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PROBABILITY INEQUALITIES FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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0. Introduction

In [1], S. V. Nagaev obtained the following estimate for large deviation probabilities for the case of identically distributed independent random variables $X_1, \cdots, X_n$ with $\mathbf{E}X_1 = 0, \mathbf{D}X_1 = 1$:

$$\mathbf{P}\left\{\sum_{i=1}^{n} X_i \geqq x\right\} \leqq n\mathbf{P}\{X_1 \geqq y\} + \exp\left\{2n\left[\frac{1}{y}\log\left(\frac{y^t}{nK_tA_t}\right)\right]^2 + 1\right\}\left\{\frac{nK_tA_t}{y^t}\right\}^{x/y},$$

where $x > 0, y > 0, A_t = \mathbf{E}|X_1|^t < \infty, t > 2$ and $K_t = 1 + e^{-t}(t+1)^{t+2}$.

This paper is devoted to improving this result and to extending it to the case of non-identically distributed independent random variables for which the existence of finite moments of some particular order is not assumed. In Section 1, certain inequalities are derived whose right-hand sides consist of two components: the sum of the probabilities of the tails and a component containing truncated moments. In Section 2, the proofs of these inequalities are given. A bilateral inequality is stated in Section 3. Special cases are considered in Section 4. Section 5 deals with examples involving the computation of the probabilities and Section 6 contains applications to the strong law of large numbers.

Let $X_1, \cdots, X_n$ be non-identically distributed independent random variables (i.r.v.) with respective distribution functions $F_1(u), \cdots, F_n(u)$. Set

$$S_n = X_1 + \cdots + X_n.$$

Throughout the following x is an arbitrary prescribed positive number, $Y = \{y_1, \cdots, y_n\}$ is a set of n positive numbers and $y \geqq \max\{y_1, \cdots, y_n\}$.

$A(t; \cdot, \cdot)$, $B^2(\cdot, \cdot)$ and $\mu(\cdot, \cdot)$ are to denote, respectively, the sum of the absolute moments of order t (specified within the parentheses), the variances and the means truncated at the levels specified within the parentheses. The letter Y designates summation over i from 1 to n of the moments truncated at levels $y_1, \cdots, y_n$. For example,

$$A(t; -Y, 0) = \sum_{i=1}^{n} \int_{-y_i}^{0} |u|^t \, dF_i(u),$$

$$B^2(-Y, Y) = \sum_{i=1}^{n} \int_{-y_i}^{y_i} u^2 \, dF_i(u), \qquad \mu(-\infty, Y) = \sum_{i=1}^{n} \int_{-\infty}^{y_i} u \, dF_i(u).$$

Theorem 3 stated in Section 1 and the results of Section 6 are joint efforts while the remaining results are due to D. Kh. Fuk.

1. Unilateral Inequalities

Theorem 1. *Let* $0 < t \leqq 1$. *Then*

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_1, \tag{1}$$

where

$$P_1 = \exp\left\{\frac{x}{y} - \frac{x}{y}\log\left(\frac{xy^{t-1}}{A(t; 0, Y)} + 1\right)\right\}. \tag{2}$$

If

$$xy^{t-1} > A(t; 0, Y), \tag{3}$$

then

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_2, \tag{4}$$

where

$$P_2 = \exp\left\{\frac{x}{y} - \frac{A(t, 0, Y)}{y^t} - \frac{x}{y}\log\left(\frac{xy^{t-1}}{A(t; 0, Y)}\right)\right\}, \tag{5}$$

and $P_2 \leqq P_1$.

Theorem 2. *Let* $1 \leqq t \leqq 2$. *Then*

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_3, \tag{6}$$

where

$$P_3 = \exp\left\{\frac{x}{y} - \left(\frac{x - \mu(-Y, Y)}{y} + \frac{A(t; -Y, Y)}{y^t}\right)\log\left(\frac{xy^{t-1}}{A(t; -Y, Y)} + 1\right)\right\}. \tag{7}$$

We now go over to the case $t \geqq 2$ and we let

$$P_4 = \exp\left\{\beta\frac{x}{y} - \left(\left(1 - \frac{\alpha}{2}\right)\frac{x}{y} - \frac{u(-Y, Y)}{y}\right)\log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right\}, \tag{8}$$

$$P_5 = \exp\left\{\left(\beta - \frac{t\alpha}{2}\right)\frac{x}{y} - \left(\beta\frac{x}{y} - \frac{\mu(-Y, Y)}{y}\right)\log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right\}, \tag{9}$$

$$P_6 = \exp\left\{-\frac{\alpha x(\alpha x/2 - \mu(-Y, Y))}{e^t B^2(-Y, Y)}\right\}. \tag{10}$$

Theorem 3. *Let $t \geqq 2, 0 < \alpha < 1$ and $\beta = 1 - \alpha$. If*

$$\max\left[t, \log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right] \leqq \frac{\alpha x y}{e^t B^2(-Y, Y)}, \tag{11}$$

then

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_4, \tag{12}$$

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_5. \tag{13}$$

If

$$\max\left[t, \log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right] > \frac{\alpha x y}{e^t B^2(-Y, Y)}, \tag{14}$$

then

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_6. \tag{15}$$

Theorem 3′. *For $t \geqq 2, 0 < \alpha < 1$ and $\beta = 1 - \alpha$, the assertions of Theorem 3 hold with the quantities $B^2(-Y, Y)$ and $\mu(-Y, Y)$ replaced by $B^2(-\infty, Y)$ and $\mu(-\infty, Y)$, respectively.*

Theorem 4. *The following inequality holds:*

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_7, \tag{16}$$

where

$$P_7 = \exp\left\{\frac{x}{y} - \left(\frac{x}{y} - \frac{\mu(-\infty, Y)}{y} + \frac{B^2(-\infty, Y)}{y^2}\right)\log\left(\frac{xy}{B^2(-\infty, Y)} + 1\right)\right\}. \tag{17}$$

Remark 1. There are inequalities for $\mathbf{P}\{S_n \leqq -x\}$ which are left-sided analogues of the inequalities obtained in Theorems 1–4. The quantities $\sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\}$, $A(t; -0, Y)$, $B^2(-\infty, Y)$, $\mu(-Y, Y)$ and $\mu(-\infty, Y)$ on the right-hand sides of the latter have to be replaced by $\sum_{i=1}^{n} \mathbf{P}\{X_i \leqq -y_i\}$,

$A(t; -Y, 0)$, $B^2(-Y, \infty)$, $-\mu(-Y, Y)$ and $-\mu(-Y, \infty)$, respectively; the quantities $A(t; -Y, Y)$ and $B^2(-Y, Y)$ remain unchanged.

REMARK 2. In the above inequalities, one can always put
(a) $\mu(-Y, Y) = 0$ if the i.r.v. are symmetrically distributed;
(b) $\mu(-\infty, Y) = 0$ (or $-\mu(-Y, \infty) = 0$) if $\mathbf{E}X_i = 0, i = 1, \cdots, n$.

REMARK 3. The first extension of S. V. Nagaev's inequality to the case of non-identically distributed variables mentioned in the introduction was apparently due to A. Bikyalis [6].

2. Proofs of Theorems 1–4

Let

$$\tilde{X}_i = \begin{cases} X_i & \text{for } X_i \leqq y_i, \\ 0 & \text{for } X_i > y_i, \end{cases} \qquad i = 1, \cdots, n,$$

$$\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i.$$

The event $\{S_n \geqq x\}$ implies the occurrence of at least one of the following two events: $\{\tilde{S}_n \neq S_n\}$ or $\{\tilde{S}_n \geqq x\}$. Therefore,

$$\mathbf{P}\{S_n \geqq x\} \leqq \mathbf{P}\{\tilde{S}_n \neq S_n\} + \mathbf{P}\{\tilde{S}_n \geqq x\}. \tag{18}$$

The random variables $\tilde{X}_i, i = 1, \cdots, n$, are independent and bounded from above and so, for any positive h,

$$\mathbf{P}\{\tilde{S}_n \geqq x\} \leqq e^{-hx}\, \mathbf{E}\, e^{h\tilde{S}_n}.$$

From this and (18), it follows that

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + e^{-hx}\, \mathbf{E}\, e^{h\tilde{S}_n}. \tag{19}$$

Our goal in proving the theorem is essentially to minimize the right-hand side of inequality (19) with respect to h.

PROOF OF THEOREM 1. Suppose $0 < t \leqq 1$. The functions $(e^{hu} - 1)/u$ and $(e^{hu} - 1)/u^t$ are increasing for $u > 0$. Therefore,

$$\begin{aligned} \mathbf{E}\, e^{h\tilde{X}_i} &\leqq \int_{-\infty}^{0} dF_i(u) + \int_{y_i}^{\infty} dF_i(u) + \int_{0}^{y_i} e^{hu}\, dF_i(u) \\ &= 1 + \int_{0}^{y_i} \frac{e^{hu} - 1}{u} u\, dF_i(u) \leqq 1 + \frac{e^{hy_i} - 1}{y_i} \int_{0}^{y_i} u\, dF_i(u) \\ &\leqq 1 + \frac{e^{hy} - 1}{y^t} \int_{0}^{y_i} u^t\, dF_i(u). \end{aligned}$$

Since the $\tilde{X}_i$, $i = 1, \cdots, n$, are independent, this implies

$$\mathbf{E}\, e^{h\tilde{S}_n} = \sum_{i=1}^{n} \mathbf{E}\, e^{h\tilde{X}_i} \leqq \prod_{i=1}^{n} \left\{1 + \frac{e^{hy} - 1}{y^t} \int_0^{y_i} u^t \, dF_i(u)\right\}$$
$$\leqq \prod_{i=1}^{n} \exp\left\{\frac{e^{hy} - 1}{y^t} \int_0^{y_i} u^t \, dF_i(u)\right\} = \exp\left\{\frac{e^{hy} - 1}{y^t} A(t; 0, Y)\right\}.$$

Hence

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq \exp\left\{\frac{e^{hy} - 1}{y^t} A(t; 0, Y) - hx\right\}. \tag{20}$$

We now set

$$h = \frac{1}{y} \log\left(\frac{xy^{t-1}}{A(t; 0, Y)} + 1\right)$$

in the right-hand side of (20). Then

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq P_1,$$

where P_1 is given by (2). This together with (19) leads to inequality (1).

The right-hand side of (20) attains a minimum value for

$$h = \frac{1}{y} \log\left(\frac{xy^{t-1}}{A(t; 0, Y)}\right),$$

where h is positive by virtue of condition (3). Inserting this value of h into the right-hand side of (20), we obtain

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq P_2,$$

where P_2 is given by (5). This together with (19) implies inequality (4). Clearly, $P_2 \leqq P_1$. Theorem 1 is proved.

Proof of Theorem 2. Suppose $1 \leqq t \leqq 2$. By virtue of the monotonicity of $u^{-2}(e^{hu} - 1 - hu)$ for $u \leqq y$ and $u^{-t}(e^{hu} - 1 - hu)$ for $u > 0$, we have

$$\mathbf{E}\, e^{h\tilde{X}_i} \leqq 1 + h \int_{|u| \leqq y_i} u \, dF_i(u) + \int_{|u| \leqq y_i} \frac{e^{hu} - 1 - hu}{u^2} u^2 \, dF_i(u)$$
$$\leqq 1 + h \int_{|u| \leqq y_i} u \, dF_i(u) + \frac{e^{hy_i} - 1 - hy_i}{y_i^2} \int_{|u| \leqq y_i} u^2 \, dF_i(u)$$
$$\leqq 1 + h \int_{|u| \leqq y_i} u \, dF_i(u) + \frac{e^{hy} - 1 - hy}{y^t} \int_{|u| \leqq y_i} |u|^t \, dF_i(u).$$

Hence,

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq \exp\{(e^{hy} - 1 - hy)y^{-t} A(t; -Y, Y) - hx + h\mu(-Y, Y)\}. \tag{21}$$

Setting

$$h = \frac{1}{y}\log\left(\frac{xy^{t-1}}{A(t;\,-Y,Y)} + 1\right) \tag{22}$$

in the right-hand side of (21), we obtain

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq P_3, \tag{23}$$

where P_3 is given by (7). Inequalities (23) and (19) imply inequality (6). Theorem 2 is proved.

PROOF OF THEOREM 3. We now proceed to the case $t \geqq 2$. Consider first the case where $hy \leqq t$, i.e., $hy_i \leqq t$ for $i = 1, \cdots, n$. We have

$$\begin{aligned}\mathbf{E}\, e^{h\tilde{X}_i} &\leqq \int_{|u|\geqq y_i} dF_i(u) + \int_{|u|\leqq y_i} e^{hu}\, dF_i(u) \\ &= 1 + h\int_{|u|\leqq y_i} u\, dF_i(u) + \frac{h^2}{2}\int_{|u|\leqq y_i} u^2 e^{h\theta u}\, dF_i(u) \\ &\leqq 1 + h\int_{|u|\leqq y_i} u\, dF_i(u) + \frac{1}{2}e^t h^2 \int_{|u|\leqq y_i} u^2\, dF_i(u), \qquad 0 \leqq \theta \leqq 1.\end{aligned} \tag{24}$$

Hence it follows that, for $hy \leqq t$,

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq \exp\{\tfrac{1}{2}e^t B^2(-Y,Y)h^2 - hx + h\mu(-Y,Y)\}. \tag{25}$$

Suppose now that $hy > t$. Let us make use of the monotonicity of $u^{-t}(e^{hu} - 1 - hu)$ for $u \geqq t/h$. Let i be an index such that $hy_i > t$. For this case, we estimate $\mathbf{E}\, e^{h\tilde{X}_i}$ as follows ($0 \leqq \theta \leqq 1$):

$$\begin{aligned}\mathbf{E}\, e^{h\tilde{X}_i} &\leqq \int_{-\infty}^{t/h} dF_i(u) + h\int_{-y_i}^{t/h} u\, dF_i(u) + \frac{h^2}{2}\int_{-y_i}^{t/h} e^{h\theta u} u^2\, dF_i(u) \\ &\quad + \int_{t/h}^{\infty} dF_i(u) + h\int_{t/h}^{y_i} u\, dF_i(u) + \int_{t/h}^{y_i} \frac{e^{hu} - 1 - hu}{u^t} u^t\, dF_i(u) \\ &\leqq 1 + h\int_{|u|\leqq y_i} u\, dF_i(u) + \frac{e^t}{2}h^2 \int_{|u|\leqq y_i} u^2\, dF_i(u) + \frac{e^{hy_i} - 1 - hy_i}{y_i^t}\int_{t/h}^{y_i} u^t\, dF_i(u) \\ &\leqq 1 + h\int_{|u|\leqq y_i} u\, dF_i(u) + \frac{e^t}{2}h^2 \int_{|u|\leqq y_i} u^2\, dF_i(u) + \frac{e^{hy} - 1 - hy}{y^t}\int_{0}^{y_i} u^t\, dF_i(u).\end{aligned} \tag{26}$$

The right-hand side of (24) is clearly no greater than the right-hand side of (26) for any positive value of h. Therefore, inequality (26) also holds for the case where $hy_i \leqq t$. Hence, for $h > 0$, relation (26) and the inequality

$$\begin{aligned}e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq \exp\{\tfrac{1}{2}e^t B^2(-Y,Y)h^2 + \frac{e^{hy} - 1 - hy}{y^t}A(t;0,Y) - hx \\ + h\mu(-Y,Y)\}\end{aligned} \tag{27}$$

which follows from it, are both valid.

Let

$$f_1(h) = \tfrac{1}{2} e^t B^2(-Y, Y)h^2 - \alpha hx, \qquad 0 < \alpha < 1,$$

$$f_2(h) = \frac{e^{hy} - 1 - hy}{y^t} A(t; 0, Y) - \beta hx, \qquad \beta = 1 - \alpha.$$

From this and (19) and (27) it follows that

$$\text{(28)} \quad \mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + \exp\{f_1(h) + f_2(h) + h\mu(-Y, Y)\}.$$

Let

$$\text{(29)} \qquad h_1 = \frac{\alpha x}{e^t B^2(-Y, Y)},$$

$$\text{(30)} \qquad h_2 = \max\left[\frac{t}{y}, \frac{1}{y} \log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right].$$

Suppose condition (11) is satisfied, i.e.,

$$\text{(31)} \qquad h_2 \leqq h_1.$$

Set $h = h_2$ and apply inequality (28). Then

$$f_1(h_2) + f_2(h_2) + h_2\mu(-Y, Y) = h_2\left(\frac{e^t}{2} B^2(-Y, Y)h_2 - x\right)$$

$$+ \frac{e^{h_2 y} - 1 - h_2 y}{y^t} A(t; 0, Y) + h_2\mu(-Y, Y)$$

$$\text{(32)} \qquad \leqq h_2\left(\frac{e^t}{2} B^2(-Y, Y)h_1 - x\right) + \beta\frac{x}{y} - \frac{A(t, 0, Y)}{y^{t-1}} h_2 + h_2 u(-Y, Y)$$

$$= \beta\frac{x}{y} - \left(\left(1 - \frac{\alpha}{2}\right)x - \mu(-Y, Y) + \frac{A(t; 0, T)}{y^{t-1}}\right)h_2$$

$$\text{(33)} \qquad \leqq \beta\frac{x}{y} - \left(\left(1 - \frac{\alpha}{2}\right)x - \mu(-Y, Y)\right)h_2 = \beta\frac{x}{y} - \frac{\alpha x}{2} h_2$$

$$- (\beta x - \mu(-Y, Y))h_2$$

$$\leqq \left(\beta - \frac{t\alpha}{2}\right)\frac{x}{y} - (\beta x - \mu(-Y, Y))h_2.$$

Replacing h_2 in (32) and (33) by the expression (30), we arrive at the respective expressions for P_4 and P_5 in (8) and (9). Inequalities (12) and (13) then follow from (28).

Suppose now that $h_2 \geqq h_1 \geqq t/y$. Clearly, $f_1(h)$ and $f_2(h)$ are convex functions, $f_1(0) = f_2(0) = 0$, $f_1(h)$ attains a minimum value for $h = h_1$ and

$f_2(h)$ for $h = h_2$. Set $h = h_1$ on the right-hand side of (28). Thus,

$$f_1(h_1) = -\frac{\alpha^2 x^2}{2\, e^t B^2(-Y, Y)}, \qquad f_2(h_1) < 0, \qquad \mu(-Y, Y)h_1 = \frac{\alpha x \mu(-Y, Y)}{e^t B^2(-Y, Y)}.$$

By virtue of (28), this implies inequality (15).

When $h_1 \leqq t/y$, it is necessary to make use of the estimate (25). Theorem 3 is proved.

Proof of Theorem 3′. As in proof of Theorem 3, it is easy to observe that, for any positive h,

$$\mathbf{E}\, e^{h\tilde{X}_i} \leqq 1 + h\int_{-\infty}^{y_i} u\, dF_i(u) + \frac{1}{2} e^t h^2 \int_{-\infty}^{y_i} u^2\, dF_i(u) + \frac{e^{hy} - 1 - hy}{y^t}\int_0^{y_i} u^t\, dF_i(u).$$

From this it is apparent that $B^2(-Y, Y)$ and $\mu(-Y, Y)$ may be replaced on the right-hand side of (27) by $B^2(-\infty, Y)$ and $\mu(-\infty, Y)$, respectively. Repeating the proof of Theorem 3, we arrive at the conclusions of Theorem 3′.

Proof of Theorem 4. Starting with (19), we can estimate $\mathbf{E}\, e^{h\tilde{S}_n}$. We have

$$\mathbf{E}\, e^{h\tilde{X}_i} \leqq 1 + h\int_{-\infty}^{y_i} u\, dF_i(u) + \int_{-\infty}^{y_i} \frac{e^{hu} - 1 - hu}{u^2} u^2\, dF_i(u)$$
$$\leqq 1 + h\int_{-\infty}^{y_i} u\, dF_i(u) + \frac{e^{hy} - 1 - hy}{y^2}\int_{-\infty}^{y_i} u^2\, dF_i(u).$$

This implies

$$e^{-hx}\mathbf{E}\, e^{h\tilde{S}_n} \leqq \exp\left\{\frac{e^{hy} - 1 - hy}{y^2} B^2(-\infty, Y) - hx + h\mu(-\infty, Y)\right\}. \tag{34}$$

Setting

$$h = \frac{1}{y}\log\left(\frac{xy}{B^2(-\infty, Y)} + 1\right)$$

in the right-hand side of this last inequality, we obtain the expression for P_7 given in (17). This together with (19) leads to inequality (16). Theorem 4 is proved.

3. Bilateral Inequality

Theorem 5. *For* $0 < t \leqq 1$.

$$\mathbf{P}\{|S_n| \geqq x\} \leqq \sum_{k=1}^{n} \mathbf{P}\{|X_k| \geqq y_k\} + P_8, \tag{35}$$

where

$$(36) \qquad P_8 = \exp\left\{\frac{x}{y} - \frac{x}{y}\log\left(\frac{xy^{t-1}}{A(t;-Y,Y)} + 1\right)\right\}.$$

If

$$(37) \qquad xy^{t-1} > A(t;-Y,Y),$$

then

$$(38) \qquad \mathbf{P}\{|S_n| \geqq x\} \leqq \sum_{k=1}^{n} \mathbf{P}\{|X_k| \geqq y_k\} + P_9,$$

where

$$(39) \qquad P_9 = \exp\left\{\frac{x}{y} - \frac{A(t;-Y,Y)}{y^t} - \frac{x}{y}\log\left(\frac{xy^{t-1}}{A(t;-Y,Y)}\right)\right\}$$

and $P_9 \leqq P_8$.

PROOF. Let

$$\bar{X}_i = \begin{cases} X_i & \text{if } |X_i| \leqq y_i, \\ 0 & \text{if } |X_i| > y_i, \end{cases} \qquad i = 1, \cdots, n,$$

$$\bar{S}_n = \sum_{i=1}^{n} \bar{X}_i.$$

Clearly,

$$\mathbf{P}\{|S_n| \geqq x\} \leqq \mathbf{P}\{\bar{S}_n \neq S_n\} + \mathbf{P}\{|\bar{S}_n| \geqq x\}.$$

Hence, it follows that

$$(40) \qquad \mathbf{P}\{|S_n| \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{|X_i| \geqq y_i\} + e^{-hx}\mathbf{E}\, e^{h|\bar{S}_n|}.$$

We estimate $\mathbf{E}e^{h|\bar{S}_n|}$ as follows. Let $0 < t \leqq 1$. Observe that $|u|^{-1}(e^{h|u|} - 1)$ attains its maximum value in the region $|u| \leqq z$ for $|u| = z$. Therefore,

$$\begin{aligned} \mathbf{E}e^{h|\bar{S}_n|} &\leqq \prod_{i=1}^{n} \mathbf{E}e^{h|\bar{X}_i|} = \prod_{i=1}^{n} \mathbf{E}\left\{1 + \frac{e^{h|\bar{X}_i|} - 1}{|\bar{X}_i|}|\bar{X}_i|\right\} \\ &\leqq \prod_{i=1}^{n}\left\{1 + \frac{e^{hy_i} - 1}{y_i}\int_{|u| \leqq y_i} |u|\, dF_i(u)\right\} \\ &\leqq \prod_{i=1}^{n}\left\{1 + \frac{e^{hy} - 1}{y^t}\int_{|u| \leqq y_i} |u|^t\, dF_i(u)\right\} \leqq \exp\left\{\frac{e^{hy} - 1}{y^t}A(t;-Y,Y)\right\}. \end{aligned}$$

Hence,

$$(41) \qquad e^{-hx}\,\mathbf{E}e^{h|\bar{S}_n|} \leqq \exp\left\{\frac{e^{hy} - 1}{y^t}A(t;-Y,Y) - hx\right\}.$$

Minimizing the right-hand side of (41) with respect to h and taking condition (37) into consideration, we obtain (39). From this and (40) follows inequality (38). If we replace h in (41) by its value given by (22), then inequality (35) will follow from (36) and (40). Clearly, $P_9 \leqq P_8$. Theorem 5 is proved.

4. Special Cases

In this section, t is to denote either the order of the truncated moments or the order of finite moments. We introduce the following notation for finite absolute moments and variances:

$$A_{t,n} = \sum_{i=1}^{n} \mathbf{E}|X_i|^t, \qquad B_n^2 = \sum_{i=1}^{n} \mathbf{D}X_i.$$

Observe that the proofs of Theorems 1–5 remain valid if in inequalities (20), (21), (27), (34) and (41) the truncated absolute moments and variances are replaced by the full absolute moments and variances, respectively. Hence, under the assumption that the absolute moments of any order exist, one can replace the truncated absolute moments in the various inequalities obtained in Theorems 1–5 by the full absolute moments.

We shall now state some consequences of Theorems 3 and 4.

Suppose first that the i.r.v. $X_1, \cdots, X_n$ satisfy the following conditions:

$$X_i \leqq L_i, \qquad \mathbf{E}X_i = 0, \qquad i = 1, \cdots, n. \tag{42}$$

Let $y_i = L_i, i = 1, \cdots, n$, and let $y = L = \max\{L_1, \cdots, L_n\}$. Then Theorem 4 implies

Corollary 1. *Suppose the i.r.v.* $X_1, \cdots, X_n$ *satisfy conditions* (42). *Then*

$$\mathbf{P}\{S_n \geqq x\} \leqq \exp\left\{\frac{x}{L} - \left(\frac{x}{L} + \frac{B_n^2}{L^2}\right)\log\left(\frac{xL}{B_n^2} + 1\right)\right\}. \tag{43}$$

We point out that inequality (43) was obtained independently by Bennett [10], [11] and Hoeffding [12],

Let $\beta = t/(t+2)$ and $\alpha = 1 - \beta$. Then inequalities (13) and (15) of Theorem 3 imply

Corollary 2. *Suppose* $t \geqq 2$, $\beta = t/(t+2)$ *and* $\alpha = 1 - \beta$. *Then*

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + \exp\left\{\max\left[-\left(\beta\frac{x}{y} - \frac{\mu(-Y, Y)}{y}\right) \times \log\left(\frac{\beta x y^{t-1}}{A(t;0,Y)} + 1\right), -\frac{\alpha x(\alpha x/2 - \mu(-Y,Y))}{e^t B^2(-Y,Y)}\right]\right\} \tag{44}$$

$$\leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + \exp\left\{-\left(\beta\frac{x}{y} - \frac{\mu(-Y, Y)}{y}\right) \times \log\left(\frac{\beta x y^{t-1}}{A(t;0,Y)} + 1\right)\right\} + \exp\left\{-\frac{\alpha x(\alpha x/2 - \mu(-Y,Y))}{e^t B^2(-Y,Y)}\right\}. \tag{45}$$

From Theorem 3′ follows

Corollary 3. *Suppose* $t \geqq 2$, $\beta = t/(t+2)$ *and* $\alpha = 1 - \beta$. *Suppose further* that $\mathbf{E}X_i = 0$, $i = 1, \cdots, n$. *Then*

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + \exp\left\{\max\left[-\beta\frac{x}{y}\log\left(\frac{\beta x y^{t-1}}{A(t;0,Y)} + 1\right),\right.\right. \tag{46}$$

$$\left.\left. -\frac{\alpha^2 x^2}{2e^t B^2(-\infty, Y)}\right]\right\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + \left(\frac{\beta x y^{t-1}}{A(t;0,Y)} + 1\right)^{-\beta x/y} + \exp\left\{-\frac{\alpha^2 x^2}{2e^t B^2(-\infty, Y)}\right\}. \tag{47}$$

Set $y_1 = \cdots = y_n = y = \beta x$ in (46) and (47). Then from Corollary 3 we obtain

Corollary 4. *Let* $\mathbf{E}X_i = 0$ *and* $\mathbf{E}|X_i|^t < \infty$, $i = 1, \cdots, n$, *for* $t \geqq 2$. *Then*

$$\mathbf{P}\{S_n \geqq x\} \leqq c_t^{(1)} A_{t,n} x^{-t} + \exp\{-c_t^{(2)} x^2 / B_n^2\}, \tag{48}$$

where $c_t^{(1)} = (1 + 2/t)^t$ *and* $c_t^{(2)} = 2(t+2)^{-1} e^{-t}$.

5. Examples

In this section, we shall give some examples involving the calculation of probabilities on the basis of the resultant inequalities. The set of n positive numbers y_i will be chosen in arbitrary fashion. When x is large, this permits us to select y so that both components on the right-hand sides of these inequalities have small values simultaneously. Some specific examples are cited and they are compared with the inequalities obtained earlier by other authors ([5], [9]–[12]).

We shall consider the case where the moments of order $0 < t \leqq 2$ exist. To make things convenient while computing the probabilities, we shall assign values to x and y of the form $x^t = a^t A_{t,n}$ and $y^t = b^t A_{t,n}$, where a and b are positive numbers. To simplify notation, we shall omit writing the subscripts on $A_{t,n}$.

1. Case $0 < t \leqq 1$. It follows from inequality (35) that

$$\mathbf{P}\{|S_n| \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{|X_i| \geqq y_i\} + P_{10}, \tag{49}$$

where

$$P_{10} = \exp\left\{\frac{x}{y} - \frac{A}{y^t} - \frac{x}{y}\log\left(\frac{x y^{t-1}}{A}\right)\right\}.$$

We wish to show that for $\gamma \geqq 1$ and $x^t > \gamma e^2 A$, it is possible to choose y so that the relation

$$P_{10} < A x^{-t}/\gamma \tag{50}$$

is satisfied.

For, suppose $0 < 2y \leqq x$. Let $z = A/xy^{t-1}$ and $x/y = c$. Observe that $0 < ze^{1-z} \leqq 1$ for all $z > 0$. Therefore, under the given conditions we have

$$P_{10} = (ze^{1-z})^c \leqq (ez)^2 = (eAc^{t-1}x^{-t})^2 < Ax^{-t} \cdot e^2 Ax^{-t} \leqq Ax^{-t}/\gamma.$$

The familiar Chebyshev inequality yields the bilateral estimate

$$\mathbf{P}\{|S_n| \geqq x\} \leqq Ax^{-t}. \tag{51}$$

In the next example we compare the estimates (49) and (51). Suppose $X_1, \cdots, X_n$ are i.r.v. with common Cauchy distribution function on the positive real line, namely,

$$F(u) = \frac{2}{\pi}\int_0^u \frac{dz}{1+z^2}, \qquad u \geqq 0.$$

For definiteness, take $t = 2/3$ and $n = 2$ and 8. The first table furnishes values of the right-hand sides of inequalities (49) and (51).

2. Case $1 \leqq t \leqq 2$. For identically distributed i.r.v., Bengt and Esseen [9] obtained the following bilateral estimate:

$$\mathbf{P}\{|S_n| \geqq x\} \leqq M(t, n)Ax^{-t}, \tag{52}$$

where

$$M(t,n) = \begin{cases} \min\{2 - n^{-1}, [1 - D(t)]^{-1}\} & \text{for } D(t) = \dfrac{13,52}{\pi(2,6)^t}\Gamma(t)\sin\dfrac{t\pi}{2} < 1, \\ 2 - n^{-1} & \text{otherwise.} \end{cases} \tag{53}$$

$\frac{x}{A^{3/2}}$	$\frac{y}{A^{3/2}}$	(51)	(49) $n = 2$	(49) $n = 8$
2	$\frac{1}{3}$	.6300	.5523	.3338
6	$\frac{3}{2}$	.3299	.1383	.0852
10	2	.2154	.0819	.0422
14	$\frac{7}{2}$	.1721	.0504	.0278
18	$\frac{9}{2}$	.1456	.0381	.0205
24	6	.1201	.0278	.0145

For symmetrically distributed summands, Theorem 2 gives

$$\mathbf{P}\{S_n \geqq x\} \leqq \sum_{i=1}^{n} \mathbf{P}\{X_i \geqq y_i\} + P_{11}, \tag{54}$$

where

$$P_{11} = \exp\left\{\frac{x}{y} - \left(\frac{x}{y} + \frac{A}{y^t}\right)\log\left(\frac{xy^{t-1}}{A} + 1\right)\right\}.$$

When $x^t > 9\gamma e^2 A$, $\gamma \geqq 1$ and $1 \leqq t \leqq 2$, the following relation holds:

$$P_{11} < A(x^t + A)^{-1}/\gamma < Ax^{-t}/\gamma. \tag{55}$$

Indeed, suppose $2 \leqq x/y = c \leqq 3$ and $z = A/xy^{t-1}$. We have

$$P_{11} = [e(1 + 1/z)^{-1-z}]^c < \left(\frac{ez}{1+z}\right)^2 < 9Ae^2x^{-t}A(x^t + A)^{-1}$$
$$\leqq A(x^t + A)^{-1}/\gamma < Ax^{-t}/\gamma.$$

Taking (55) into account, one can hope to obtain, for large x, better estimates than those given by inequality (52) and Cantelli's inequality

$$\mathbf{P}\{S_n \geqq x\} \leqq \frac{B_n^2}{x^2 + B_n^2} \tag{56}$$

by choosing the values of y in inequality (54) suitably.

Let $X_1, \cdots, X_n$ be i.r.v. with common distribution function $F(u) = \frac{1}{2}(1 + u(1 + u^2)^{-1/2})$. It is not too hard to see that $X_1, \cdots, X_n$ are symmetrically distributed, $\mathbf{D}X_1 = \infty$ and $\mathbf{E}|X_1|^t < \infty$ for $0 < t < 2$. Choose $t = 9/5$ and $n = 2$ or 10 (for $t = 9/5$, $M(t, n) \approx 1.285$ in (53)). We obtain the following table of values for the right-hand sides of (52) and (54):

$\frac{x}{A^{5/9}}$	$\frac{y}{A^{5/9}}$	(52)	(54) $n = 2$	(54) $n = 10$
2	$\frac{2}{3}$	.36903	.29552	.28999
4	1	.10598	.04548	.05006
8	3	.03043	.00763	.00689
14	6	.01111	.00177	.00161
22	9	.00493	.00048	.00047

We now give one further example demonstrating the accuracy of the estimates (54) and (56). Let $X_1, \cdots, X_n$ be i.r.v. with common distribution function

$$F(u) = \frac{2}{\pi}\int_{-\infty}^{u} \frac{dz}{(1 + z^2)^2}.$$

It is not hard to see that $\mathbf{E}X_1 = 0$ and $\mathbf{D}X_1 = 1$. We have the following table:

$\frac{x}{B_n}$	$\frac{y}{B_n}$	(56)	(54) $n = 4$	(54) $n = 16$
2	1	.20000	.35474	.33058
4	$\frac{4}{3}$	.05882	.06533	.04949
8	3	.01538	.00424	.00372
12	5	.00689	.00129	.00091
16	6	.00389	.00052	.00038
24	12	.00173	.00009	.00004

As Hoeffding pointed out in [12], inequality (43) refines the following inequality due to Yu. V. Prokhorov [5]:

$$\mathbf{P}\{S_n \geqq x\} \leqq \exp\left\{-\frac{x}{2L}\arcsin\left(\frac{xL}{2B_n^2}\right)\right\}.$$

As Yu. V. Prokhorov pointed out, this estimate improves the inequality of S. N. Bernstein and A. N. Kolmogorov. Inequality (43) also refines the familiar Cantelli inequality (56).

6. Applications to the Strong Law of Large Numbers

Let

$$\xi_1, \xi_2, \cdots, \xi_n, \cdots \tag{57}$$

be a sequence of symmetrically distributed independent random variables. It is known that it does not restrict the generality if one requires symmetry when studying conditions for the applicability of the strong law of large numbers (see [2], [3]). Suppose that the variables in the sequence (57) are centered about their means, i.e., $\mathbf{E}\xi_k = 0$ for all $k \geqq 1$.

We partition the sequence into classes by including in the r-th class the random variables ξ_k with $k \in I_r = \{2^r, 2^r + 1, \cdots, 2^{r+1} - 1\}$, $r \geqq 0$.

Let $\{\delta_r\}$ be a sequence of positive numbers. Introduce the following notation (the summation is everywhere with respect to $k \in I_r$):

$$\begin{gathered}\chi_r = \sum 2^{-r}\xi_k, \qquad K(t, \delta_r, r) = \sum 2^{-tr}\int_0^{2^r\delta_r} u^t\, dF_k(u),\\ K_{t,r} = \sum 2^{-tr}\mathbf{E}|\xi_k|^t,\\ H(\delta_r, r) = \sum 2^{-2r}\int_{-2^r\delta_r}^{2^r\delta_r} u^2\, dF_k(u), \qquad H_r = \sum 2^{-2r}\mathbf{D}\xi_k.\end{gathered} \tag{58}$$

The following condition (due to Yu. V. Prokhorov [2]):

$$\sum_{r=1}^{\infty} \mathbf{P}\{\chi_r \geqq \varepsilon\} < \infty, \qquad \forall\, \varepsilon > 0 \tag{59}$$

is necessary and sufficient for the sequence (57) to obey the strong law of large numbers, i.e., for

$$n^{-1}(\xi_1 + \cdots + \xi_n) \to 0 \quad \text{a.e.}$$

For $t \geqq 2$, inequality (45) leads to

$$\begin{aligned}\mathbf{P}\{\chi_r \geqq \varepsilon\} < \sum_{k\in I_r} \mathbf{P}\{2^{-r}\xi_k \geqq \delta_r\} + (\varepsilon_1\delta_r^{t-1}/K(t, \delta_r, r) + 1)^{-\varepsilon_1/\delta_r}\\ + \exp\{-\varepsilon_2/H(\delta_r, r)\},\end{aligned} \tag{60}$$

where $\varepsilon_1 = \varepsilon t(t + 2)^{-1}$ and $\varepsilon_2 = 2\varepsilon^2 e^{-t}(t + 2)^{-2}$.

It is not hard to see that if the series

$$\sum_{r=1}^{\infty} (\varepsilon_1 \delta_r^{t-1}/K(t, \delta_r, r) + 1)^{-\varepsilon_1/\delta_r} \quad \text{and} \quad \sum_{r=1}^{\infty} \exp\{-\varepsilon_2/H(\delta_r, r)\}$$

are convergent for all positive ε, then the series obtained by replacing ε_1 and ε_2 by ε are also convergent for all positive ε, and conversely. Therefore, when using condition (59), we can substitute ε for ε_1 and ε_2 on the right-hand side of (60). Thus, we have

Theorem 6. *Let $\{\delta_r\}$ be a sequence of positive numbers such that the following conditions hold for any positive ε:*

$$\sum_{r=1}^{\infty} \sum_{k \in I_r} \mathbf{P}\{2^{-r}\xi_k \geqq \delta_r\} < \infty, \tag{61}$$

$$\sum_{r=1}^{\infty} (\varepsilon \delta_r^{t-1}/K(r, \delta_r, r) + 1)^{-\varepsilon/\delta_r} < \infty, \qquad t \geqq 2, \tag{62}$$

$$\sum_{r=1}^{\infty} \exp\{-\varepsilon/H(\delta_r, r)\} < \infty. \tag{63}$$

Then the sequence (57) *obeys the strong law of large numbers.*

Remark 1. Denote $H(\delta_r, r)$ by φ_r. Then for $t = 2$, condition (62) can be rewritten in the following form:

$$\sum_{r=1}^{\infty} (\varepsilon/\varphi_r + 1)^{-\varepsilon\varphi_r/H(\delta_r, r)} < \infty.$$

From this we see that if $\varphi_r \to 0$ as $r \to \infty$, then (62) implies (63). But if $\liminf_{r\to\infty} \varphi_r > 0$, then (62) and (63) are equivalent.

If the absolute moments of order $t \geqq 2$ exist, then the following theorem is valid.

Theorem 6′. *Suppose the sequence* (57) *is such that $\mathbf{E}|\xi_k|^t < \infty$ for $t \geqq 2$ and for all $k \geqq 1$. Then the fulfillment of condition* (61) *and the following conditions*:

$$\sum_{r=1}^{\infty} (\varepsilon \delta_r^{t-1}/K_{t,r} + 1)^{-\varepsilon/\delta_r} < \infty, \tag{64}$$

$$\sum_{r=1}^{\infty} e^{-\varepsilon/H_r} < \infty, \qquad \forall\, \varepsilon > 0, \tag{65}$$

where $K_{t,r}$ and H_r are given by (58), *is sufficient for the strong law of large numbers.*

We state now some corollaries to Theorem 6′. Setting $\delta_r = \varepsilon\beta^{-1}$, $\beta \geqq 1$, we obtain

Corollary 1. *If $t \geqq 2$ and there exists a constant $\beta \geqq 1$ such that for any positive ε conditions* (65) *and*

$$\sum_{k=1}^{\infty} \mathbf{P}\{\xi_k \geqq k\varepsilon\} < \infty, \tag{66}$$

$$\sum_{r=1}^{\infty} (K_{t,r})^{\beta} < \infty \tag{67}$$

hold, then the sequence (57) *obeys the strong law of large numbers.*

It is well-known that condition (66) is necessary for the strong law of large numbers.

Corollary 2. *If condition* (66) *holds and there exists a $\beta \geqq 1$ such that*

$$\sum_{r=1}^{\infty} H_r^{\beta} < \infty, \tag{68}$$

then the strong law of large numbers holds.

REMARK 2. *Let $\{b_n\}$ be a non-decreasing sequence of positive numbers with $b_n \to \infty$.*

V. A. Egorov [7] showed recently that if

$$\sum_{n=1}^{\infty} \mathbf{P}(|\xi_n| > \varepsilon b_n) < \infty \quad \text{and} \quad \sum_{n=k}^{\infty} \frac{\sigma_n^2}{b_n^{2k}} \sum \sigma_{j_1}^2 \cdots \sigma_{j_{k-1}}^2 < \infty, \qquad k \geqq 2, \tag{69}$$

where $\sigma_j^2 = \mathbf{D}\xi_j$ and the inner summation extends over all $j_1, \cdots, j_{k-1}$ satisfying $1 \leqq j_1 < \cdots < j_{k-1} \leqq n-1$, then $(b_n)^{-1} \sum_{j=1}^{n} \xi_j \to 0$ a.e.

It is interesting to compare conditions (68) and (69) when $b_n = n$.

We first give an example to show that generally speaking (68) does not follow from (69).

For, let $n_j = j2^{(j^2-j)/2}$ and assume $\sigma_n^2 = 2^{j^2}$ if $n = n_j$ and vanishes for other values of n. It is not hard to see that

$$\sum_{n=1}^{n_j-1} \sigma_n^2 < (j-1)2^{(j-1)^2}.$$

Hence, the sum of the series occurring in condition (69) does not exceed $2\sum_{j=1}^{\infty} j^{-3}$ for $k = 2$. At the same time, $\sigma_{n_j}^2/n_j^2 \to \infty$ as $j \to \infty$ from which it follows that condition (68) cannot be satisfied no matter what $\beta \geqq 1$.

In this example, the sequence $\{\sigma_n^2\}$ contains large gaps. But if σ_n^2 varies regularly as $n \to \infty$, then the case when $\lim_{r\to\infty} B_{2^{r+1}}^2/B_{2^r}^2 < \infty$ with $B_n^2 = \sum_{i=1}^{n} \sigma_i^2$ and, in addition, $\liminf_{n\to\infty} B_n^{(k)}/B_n^{2k} > 0$, where $B_n^{(k)} = \sum \sigma_{j_1}^2 \cdots \sigma_{j_k}^2$ (the summation being over those $j_1, \cdots, j_k$ which satisfy $1 \leqq j_1 < \cdots < j_k < n$) is typical. In that event.

$$\sum_{r=1}^{\infty} H_r^k = \sum_{r=1}^{\infty} 2^{-2^r} \sum_{n\in I_r} \sigma_n^2 H_r^{k-1} < L_1 \sum_{n=1}^{\infty} \frac{\sigma_n^2 B_n^{2k-2}}{n^{2k}} < L_2 \sum_{n=1}^{\infty} \frac{\sigma_n^2 B_n^{(k-1)}}{n^{2k}},$$

where L_1 and L_2 are constants, or in other words, from condition (69) follows condition (68).

Setting $\beta = 1$ in (67) and using Chebyshev's inequality, we arrive at

Corollary 3. *Condition* (65) *and the condition*

$$\sum_{k=1}^{\infty} \frac{\mathbf{E}|\xi_k|^t}{k^t} < \infty, \qquad t > 2, \tag{70}$$

are sufficient for the strong law of large numbers.

REMARK 3. When $t \geqq 2$, Yu. V. Prokhorov [2] (and Brunk [8] for even t) showed that the condition

$$\sum_{k=1}^{\infty} \frac{\mathbf{E}|\xi_k|^t}{k^{t/2+1}} < \infty \tag{71}$$

is sufficient for the strong law of large numbers. Since

$$\left(\frac{1}{2^r} \sum_{n \in I_r} \mathbf{D}\xi_n\right)^{t/2} \leqq \frac{1}{2^r} \sum_{n \in I_r} \mathbf{E}|\xi_n|^t,$$

condition (71) implies conditions (65) and (70) under which Corollary 3 is valid.

Corollary 4. *Let* $\xi_k = O(k/\varphi(k))$, *where* $\varphi(k) \uparrow \infty$ *as* $k \to \infty$. *Then the condition*

$$\sum_{r=1}^{\infty} (\varepsilon/\varphi(2^r) K_{t,r} + 1)^{-\varepsilon\varphi(2^r)} < \infty, \qquad \forall\, \varepsilon > 0, \qquad t \geqq 2, \tag{72}$$

is sufficient for the strong law of large numbers.

REMARK 4. Yu. V. Prokhorov proved in [4] that the condition

$$\sum_{r=1}^{\infty} e^{-\varepsilon/H_r} < \infty, \qquad \forall\, \varepsilon > 0 \tag{73}$$

is necessary and sufficient for the strong law of large numbers if $\varphi(k) = \log\log k$. We can show that in this case condition (72) is necessary and sufficient for all $t \geqq 2$. To this end, we merely have to prove that (72) follows from (73) for $t = 2$, since

$$K_{t,r} = K(t, 1, r) \leqq H(1, r) = H_r,$$

if $\xi_n < n$, $\forall\, n \in I_r$.

Without loss of generality, we may assume that $\varphi(2^r) = \log r$. Set $\delta_r = 1/\log r$. If $\varepsilon\delta_r/H_r \geqq e^{2/\varepsilon} - 1$, then

$$(\varepsilon\delta_r/H_r + 1)^{-\varepsilon/\delta_r} \leqq r^{-2}.$$

But if $\varepsilon\delta_r/H_r < e^{2/\varepsilon} - 1$, then the inequality $\log(x + 1) \geqq x/(x + 1)$ leads to

$$(\varepsilon/\delta_r)\log(\varepsilon\delta_r/H_r + 1) > \varepsilon^2 e^{-2/\varepsilon}/H_r.$$

Thus

$$\sum_{r=1}^{\infty} (\varepsilon\delta_r/H_r + 1)^{-\varepsilon/\delta_r} < \sum_{r=1}^{\infty} r^{-2} + \sum_{r=1}^{\infty} \exp\{-\varepsilon^2 e^{-2/\varepsilon}/H_r\}.$$

REMARK 5. Under conditions (67) and (68), $K_{t,r}$ and H_r may be replaced, respectively, by $K(t, 1, r)$ and $H(1, r)$ since there is no loss of generality in assuming that $|\xi_n| < n$.

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