of the Petrov inequality (see [9, Theorem 16, p. 81]).

Denote $C_\rho = f'''(\rho)$. If $C = 0$, then the process is binary and $C_\rho = 0$ for all $\rho \geq 0$.

COROLLARY 2. If $C > 0$, $1 < \rho < R$, and

$$(18) \quad 2(Bn + 1) < k \leq \left(\frac{B}{C_\rho} \wedge (\rho - 1)\right) \left(1 + \frac{Bn}{2}\right)^2,$$

then

$$(19) \quad \mathbf{P}\{M_n \geq k\} < \frac{6.5k}{(Bn + 2)^2} \exp\left\{-\frac{2k}{Bn + 2} + \frac{8C_\rho k^2 n}{(Bn + 2)^4} + 1\right\}.$$

If $C = 0$ and $k > 2(Bn + 1)$, then

$$(20) \quad \mathbf{P}\{M_n \geq k\} < \frac{6.5k}{(Bn + 2)^2} \exp\left\{-\frac{2k}{Bn + 2} + 1\right\}.$$

If condition (18) is fulfilled, then the second summand in the exponent in (19) is negligible for $k = o(n^{3/2})$. Thus, for $k = o(n^{3/2})$ we can rewrite the bound (19) as follows:

$$(21) \quad \mathbf{P}\{M_n \geq k\} \leq \frac{(6.5e)k}{B^2 n^2} \exp\left\{-\frac{2k}{Bn}\right\} (1 + o(1)).$$

It is proved in [10] under condition $R > 1$ that for $k = o(n^2/\log n)$,

$$(22) \quad \mathbf{P}\{Z_n \geq k\} = \frac{2}{Bn} \exp\left\{-\frac{2k}{Bn}\right\} (1 + o(1)).$$

Bound (21) differs from the right-hand side of (22) by the factor $(3.25e)k/Bn$. The same relation takes place between (20) and (22), but in the larger domain $k = o(n^2/\log n)$, i.e., if (22) holds.

The conditional distribution $\mathbf{P}\{Z_n < x \mid Z_n > 0\}$ is approximated by the exponential distribution $F_n(x)$ with parameter $Bn/2$. The generating function of this distribution is

$$\widehat{F}_n(h) := \int_0^\infty e^{hx} dF_n(x) = \frac{1}{1 - hBn/2}.$$

Let us estimate $F_n(x)$ with the aid of the inequality

$$1 - F_n(x) < e^{-hx} \widehat{F}_n(h).$$

It is easily seen that

$$\min_h e^{-hx} \widehat{F}_n(h) = \exp\left\{-\frac{2k}{Bn} + \log \frac{2ex}{Bn}\right\};$$

i.e., the bound

$$e^{-2x/Bn} = 1 - F_n(x) < \frac{2ex}{Bn} e^{-2x/Bn}$$

holds. The additional factor $2ex/Bn$ here is almost the same as in (21). Note that for the binary process it coincides with the excessive factor in (15). Therefore, the bound (5) is optimal in some sense.

We now proceed to the case when Cramèr's condition fails.

THEOREM 2. *Assume that $B_r < \infty$ for some $r \in (1, 2]$. Then for every N such that*

$$(23) \quad N^{r-1} > eB_r n,$$

the inequality

$$(24) \quad \mathbf{P}\{M_n \geq k\} \leq \frac{3}{2N} \log\left(\frac{eN^{r-1}}{B_r n}\right) \left[\left(\frac{N^{r-1}}{eB_r n}\right)^{k/N} - 1\right]^{-1} + n\mathbf{P}\{\xi > N\}$$

holds.

Note that the following theorem does not assume the existence of moments of the random variable ξ of orders higher than one.

THEOREM 3. *Let $r \geq 2$. Then for all $N \geq 1$ and $y_0 > 0$ the following inequality is valid:*

$$(25) \quad \mathbf{P}\{M_n \geq k\} \leq \left(y_0 + \frac{1}{N}\right) \left[\left(1 + \frac{1}{1/y_0 + e^r \overline{B}_n/2 + n\overline{\beta}_r e^{y_0 N}/N^{r-2}}\right)^k - 1\right]^{-1} + n\mathbf{P}\{\xi > N\}.$$

To prove Theorems 2 and 3 we use the truncation method with the subsequent estimation of generating functions of truncated random variables. This approach was used earlier to deduce probability inequalities for sums of independent random variables. The most general results in this direction can be found in the paper of Fuk and Nagaev [11]. In this work the finiteness of any moments is not assumed and all bounds are expressed in terms of truncated moments and tail probabilities of summands.

The first summand in (25) corresponds to the limit theorem for the critical Galton–Watson process, and the second corresponds to the probability of attaining a high level as a result of one big jump, i.e., at the expense of the appearance of the particle with a large number of offspring. Inequality (25) is, in some sense, intermediate. Its right-hand side contains free parameters. Finding their optimal values is a sufficiently complicated problem. The next theorem illustrates how the parameter y_0 can be chosen.

THEOREM 4. *Suppose that B_r is finite for some $r > 2$. Then for all $n \geq 1$, $N \geq 1$, and $k \geq Bn$,*

$$(26) \quad \mathbf{P}\{M_n \geq k\} \leq 2(r+1)e^{r+1} \frac{(r-2)\log(2N)+1}{N} \left(1 + \frac{1}{(r+1)e^r(Bn \vee 1)}\right)^{-k} + 2 \frac{r+1}{rBn} e^{1-\gamma k/N} \left(\frac{B_r}{BN^{r-2}}\right)^{\gamma k/N} + n \mathbf{P}\{\xi > N\},$$

where $\gamma = r(2r+1)/(2(r+1)^2)$.

COROLLARY 3. *For arbitrary $n \geq 1$ and $k \geq Bn$,*

$$(27) \quad \mathbf{P}\{M_n \geq k\} \leq 4(r+1)^2 e^{r+1} \frac{\log(2k) + (r-2)^{-1}}{k} \left(1 + \frac{1}{(r+1)e^r(Bn \vee 1)}\right)^{-k} + \left(nB_r + 2e \frac{r+1}{rBn} \left(\frac{B_r}{eB}\right)^{r/(r-2)}\right) \frac{C(r)}{k^r},$$

where

$$C(r) = \left(\frac{r+1}{r-2}\right)^r \left(\frac{2r+2}{2r+1}\right)^r.$$

Obviously, $C(r)$ decreases if $r > 2$ and $\lim_{r \downarrow 2} C(r) = \infty$.

Bounds (26) and (27) are valid for $k \geq Bn$. In the case when $k < Bn$, the sufficiently precise bound can be derived from the Doob inequality (7). Indeed, from the simple inequality $\mathbf{P}\{M_n \geq k\} \geq \mathbf{P}\{Z_{[k/B]} \geq k\}$ and the limit theorem for the critical Galton–Watson process we conclude that

$$\liminf_{k \rightarrow \infty} k \mathbf{P}\{M_n \geq k\} \geq 2e^{-2}$$

as $k \rightarrow \infty$ and $k < Bn$. On the other hand, by (6),

$$k \mathbf{P}\{M_n \geq k\} \leq 1.$$

It turns out that there exists another approach which is based on the Fuk probability inequalities for martingales [12]. Note that the results of Fuk cannot be applied to the process Z_n (which is a martingale) since the conditional moments $\mathbf{E}\{|Z_{n+1} - Z_n|^t \mid Z_n = k\}$ are not bounded in k . But this condition is fulfilled for the

process $W_n = \sqrt{Z_n}$ which is a supermartingale. It is easy to verify, repeating word for word all Fuk's arguments, that his inequalities are valid for supermartingales. As a result we have the following theorem.

THEOREM 5. *If $B_r < \infty$ for some $r > 2$, then for all $k \geq Bn$,*

$$(28) \quad \mathbf{P}\{M_n \geq k\} \leq \exp\left(-\frac{k}{l(r)Bn}\right) + \frac{G(r, B, B_r)n}{k^r},$$

where $l(r) = 2r^2e^{2r-2}$, and

$$G(r, B, B_r) = \left(\frac{r}{r-2}\right)^{2r} \left(3(r+1)^r(B_r + e^{r+1}B^r) + B^{-1/(r-2)}\left(\frac{3}{2}r^{r-1}((2r-2)B_r + e^rB^{r-1})\right)^{(r-1)/(r-2)}\right).$$

Further, we compare the bounds deduced by different methods. Letting $r = 3$ in (27) and (28), we get, respectively,

$$\mathbf{P}\{M_n \geq k\} \leq 64e^4 \frac{\log(2k) + 1}{k} \left(1 + \frac{1}{4e^3Bn}\right)^{-k} + \left(nB_3 + \frac{8B_3^3}{3e^2B^4n}\right) \frac{32^3}{7^3k^3}$$

and

$$\mathbf{P}\{M_n \geq k\} \leq \exp\left(-\frac{k}{18e^4Bn}\right) + \frac{3^6}{k^3} \left(192(B_3 + e^4B^3) + \frac{27^2}{16B}(4B_3 + e^3B^2)^2\right)n.$$

Note that the first term in the right-hand side of the second inequality does not contain the factor converging to zero. The second terms are of the same order of decreasing in k , but they depend on moments in different ways.

2. Proofs of the main results.

Proof of Theorem 1. For every $h \geq 0$ we define the random variable $Y_n(h)$, $n \geq 1$, by the equality $Y_n(h) = e^{hZ_n} - 1$. It is easy to check that this sequence is a submartingale. Applying the Doob inequality, we have

$$(29) \quad \mathbf{P}\{M_n \geq k\} = \mathbf{P}\left\{\max_{i \leq n} Y_i(h) \geq e^{hk} - 1\right\} \leq \frac{\mathbf{E}Y_n(h)}{e^{hk} - 1} = \frac{f_n(e^h) - 1}{e^{hk} - 1}.$$

Consider now the sequence of real numbers which are defined by the equalities

$$(30) \quad y_{n-1} = y_n + \frac{B_0}{2} y_n^2, \quad y_0 > 0, \quad n \geq 1,$$

on every step y_n being taken as the positive solution of the equation $y_{n-1} = y + (B_0/2)y^2$. For this sequence the following inequalities are obtained in [1]:

$$(31) \quad \frac{1}{y_n} < \frac{1}{y_0} + \frac{B_0n}{2},$$

$$(32) \quad f_n(1 + y_n) < 1 + y_0.$$

Letting $h = \log(1 + y_n)$ in (29) and taking into account (31), (32), we arrive at the desired result.

Proof of Corollary 1. Put $y_0 = 2/(n\Omega_\rho)$ in the inequality of Theorem 1. For this value $1 + y_0 \leq \rho$, according to (16), and consequently, $B_0 \leq \Omega_\rho$. This means, in its turn, that $(1 + 1/y_0 + B_0n/2)^{-1} \geq n\Omega_\rho + 1$. Letting $a_0 = B_0n/2$, we have

$$(33) \quad \begin{aligned} \left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^{-1} &= \frac{1 + a_0y_0}{1 + (a_0 + 1)y_0} = 1 - \frac{y_0}{1 + (a_0 + 1)y_0} \\ &= 1 - \frac{1}{1 + 1/y_0 + B_0n/2} < 1 - \frac{1}{n\Omega_\rho + 1}. \end{aligned}$$

Hence,

$$\left(1 + \frac{1}{1/y_0 + a_0}\right)^{-k} \leq \exp\left\{-\frac{k}{n\Omega_\rho + 1}\right\} < e^{-1}.$$

From these bounds and equality $(x - 1)^{-1} = x^{-1}/(1 - x^{-1})$ we conclude that

$$\left[\left(1 + \frac{1}{1/y_0 + a_0}\right)^k - 1\right]^{-1} < \frac{1}{1 - e^{-1}} \exp\left\{-\frac{k}{n\Omega_\rho + 1}\right\}.$$

Substituting this bound in the right-hand side of (5), we obtain the desired result.

Proof of Corollary 2. First assume that $C_\rho > 0$. Let $y_0 = k/(a + 1)^2$, where $a = Bn/2$. By the Taylor formula for $y_0 \leq \rho - 1$,

$$(34) \quad B_0 = f''(1 + y_0) = B + y_0 f'''(1 + \theta y_0), \quad \theta \in (0, 1).$$

It is easily seen that under condition (18),

$$(35) \quad B_0 < 2B.$$

Using (33), we have

$$(36) \quad \begin{aligned} \left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^{-1} &= 1 - \frac{y_0}{1 + (1 + a_0)y_0} \\ &= 1 - \frac{1}{1 + a_0} + \frac{1}{(1 + a_0)(1 + (1 + a_0)y_0)} \end{aligned}$$

(a_0 is defined in the proof of Corollary 1). If (18) is fulfilled, then, in view of (35),

$$(37) \quad a_0 + 1 = \frac{B_0n}{2} + 1 < Bn + 1.$$

On the other hand, according to the choice of y_0 ,

$$(38) \quad \frac{1}{(1 + a_0)(1 + (1 + a_0)y_0)} < \frac{1}{k}.$$

It follows from (36)–(38) that

$$\left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^{-k} < e^{-1}$$

if $k > 2(Bn + 1)$.

Therefore,

$$\left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^k - 1 > (1 - e^{-1}) \left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^k.$$

Hence, because of (36) and (38),

$$\begin{aligned} \left[\left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^k - 1\right]^{-1} &< \frac{1}{1 - e^{-1}} \left(1 + \frac{1}{1/y_0 + B_0n/2}\right)^{-k} \\ (39) \qquad \qquad \qquad &< 1.6 \exp\left\{-\frac{k}{1 + a_0} + 1\right\}. \end{aligned}$$

In view of (34),

$$a_0 + 1 < a + 1 + \frac{C_\rho kn}{2(a + 1)^2}.$$

Consequently,

$$(40) \qquad \qquad \qquad \frac{1}{1 + a_0} > \frac{1}{1 + a} - \frac{C_\rho kn}{2(a + 1)^4}.$$

From (5), (39), and (40) we get (19).

If $C = 0$, i.e., in the case of the binary process, then $B_0 = B$, $a_0 = a$. Thus, instead of (37) we have the equality $a_0 + 1 = Bn/2 + 1$. As a result, inequality (39) holds for $k > 2(Bn + 1)$. Now (20) follows from (5), (39).

Proofs of Theorems 2 and 3. Fix $N \geq 1$. Let $\tilde{f}(s)$ be the truncation of the function $f(s)$ on the level N , i.e.,

$$\tilde{f}(s) = \sum_{0 \leq k \leq N} p_k s^k.$$

Let x_0 be the minimal positive root of the equation $x = \tilde{f}(x)$.

For every $n \geq 1$ denote by A_n the event that every particle in the first n generations (including the zeroth) contains no more than N offspring. The probability of the event $\{M_n \geq k\}$ can be bounded in the following way:

$$(41) \qquad \qquad \mathbf{P}\{M_n \geq k\} \leq \mathbf{P}\{M_n \geq k; A_n\} + \mathbf{P}(\bar{A}_n),$$

where \bar{A}_n is complementary to A_n .

It is easily seen that

$$(42) \qquad \qquad \mathbf{P}(A_n) = \tilde{f}_n(1),$$

where $\tilde{f}_n(s)$ is the n th iteration of the function $\tilde{f}(s)$.

Since $\tilde{f}'(s) \leq 1$ for all $s \in [0, 1]$, we get

$$\begin{aligned} \tilde{f}_j(1) - \tilde{f}_{j+1}(1) &= \tilde{f}(\tilde{f}_{j-1}(1)) - \tilde{f}(\tilde{f}_j(1)) \leq \sup_{s \in [0, 1]} \tilde{f}'(s) (\tilde{f}_{j-1}(1) - \tilde{f}_j(1)) \\ &\leq \tilde{f}_{j-1}(1) - \tilde{f}_j(1) \leq \dots \leq 1 - \tilde{f}(1) = \mathbf{P}\{\xi > N\}. \end{aligned}$$

Consequently,

$$(43) \quad \mathbf{P}(\bar{A}_n) = 1 - \tilde{f}_n(1) = \sum_{j=0}^{n-1} (\tilde{f}_j(1) - \tilde{f}_{j+1}(1)) \leq n\mathbf{P}\{\xi > N\}.$$

Remark. Inequality (43) can also be obtained, using the following arguments which were proposed by a referee:

$$\mathbf{P}(\bar{A}_n) = \mathbf{P}\left(\bigcup_{k=0}^{n-1} \bigcup_{i=1}^{Z_k} \{\xi_{i,k} > N\}\right) \leq \sum_{k=0}^{n-1} \mathbf{E} \sum_{i=1}^{Z_k} \mathbf{P}\{\xi_{i,k} > N\} = n\mathbf{P}\{\xi > N\},$$

where $\{\xi_{i,k}\}$ are independent copies of ξ .

Since $\tilde{f}_n(1)$ is nonincreasing and bounded, there exists $\lim_{n \rightarrow \infty} \tilde{f}_n(1) = x^* \leq 1$ and $x^* = \tilde{f}(x^*)$. Since the equation $x = \tilde{f}(x)$ has a unique solution on the interval $[0, 1]$, $x^* = x_0$. Therefore, $\tilde{f}_n(1) \downarrow x_0$ as $n \rightarrow \infty$. Hence, by virtue of (42),

$$(44) \quad \mathbf{P}(\bar{A}_n) \uparrow 1 - x_0.$$

Noting that the function $\tilde{f}'(s)$ is nondecreasing, we arrive at the inequality

$$1 - x_0 = 1 - \tilde{f}(x_0) = 1 - \tilde{f}(1) + \tilde{f}(1) - \tilde{f}(x_0) \leq 1 - \tilde{f}(1) + \tilde{f}'(1)(1 - x_0).$$

Hence, using the equalities $\sum_{i=0}^{\infty} p_i = \sum_{i=1}^{\infty} ip_i = 1$, we get the bound

$$(45) \quad 1 - x_0 \leq \frac{1 - \tilde{f}(1)}{1 - \tilde{f}'(1)} = \frac{\sum_{i>N} p_i}{\sum_{i>N} ip_i} < \frac{1}{N}$$

if $\mathbf{P}\{\xi > N\} = \sum_{i>N} p_i > 0$. If $\mathbf{P}\{\xi > N\} = 0$, then $x_0 = 1$, and relation (45) remains valid.

To estimate the first summand in (41) we need the following lemma.

LEMMA 1. For every $h > 0$,

$$(46) \quad \mathbf{P}\{M_n \geq k; A_n\} \leq \frac{\max(\tilde{f}_n(e^h), e^h) - x_0}{e^{hk} - 1}.$$

Proof. For every $i \geq 1$, define

$$X_i = e^{hZ_i} I(A_i), \quad Y_0 = e^h.$$

It follows from the definition of A_i that $A_{i+1} \subset A_i$ for all i . This means that zero is the absorbing state of the process X_i . Therefore,

$$(47) \quad \mathbf{E}\{X_{i+1} \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}, X_i = 0\} = \mathbf{E}\{X_{i+1} \mid X_i = 0\} = 0.$$

If the event $\{X_i = e^{hj}\}$ occurs, then the events $\{I(A_i) = 1\}$ and $\{Z_i = j\}$ also occur. In this case,

$$I(A_{i+1}) = I(\xi_l \leq N, l = 1, \dots, j),$$

where $\{\xi_i\}$ is the sequence of independent random variables with the common distribution $\{p_k\}$.

Hence, for every $j \geq 0$ we have the equality

$$\begin{aligned} & \mathbf{E}\{X_{i+1} \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}, X_i = e^{hj}\} = \mathbf{E}\{X_{i+1} \mid X_i = e^{hj}\} \\ (48) \quad & = \mathbf{E}\{e^{h(\xi_1 + \dots + \xi_j)} I(\xi_l \leq N, l = 1, \dots, j) \mid Z_i = j\} = (\tilde{f}(e^h))^j. \end{aligned}$$

From (47) and (48) we conclude that the sequence X_i is a supermartingale if h satisfies the condition $\tilde{f}(e^h) \leq e^h$, and it is a submartingale if $\tilde{f}(e^h) \geq e^h$.

In the first case we will use the following well-known inequality (see, for example, [13]):

$$\lambda \mathbf{P}\left\{ \sup_{0 \leq i \leq n} Y_i \geq \lambda \right\} \leq \mathbf{E}Y_0 - \mathbf{E}\left\{ Y_n; \sup_{0 \leq i \leq n} Y_i < \lambda \right\}.$$

Here Y_i is a supermartingale and λ is an arbitrary positive number.

Let $\lambda = e^{hk}$, $Y_i = X_i$, $\mathbf{E}Y_0 = e^h$. Therefore,

$$e^{hk} \mathbf{P}\left\{ \sup_{0 \leq i \leq n} X_i \geq e^{hk} \right\} \leq e^h - \mathbf{E}\{X_n; \sup_{0 \leq i \leq n} X_i < e^{hk}\}.$$

Expectation in the right-hand side of this inequality can be bounded in the following manner:

$$\begin{aligned} & \mathbf{E}\left\{ X_n; \sup_{0 \leq i \leq n} X_i < e^{hk} \right\} \geq \mathbf{P}\left\{ A_n; \sup_{0 \leq i \leq n} X_i < e^{hk} \right\} \\ & = \mathbf{P}(A_n) - \mathbf{P}\left\{ A_n; \sup_{0 \leq i \leq n} X_i \geq e^{hk} \right\} \geq \mathbf{P}(A_n) - \mathbf{P}\left\{ \sup_{0 \leq i \leq n} X_i \geq e^{hk} \right\}. \end{aligned}$$

As a result, we have

$$(e^{hk} - 1) \mathbf{P}\left\{ \sup_{0 \leq i \leq n} X_i \geq e^{hk} \right\} \leq e^h - \mathbf{P}(A_n).$$

Hence, taking into account (44), we obtain the bound

$$(49) \quad \mathbf{P}\left\{ \sup_{0 \leq i \leq n} X_i \geq e^{hk} \right\} \leq \frac{e^h - x_0}{e^{hk} - 1}.$$

Further,

$$\mathbf{P}\{M_n \geq k; A_n\} = \mathbf{P}\left\{ I(A_n) \sup_{0 \leq i \leq n} e^{hZ_i} \geq e^{hk} \right\} \leq \mathbf{P}\left\{ \sup_{0 \leq i \leq n} X_i \geq e^{hk} \right\}.$$

Hence, in view of (49),

$$(50) \quad \mathbf{P}\{M_n \geq k; A_n\} \leq \frac{e^h - x_0}{e^{hk} - 1}$$

if h satisfies the condition $\tilde{f}(e^h) \leq e^h$.

In the case $\tilde{f}(e^h) \geq e^h$, we apply the inequality (see [8])

$$\lambda \mathbf{P}\left\{ \sup_{0 \leq i \leq n} Y_i \geq \lambda \right\} \leq \mathbf{E}\left\{ Y_n; \sup_{0 \leq i \leq n} Y_i \geq \lambda \right\}.$$

Here Y_i is a submartingale and λ is a positive number.

Subtracting $\mathbf{P}\{\sup_{0 \leq i \leq n} Y_i \geq \lambda\}$ from both sides we have

$$\begin{aligned} (\lambda - 1) \mathbf{P}\left\{\sup_{0 \leq i \leq n} Y_i \geq \lambda\right\} &\leq \mathbf{E}\left\{Y_n - 1; \sup_{0 \leq i \leq n} Y_i \geq \lambda\right\} \\ &= \mathbf{E}(Y_n - 1) - \mathbf{E}\left\{Y_n - 1; \sup_{0 \leq i \leq n} Y_i < \lambda\right\}. \end{aligned}$$

Assume that Y_i is nonnegative. Then

$$-\mathbf{E}\left\{Y_n - 1; \sup_{0 \leq i \leq n} Y_i < \lambda\right\} \leq -\mathbf{E}\{Y_n - 1; Y_n < 1\} \leq \mathbf{P}\{Y_n < 1\}.$$

Thus,

$$(\lambda - 1) \mathbf{P}\left\{\sup_{0 \leq i \leq n} Y_i \geq \lambda\right\} \leq \mathbf{E}(Y_n - 1) + \mathbf{P}\{Y_n < 1\}.$$

Letting here $Y_i = X_i$, $\lambda = e^{hk}$, we arrive at the bound

$$(e^{hk} - 1) \mathbf{P}\left\{\sup_{0 \leq i \leq n} X_i \geq e^{hk}\right\} \leq \tilde{f}_n(e^h) - 1 + \mathbf{P}\{X_n = 0\}.$$

Noting that $\mathbf{P}\{X_n = 0\} = \mathbf{P}(\bar{A}_n)$ and using (44), we get

$$(51) \quad \mathbf{P}\{M_n \geq k; A_n\} \leq \mathbf{P}\left\{\sup_{0 \leq i \leq n} X_i \geq e^{hk}\right\} \leq \frac{\tilde{f}_n(e^h) - x_0}{e^{hk} - 1},$$

where h satisfies $\tilde{f}(e^h) \geq e^h$.

It should be noted that bounds (50) and (51) coincide when h is such that $e^h = \tilde{f}(e^h)$ because in this case $\tilde{f}_n(e^h) = e^h$ for every n . Let us denote by h_0 the positive root of $e^h = \tilde{f}(e^h)$, i.e., h_0 is the fixed point of the mapping $\tilde{f}(e^h)$. The statement of the lemma can be interpreted as follows: If $h \leq h_0$, then to bound $\mathbf{P}\{M_n \geq k; A_n\}$ we use inequality (50); otherwise we use inequality (51).

Let us now prove Theorem 2. Put

$$y_0 = \frac{1}{N} \log \frac{N^{r-1}}{nB_r}$$

and consider the recurrent sequence

$$y_{n-1} = y_n + B_r \frac{e^{y_n N}}{N^r}.$$

Obviously, y_n decreases. Therefore,

$$y_n > y_{n-1} - B_r \frac{e^{y_0 N}}{N^r}.$$

Summing up these inequalities, we have

$$(52) \quad y_n > y_0 - nB_r \frac{e^{y_0 N}}{N^r}.$$

It follows from (23) that $y_0 > 1/N$. On the other hand, by the definition of y_0 ,

$$(53) \quad nB_r \frac{e^{y_0 N}}{N^r} = \frac{1}{N}.$$

Going back to (52), we see that $y_n > 0$.

In [11] the following inequality is obtained:

$$\mathbf{E}\{e^{y\xi}; \xi \leq N\} \leq 1 + y \int_0^N u d\mathbf{P}\{\xi < u\} + \frac{e^{yN} - 1 - yN}{N^r} \int_0^N u^r d\mathbf{P}\{\xi < u\}.$$

Since

$$\int_0^N u d\mathbf{P}\{\xi < u\} \leq 1, \quad \int_0^N u^r d\mathbf{P}\{\xi < u\} \leq B_r < \infty,$$

we have

$$\tilde{f}(e^y) = \mathbf{E}\{e^{y\xi}; \xi \leq N\} \leq 1 + y + B_r \frac{e^{yN}}{N^r}.$$

Letting $y = y_n$, we get

$$\tilde{f}(e^{y_n}) \leq 1 + y_n + B_r \frac{e^{y_n N}}{N^r} = 1 + y_{n-1} \leq e^{y_{n-1}}.$$

Therefore,

$$(54) \quad \tilde{f}_n(e^{y_n}) \leq 1 + y_0 \leq e^{y_0}.$$

Hence, putting $h = y_n$ in Lemma 1, we get the bound

$$(55) \quad \mathbf{P}\{M_n \geq k; A_n\} \leq \frac{e^{y_0} - x_0}{e^{y_n k} - 1}.$$

The assumption $p_0 > 0$ and the criticality of the considered process mean that ξ has a nondegenerate distribution. By the Jensen inequality, $B_r > (\mathbf{E}\xi)^r = 1$ for $r > 1$. From this bound and (23), we conclude that

$$y_0 < \frac{\log N^{r-1}}{N} < \frac{\log N}{N} \leq e^{-1}.$$

By the formula of finite differences, $e^{y_0} < 1 + e^{1/e}y_0 < 1 + 3y_0/2$. This inequality and (45) imply

$$(56) \quad e^{y_0} - x_0 < \frac{3}{2}y_0 + 1 - x_0 < \frac{3}{2}y_0 + \frac{1}{N} < \frac{3}{2}\left(y_0 + \frac{1}{N}\right).$$

It follows from the definition of y_0 and relations (52), (53) that

$$(57) \quad y_n > \frac{1}{N} \log \frac{N^{r-1}}{B_r n} - \frac{1}{N} = \frac{1}{N} \log \frac{N^{r-1}}{eB_r n}.$$

Substituting bounds (57) and (56) in (55), we obtain

$$\mathbf{P}\{M_n \geq k; A_n\} \leq \frac{3}{2N} \log \left(\frac{eN^{r-1}}{B_r n} \right) \left[\left(\frac{N^{r-1}}{eB_r n} \right)^{k/N} - 1 \right]^{-1}.$$

The statement of Theorem 2 follows from the last inequality and relations (41) and (43).

Let us turn now to the proof of Theorem 3. By the definition of $\tilde{f}(s)$,

$$\tilde{f}(1+y) = \sum_{0 \leq k \leq N} p_k (1+y)^k.$$

It follows from the Taylor formula that

$$(1+y)^k \leq 1 + ky + \frac{k(k-1)}{2} y^2 (1+y)^{k-2} \leq 1 + ky + \frac{k(k-1)}{2} y^2 e^{yk}.$$

Therefore, we have the bound

$$\begin{aligned} \tilde{f}(1+y) &\leq \sum_{0 \leq k \leq N} p_k + y \sum_{0 \leq k \leq N} kp_k + y^2 \sum_{0 \leq k \leq N} p_k \frac{k(k-1)}{2} e^{yk} \\ (58) \quad &\leq 1 + y + y^2 S(y), \end{aligned}$$

where

$$\begin{aligned} S(y) &= \sum_{0 \leq k \leq N} p_k \frac{k(k-1)}{2} e^{yk} \\ &= \sum_{0 \leq k \leq r/y} p_k \frac{k(k-1)}{2} e^{yk} + \sum_{r/y < k \leq N} p_k \frac{k(k-1)}{2} e^{yk} \equiv S_1(y) + S_2(y). \end{aligned}$$

Note that if $y < r/N$, then the second term in the right-hand side of this representation equals zero.

Since $e^{yk} \leq e^r$ for $k \leq r/y$,

$$(59) \quad S_1(y) \leq e^r \sum_{0 \leq k \leq r/y} p_k \frac{k(k-1)}{2} \leq \frac{e^r \bar{B}}{2}.$$

Note that $x^{-r+2}e^x$ increases if $x \geq r-2$. Hence, for $z \geq r-2$ we have the inequality

$$e^x \leq e^z \left(\frac{x}{z} \right)^{r-2}, \quad x \in [r-2, z],$$

and consequently,

$$(60) \quad S_2(y) = \sum_{r/y < k \leq N} p_k \frac{k(k-1)}{2} e^{yk} \leq \sum_{r/y < k \leq N} p_k e^{yN} \frac{k(k-1)}{2} \left(\frac{k}{N} \right)^{r-2} \leq \frac{e^{yN}}{N^{r-2}} \bar{\beta}_r.$$

Collecting bounds (58)–(60), we conclude that

$$(61) \quad \tilde{f}(1+y) \leq 1 + y + y^2 e^r \frac{\bar{B}}{2} + y^2 \bar{\beta}_r \frac{e^{yN}}{N^{r-2}}.$$

Let the sequence y_n be defined by the equality

$$(62) \quad y_{n-1} = y_n + y_n^2 e^r \frac{\bar{B}}{2} + y_n^2 \bar{\beta}_r \frac{e^{y_n N}}{N^{r-2}}.$$

It is easy to check that $x + x^2 e^r \bar{B}/2 + x^2 \bar{\beta}_r e^{xN}/N^{r-2}$ has an inverse function on $x \geq 0$, with the latter being positive. Hence, we conclude that $y_n > 0$ for every n . Note also that the sequence y_n is nonincreasing.

Dividing both parts of (62) by $y_n y_{n-1}$, we arrive at the bound

$$\frac{1}{y_n} \leq \frac{1}{y_{n-1}} + e^r \frac{\bar{B}}{2} + \bar{\beta}_r \frac{e^{y_0 N}}{N^{r-2}} \leq \dots \leq \frac{1}{y_0} + e^r \frac{\bar{B}}{2} n + \bar{\beta}_r \frac{e^{y_0 N}}{N^{r-2}} n.$$

Consequently,

$$(63) \quad y_n \geq \left(\frac{1}{y_0} + e^r \frac{\bar{B}}{2} n + \bar{\beta}_r \frac{e^{y_0 N}}{N^{r-2}} n \right)^{-1}.$$

Comparing (62) and (61), we verify that $\tilde{f}(1 + y_n) \leq 1 + y_{n-1}$. Hence,

$$(64) \quad \tilde{f}_n(1 + y_n) = \tilde{f}_{n-1}(\tilde{f}(1 + y_n)) \leq \tilde{f}_{n-1}(1 + y_{n-1}) \leq \dots \leq 1 + y_0.$$

Letting $h = \log(1 + y_n)$ in Lemma 1 and taking into account (64), we have

$$\mathbf{P}\{M_n \geq k; A_n\} \leq \frac{y_0 + 1 - x_0}{(1 + y_n)^k - 1}.$$

Using (45) and (63), we obtain the bound

$$(65) \quad \mathbf{P}\{M_n \geq k; A_n\} \leq \left(y_0 + \frac{1}{N} \right) \left[\left(1 + \frac{1}{1/y_0 + e^r \bar{B}n/2 + n \bar{\beta}_r e^{y_0 N}/N^{r-2}} \right)^k - 1 \right]^{-1}.$$

Combining (41), (43), and (65), we get the desired result.

Proof of Theorem 4. It is easily seen that the truncated moments \bar{B} , $\bar{\beta}_r$ are simultaneously positive or equal to zero. First we assume that $\bar{B} > 0$ and $\bar{\beta}_r > 0$. Let

$$y_0 = \frac{1}{N} \log \frac{\bar{B} N^{r-2}}{\bar{\beta}_r} + \frac{1}{N}.$$

It is easily seen that

$$(66) \quad \frac{1}{N} < y_0 < \frac{(r-2) \log(2N) + 1}{N},$$

$$(67) \quad \bar{\beta}_r \frac{e^{y_0 N}}{N^{r-2}} = e \bar{B} < e^r \frac{B}{2}.$$

Replacing \bar{B} with B in the inequality of Theorem 3 and taking into account (66), (67), we have

$$(68) \quad \mathbf{P}\{M_n \geq k\} \leq 2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn} \right)^k - 1 \right]^{-1} + n \mathbf{P}\{\xi > N\}.$$

Obviously, $(1 + x)^k \geq (1 + 1/k)^k \geq 2$ for all k and $x \geq 1/k$. Hence, for $x \geq 1/k$ we have the inequality

$$\frac{1}{(1 + x)^k - 1} = \frac{(1 + x)^{-k}}{1 - (1 + x)^{-k}} \leq 2(1 + x)^{-k}.$$

On the other hand, if $x < 1/k$, then

$$\frac{1}{(1+x)^k - 1} \leq \frac{1}{kx} \leq \left(1 + \frac{1}{k}\right)^k \frac{(1+x)^{-k}}{kx} \leq e \frac{(1+x)^{-k}}{kx}.$$

Summarizing, we get

$$\frac{1}{(1+x)^k - 1} \leq 2 \max\left(1, \frac{e}{2kx}\right) \frac{1}{(1+x)^k}.$$

Applying this bound to the first term in the right-hand side of (68), we arrive at the inequality

$$(69) \quad \begin{aligned} & 2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn}\right)^k - 1 \right]^{-1} \\ & \leq 4y_0 \max\left\{1, \frac{e(1/y_0 + e^r Bn)}{2k}\right\} \left(1 + \frac{1}{1/y_0 + e^r Bn}\right)^{-k}. \end{aligned}$$

If $1/y_0 \leq re^r(Bn \vee 1)$, then

$$\begin{aligned} \max\left\{1, \frac{e(1/y_0 + e^r Bn)}{k}\right\} & \leq \max\left\{1, \frac{(r+1)e^{r+1}(Bn \vee 1)}{2k}\right\} \\ & \leq \frac{(r+1)e^{r+1}}{2} \max\left\{1, \frac{Bn}{k}\right\}. \end{aligned}$$

Therefore, for $k \geq Bn$ and $1/y_0 \leq re^r(Bn \vee 1)$ the following bound is valid:

$$\max\left\{1, \frac{e(1/y_0 + e^r Bn)}{2k}\right\} \leq \frac{(r+1)e^{r+1}}{2}.$$

Consequently,

$$2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn}\right)^k - 1 \right]^{-1} \leq 2(r+1)e^{r+1}y_0 \left(1 + \frac{1}{(r+1)e^r(Bn \vee 1)}\right)^{-k}$$

if $k \geq Bn$ and $1/y_0 \leq re^r(Bn \vee 1)$.

Invoking (66), we obtain for $k \geq Bn$ and $1/y_0 \leq re^r(Bn \vee 1)$ the inequality

$$(70) \quad \begin{aligned} & 2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn}\right)^k - 1 \right]^{-1} \\ & \leq 2(r+1)e^{r+1} \frac{(r-2)\log(2N) + 1}{N} \left(1 + \frac{1}{(r+1)e^r(Bn \vee 1)}\right)^{-k}. \end{aligned}$$

If $1/y_0 > re^r(Bn \vee 1)$, then

$$(71) \quad \begin{aligned} & y_0 \max\left\{1, \frac{e(1/y_0 + e^r Bn)}{2k}\right\} \leq y_0 \max\left\{1, \frac{e(r+1)}{2ry_0k}\right\} \leq \frac{e(r+1)}{2r} \max\left\{y_0, \frac{1}{k}\right\} \\ & \leq \frac{e(r+1)}{2r} \max\left\{\frac{1}{re^r Bn}, \frac{1}{k}\right\} \leq \frac{e(r+1)}{2rBn} \max\left\{1, \frac{Bn}{k}\right\}. \end{aligned}$$

It follows from the condition $1/y_0 > re^r(Bn \vee 1)$ that $e^r Bn < 1/(ry_0)$. Thus,

$$(72) \quad \left(1 + \frac{1}{1/y_0 + e^r Bn}\right)^{-k} < \left(1 + \frac{ry_0}{r+1}\right)^{-k}.$$

Applying (71) and (72) to bound the right-hand side of (69), we get

$$(73) \quad 2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn} \right)^k - 1 \right]^{-1} \leq 2e \frac{r+1}{rBn} \left(1 + \frac{ry_0}{r+1} \right)^{-k}$$

for $k \geq Bn, 1/y_0 > re^r(Bn \vee 1)$.

Obviously, $\log(1+x) \geq x - x^2/2$ if $x > 0$. Consequently,

$$(1+x)^{-k} \leq \exp \left(-kx \left(1 - \frac{x}{2} \right) \right)$$

for all $x > 0$. Hence, letting $x = ry_0/(r+1)$ and taking into account that $y_0 < 1/r$ in the considered case, we derive

$$(74) \quad \left(1 + \frac{ry_0}{r+1} \right)^{-k} \leq \exp \left(-\frac{kry_0}{r+1} \left(1 - \frac{1}{2(r+1)} \right) \right).$$

It follows from (73) and (74) that

$$(75) \quad 2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn} \right)^k - 1 \right]^{-1} \leq 2e \frac{r+1}{rBn} \exp(-\gamma ky_0),$$

where $\gamma = r(2r+1)/(2(r+1)^2)$. This bound is valid for $k \geq Bn, 1/y_0 > re^r(Bn \vee 1)$.

Substituting the chosen value instead of y_0 , we have

$$\exp(-\gamma ky_0) = e^{-\gamma k/N} \left(\frac{\bar{\beta}_r}{BN^{r-2}} \right)^{\gamma k/N}.$$

Note that

$$\frac{\bar{\beta}_r}{B} \leq \frac{B_r - B_{r-1}}{B} < \frac{B_r}{B}.$$

Therefore,

$$(76) \quad \exp(-\gamma ky_0) < e^{-\gamma k/N} \left(\frac{B_r}{BN^{r-2}} \right)^{\gamma k/N}.$$

It follows from (70), (75), and (76) that for $k \geq Bn$,

$$(77) \quad 2y_0 \left[\left(1 + \frac{1}{1/y_0 + e^r Bn} \right)^k - 1 \right]^{-1} \leq 2(r+1) e^{r+1} \frac{(r-2) \log(2N) + 1}{N} \\ \times \left(1 + \frac{1}{(r+1) e^r (Bn \vee 1)} \right)^{-k} + 2 \frac{r+1}{rBn} e^{1-\gamma k/N} \left(\frac{B_r}{BN^{r-2}} \right)^{\gamma k/N}.$$

Combining (68) and (77), we get the desired result.

Assume now that $\bar{B} = \bar{\beta}_r = 0$. Then the inequality of Theorem 3 takes the following form:

$$\mathbf{P}\{M_n \geq k\} \leq \left(y_0 + \frac{1}{N} \right) \frac{1}{(1+y_0)^k - 1} + n\mathbf{P}\{\xi > N\}.$$

Turning y_0 to infinity, we obtain for $k > 1$ the bound

$$\mathbf{P}\{M_n \geq k\} \leq n\mathbf{P}\{\xi > N\}.$$

If $k = 1$, then independently of the values of the truncated moments, the right-hand side of (25), and consequently, the right-hand side of (26), is bigger than one, whereas $\mathbf{P}\{M_n \geq 1\} = 1$. Thus, the proof of the theorem is completed.

Proof of Corollary 3. Let $N = k(r - 2)(2r + 1)/(2(r + 1)^2)$. If $N < 1$, then $nB_r/N^r > 1$, and consequently, bound (27) is trivial. In the case of $N \geq 1$, we use bound (26). Estimating $\mathbf{P}\{\xi > N\}$ by the Chebyshev inequality, we get the desired result.

Proof of Theorem 5. Let $S_k = \sum_{i=1}^k \eta_i$, where $\{\eta_i\}$ are independent copies of the random variable $\eta \equiv \xi - 1$.

LEMMA 2. For every $t \geq 2$, the following bounds are valid:

$$(78) \quad \mathbf{E}\left\{(\sqrt{k + S_k} - \sqrt{k})^t; S_k > 0\right\} \leq \frac{3}{2} \left(\frac{t}{2} + 1\right)^{t/2} \left(t\mathbf{E}\xi^{t/2+1} + e^{t/2+1}B^{t/2}\right),$$

$$(79) \quad \mathbf{E}\left\{(\sqrt{k + S_k} - \sqrt{k})^t; S_k > 0\right\} \leq 3 \left(\frac{t}{2} + 1\right)^{t/2} \left(k\mathbf{E}\xi^{t/2} + e^{t/2+1}B^{t/2}\right).$$

Proof. Using the inequalities

$$\sqrt{x + y} - \sqrt{x} \leq \frac{y}{\sqrt{x}}, \quad \sqrt{x + y} - \sqrt{x} \leq \sqrt{y}, \quad x > 0, \quad y > 0,$$

we get

$$\begin{aligned} & \mathbf{E}\left\{(\sqrt{k + S_k} - \sqrt{k})^t; S_k > 0\right\} \\ &= \mathbf{E}\left\{(\sqrt{k + S_k} - \sqrt{k})^t; 0 < S_k < k\right\} + \mathbf{E}\left\{(\sqrt{k + S_k} - \sqrt{k})^t; S_k \geq k\right\} \\ (80) \quad & \leq k^{-t/2}\mathbf{E}\{S_k^t; 0 < S_k < k\} + \mathbf{E}\{S_k^{t/2}; S_k \geq k\}. \end{aligned}$$

Integrating by parts, we arrive at the equalities

$$\begin{aligned} \mathbf{E}\{S_k^t; 0 < S_k < k\} &= -k^t\mathbf{P}\{S_k \geq k\} + t \int_0^k x^{t-1}\mathbf{P}\{S_k \geq x\} dx, \\ \mathbf{E}\{S_k^{t/2}; S_k \geq k\} &= k^{t/2}\mathbf{P}\{S_k \geq k\} + \frac{t}{2} \int_k^\infty x^{t/2-1}\mathbf{P}\{S_k \geq x\} dx. \end{aligned}$$

Substituting these expressions into (80), we have

$$\begin{aligned} \mathbf{E}\left\{(\sqrt{k + S_k} - \sqrt{k})^t; S_k > 0\right\} &\leq k^{-t/2}t \int_0^k x^{t-1}\mathbf{P}\{S_k \geq x\} dx \\ (81) \quad &+ \frac{t}{2} \int_k^\infty x^{t/2-1}\mathbf{P}\{S_k \geq x\} dx. \end{aligned}$$

Consider the first term in (81). Theorem 4 in [11] implies

$$(82) \quad \mathbf{P}\{S_k \geq x\} \leq k\mathbf{P}\{\eta \geq y\} + \exp\left(\frac{x}{y} - \frac{x}{y} \log\left(\frac{xy}{Bk} + 1\right)\right).$$

If $x/y = \rho$, where ρ is an arbitrary positive number, then because of (82),

$$\begin{aligned} t \int_0^k x^{t-1}\mathbf{P}\{S_k \geq x\} dx &\leq kt \int_0^k x^{t-1}\mathbf{P}\left\{\eta \geq \frac{x}{\rho}\right\} dx + te^\rho \int_0^k x^{t-1} \left(1 + \frac{x^2}{\rho Bk}\right)^{-\rho} dx \\ &\leq kt\rho^t \int_0^{k/\rho} x^{t-1}\mathbf{P}\{\eta \geq x\} dx + te^\rho(\rho Bk)^{t/2} \int_0^\infty x^{t-1}(1 + x^2)^{-\rho} dx. \end{aligned}$$

Now setting $\rho = t/2 + 1$ and taking into account that

$$\int_0^\infty x^{t-1}(1+x^2)^{-t/2-1} dx = \frac{1}{t}$$

(see, for example, [14]), we obtain the inequality

$$(83) \quad \begin{aligned} t \int_0^k x^{t-1} \mathbf{P}\{S_k \geq x\} dx &\leq t \left(\frac{t}{2} + 1\right)^t \int_0^{2k/(t+2)} x^{t-1} \mathbf{P}\{\eta \geq x\} dx \\ &+ \left(\frac{t}{2} + 1\right)^{t/2} e^{t/2+1} B^{t/2} k^{t/2}. \end{aligned}$$

Obviously,

$$\int_0^z x^{t-1} \mathbf{P}\{\eta \geq x\} dx \leq z^s \int_0^z x^{t-s-1} \mathbf{P}\{\eta \geq x\} dx$$

for all $0 \leq s < t$. Furthermore, by the definition of η ,

$$\int_0^z x^{t-s-1} \mathbf{P}\{\eta \geq x\} dx \leq \int_0^z x^{t-s-1} \mathbf{P}\{\xi \geq x\} dx \leq \frac{1}{t-s} \mathbf{E}\xi^{t-s}.$$

As a result, we get the inequality

$$\int_0^z x^{t-1} \mathbf{P}\{\eta \geq x\} dx \leq \frac{z^s}{t-s} \mathbf{E}\xi^{t-s}.$$

Letting $z = 2k/(t+2)$ here, we have, for $s = t/2 - 1$ and $s = t/2$, respectively,

$$\begin{aligned} \int_0^{2k/(t+2)} x^{t-1} \mathbf{P}\{\eta \geq x\} dx &\leq \frac{1}{t/2+1} \left(\frac{t}{2} + 1\right)^{-t/2+1} k^{t/2-1} \mathbf{E}\xi^{t/2+1}, \\ \int_0^{2k/(t+2)} x^{t-1} \mathbf{P}\{\eta \geq x\} dx &\leq \frac{2}{t} \left(\frac{t}{2} + 1\right)^{-t/2} k^{t/2} \mathbf{E}\xi^{t/2}. \end{aligned}$$

Going back to (83), we obtain the inequalities

$$(84) \quad t \int_0^k x^{t-1} \mathbf{P}\{S_k \geq x\} dx \leq \left(\frac{t}{2} + 1\right)^{t/2} k^{t/2} (t \mathbf{E}\xi^{t/2+1} + e^{t/2+1} B^{t/2}),$$

$$(85) \quad t \int_0^k x^{t-1} \mathbf{P}\{S_k \geq x\} dx \leq \left(\frac{t}{2} + 1\right)^{t/2} k^{t/2} (2 \mathbf{E}\xi^{t/2} k + e^{t/2+1} B^{t/2}).$$

Now we bound the second term in the right-hand side of (81). Letting $y = 2x/t$ in (82), we get

$$\begin{aligned} \frac{t}{2} \int_k^\infty x^{t/2-1} \mathbf{P}\{S_k \geq x\} dx &\leq \frac{t}{2} k \int_k^\infty x^{t/2-1} \mathbf{P}\left\{\eta \geq \frac{2x}{t}\right\} dx \\ &+ \frac{t}{2} e^{t/2} \int_k^\infty x^{t/2-1} \left(\frac{2x^2}{tBk} + 1\right)^{-t/2} dx. \end{aligned}$$

The second summand in this inequality can be bounded in the following way:

$$\begin{aligned} \frac{t}{2} e^{t/2} \int_k^\infty x^{t/2-1} \left(\frac{2x^2}{tBk} + 1\right)^{-t/2} dx &\leq \frac{t}{2} e^{t/2} \int_k^\infty x^{t/2-1} \left(\frac{2x^2}{tBk}\right)^{-t/2} dx \\ &= \left(\frac{t}{2}\right)^{t/2+1} e^{t/2} B^{t/2} k^{t/2} \int_k^\infty x^{-t/2-1} dx = \left(\frac{t}{2}\right)^{t/2} e^{t/2} B^{t/2}. \end{aligned}$$

We bound the first summand using two different approaches. First,

$$\begin{aligned} \frac{t}{2} \int_k^\infty x^{t/2-1} \mathbf{P}\left\{\eta \geq \frac{2x}{t}\right\} dx &\leq \frac{t}{2k} \int_k^\infty x^{t/2} \mathbf{P}\left\{\eta \geq \frac{2x}{t}\right\} dx \\ &\leq \frac{1}{k} \left(\frac{t}{2}\right)^{t/2+1} \mathbf{E}\{\eta^{t/2+1}; \eta > 0\} \leq \frac{t}{2k} \left(\frac{t}{2} + 1\right)^{t/2} \mathbf{E}\xi^{t/2+1}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{t}{2} \int_k^\infty x^{t/2-1} \mathbf{P}\left\{\eta \geq \frac{2x}{t}\right\} dx &\leq \left(\frac{t}{2}\right)^{t/2+1} \int_{2k/t}^\infty x^{t/2-1} \mathbf{P}\{\eta \geq x\} dx \\ &\leq \left(\frac{t}{2}\right)^{t/2} \mathbf{E}\{\eta^{t/2}; \eta > 0\} \leq \left(\frac{t}{2} + 1\right)^{t/2} \mathbf{E}\xi^{t/2}. \end{aligned}$$

As a result, we have

$$(86) \quad k \frac{t}{2} \int_k^\infty x^{t/2-1} \mathbf{P}\{S_k \geq x\} dx \leq \left(\frac{t}{2} + 1\right)^{t/2} \left(\frac{t}{2} \mathbf{E}\xi^{t/2+1} + e^{t/2} B^{t/2}\right),$$

$$(87) \quad k \frac{t}{2} \int_k^\infty x^{t/2-1} \mathbf{P}\{S_k \geq x\} dx \leq \left(\frac{t}{2} + 1\right)^{t/2} (k \mathbf{E}\xi^{t/2} + e^{t/2} B^{t/2}).$$

Combining (81), (84), and (86), we arrive at inequality (78). Respectively, inequality (79) follows from (81), (85), and (87). The lemma is proved.

Now we continue the proof of Theorem 5. We introduce the following random variables:

$$W_n = \sqrt{Z_n}, \quad X_n = W_n - W_{n-1}.$$

It follows from the Jensen inequality that the sequence W_n is a supermartingale. It is easily seen that

$$\mathbf{E}\{X_n^2 \mid Z_{n-1} = k\} = \mathbf{E}(\sqrt{k + S_k} - \sqrt{k})^2 \leq \frac{\mathbf{E}S_k^2}{k} = B.$$

Furthermore, by the first inequality in Lemma 2 with $t = 2(r - 1)$,

$$\begin{aligned} \mathbf{E}\{X_n^t I(X_n > 0) \mid Z_{n-1} = k\} &= \mathbf{E}\{(\sqrt{k + S_k} - \sqrt{k})^t; S_k > 0\} \\ &\leq \frac{3}{2} r^{r-1} ((2r - 2) B_r + e^r B^{r-1}) \equiv H_1. \end{aligned}$$

Thus, we have shown that all conditions of Theorem 2 in [12] are fulfilled with $t = 2(r - 1)$. By the corollary from this theorem,

$$\mathbf{P}\left\{\max_{k \leq n} W_k \geq x\right\} \leq \exp\left(-\frac{x^2}{l(r) B n}\right) + \left(\frac{n H_1}{x y^{2r-3}}\right)^{\beta x/y} + \sum_{i=1}^n \mathbf{P}\{X_i \geq y\},$$

where $\beta = 1 - 1/r$, $l(r) = 2r^2 e^{2r-2}$.

Applying (79), we conclude that for every i ,

$$\mathbf{E}\{X_i^{2r}; X_i > 0\} = \mathbf{E}\mathbf{E}\{X_i^{2r} I(X_i > 0) \mid Z_{i-1}\} \leq 3(r + 1)^r (B_r + e^{r+1} B^r) \equiv H_2.$$

Hence,

$$\sum_{i=1}^n \mathbf{P}\{X_i \geq y\} \leq \frac{nH_2}{y^{2r}}.$$

Therefore,

$$\mathbf{P}\left\{\max_{k \leq n} W_k \geq x\right\} \leq \exp\left(-\frac{x^2}{l(r)Bn}\right) + \left(\frac{nH_1}{xy^{2r-3}}\right)^{\beta x/y} + \frac{nH_2}{y^{2r}}.$$

Setting $y = (r-2)x/r$, we arrive at the bound

$$\begin{aligned} & \mathbf{P}\left\{\max_{k \leq n} W_k \geq x\right\} \\ & \leq \exp\left(-\frac{x^2}{l(r)Bn}\right) + \left(\frac{r}{r-2}\right)^{2r} \left(H_2 + H_1^{(r-1)/(r-2)} \left(\frac{n}{x^2}\right)^{1/r-2}\right) \frac{n}{x^{2r}}. \end{aligned}$$

Noting that

$$\mathbf{P}\left\{\max_{i \leq n} W_i \geq \sqrt{k}\right\} = \mathbf{P}\left\{\max_{i \leq n} Z_i \geq k\right\},$$

we get for $k \geq Bn$ the desired inequality.

Note that similar transformations of random processes were used earlier, but only to find recurrence conditions for random walks. In the fundamental work of Lamperti [15] the functions $\log x$ and x^2 were used, and in [16], [17] x^α are applied, with $\alpha < 2$ and $1 \leq \alpha \leq 2$, respectively.

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