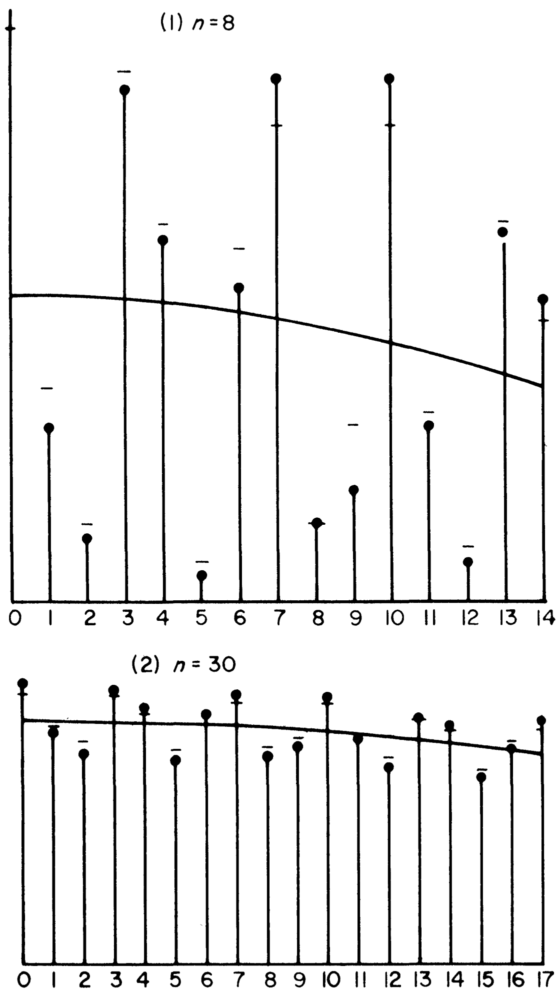

Fig. 2

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AN ESTIMATE FOR THE CONVERGENCE RATE FOR THE ABSORPTION PROBABILITY

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(Translated by B. Seckler)

Let ξ_1, $i = 1, \cdots \infty$, be a sequence of identically distributed independent random variables with distribution function $F(x)$ such that $\mathbf{E}\xi_1 = m$ and $\mathbf{E}\xi_1^2 = 2\lambda^2$. Let $c_3 = \mathbf{E}|\xi_1|^3$.

Let $\zeta_n = \sum_{i=1}^n \xi_i$ and let n_x be the smallest index n such that $\zeta_n + x \notin (a, b)$, where (a, b) is a finite interval of the real line. Set

$$P(x) = \mathbf{P}\{\zeta_{n_x} + x \geqq b\}, x \in (a, b), \quad \text{and} \quad P(x) = 0, x \notin (a, b).$$

It is known that

$$\lim_{\lambda\to 0} P(x) = v(x), \quad x \in (a, b),$$

where $v(x)$ is the solution of the equation

$$v'' + \alpha v' = 0,$$

satisfying the boundary conditions $v(a) = 0$ and $v(b) = 1$ providing

$$\lim_{\lambda\to 0} m\lambda^{-2} = \alpha \quad \text{and} \quad \int_{|x|>\varepsilon} x^2\, dF(x) = o(\lambda^2)$$

for any $\varepsilon > 0$ (see, for example, [1], Chapter 3). The aim of this paper is to estimate the rate at which $P(x)$ converges.

Theorem. *There exists an absolute constant L such that*

$$\sup_{a<x<b} |P(x) - u(x)| < \frac{Lc_3}{(b-a)\lambda^2}\left(1 + \frac{|m|}{\lambda^2}(b-a)\right),$$

where $u(x)$ is the solution of the equation

$$u'' + \frac{m}{\lambda^2}u' = 0,$$

satisfying the boundary conditions $u(a) = 0$ and $u(b) = 1$.

This result was obtained for $m = 0$ in [2].

Proof. As in [2], we denote by $S(p(x))$ the operator carrying an absolutely integrable function $p(x)$ into its Fourier–Lebesgue transform $\int_{-\infty}^{\infty} e^{itx}p(x)\,dx$ and by $S_1(F(x))$ the operator carrying a function bounded variation into its Fourier-Stieltjes transform $\int_{-\infty}^{\infty} e^{itx}\,dF(x)$. The symbol S_1^{-1} will be used to denote the operator inverse to S_1.

Without loss of generality, we may assume $F(x)$ to be continuous, $m > 0$, $a = 0$ and $b = 1$. Let $\varphi(t) = S(P)$ and $f(t) = S_1(F)$. Clearly,

$$P(x) = 1 - F(1-x) + \int_0^1 P(y)\,dF(y-x). \tag{1}$$

This equation may be written in terms of Fourier transforms as

$$\varphi(t)(1 - f(-t)) = \int_0^1 (1 - F(1-x))\,e^{itx}\,dx + S(\Phi), \tag{2}$$

where

$$\Phi(x) = \begin{cases} -\int_0^1 P(y)\,dF(y-x), & x \notin (0,1), \\ 0. & x \in (0,1). \end{cases}$$

Let

$$f_2(t) = \frac{1 - f(-t) - imt}{\lambda^2 t^2}, \qquad g(t) = \frac{1 - f(-t)}{imt + \lambda^2 t^2}, \qquad \bar{F}(x) = S_1^{-1}(f(-t)).$$

It is not hard to see that

$$\varphi(t)(1 - f(-t)) = (imt + \lambda^2 t^2)g(t)\varphi(t) = S_1\left(-\left(m + \lambda^2\frac{d}{dx}\right)P * G\right), \tag{3}$$

where

$$P * G(x) = \int_0^1 P(x-y)\,dG(y), \qquad G(x) = S_1^{-1}(g(t)).$$

From (3) and (2), it follows that $P * G(x)$ satisfies the equation

$$\lambda^2\frac{d^2y}{dx^2} + m\frac{dy}{dx} = F(1-x) - 1 \tag{4}$$

for $0 < x < 1$. Clearly,

$$(m - \lambda^2 it)g(t) = \frac{1 - f(-t)}{it}.$$

Hence we find

$$\left(m\frac{d}{dx} + \lambda^2\frac{d^2}{dx^2}\right)G(x) = \begin{cases} -F(-x), & x > 0, \\ -F(-x) + 1, & x < 0, \end{cases}$$

and therefore

$$\left(m\frac{d}{dx} + \lambda^2\frac{d^2}{dx^2}\right)G(x - 1) = \begin{cases} -F(1 - x) + 1, & x < 1, \\ -F(1 - x), & x > 1. \end{cases} \tag{5}$$

Thus, $-G(x - 1)$, $x < 1$, is the solution of equation (4). Therefore,

$$P * G(x) + G(x - 1) = w(x), \qquad 0 < x < 1, \tag{6}$$

where $w(x)$ satisfies the equation

$$\lambda^2\frac{d^2y}{dx^2} + m\frac{dy}{dx} = 0, \qquad 0 < x < 1. \tag{7}$$

Clearly, $G(x - 1) = E * G(x)$, where

$$E(x) = \begin{cases} 1, & x \geqq 1, \\ 0, & x < 1. \end{cases}$$

Let

$$P^*(x) = P(x) + E(x).$$

Then equation (6) may be rewritten in the following form:

$$P^* * G(x) = w(x), \qquad 0 < x < 1. \tag{8}$$

Let $\bar{w}(x) = P^* * G(x) - w(x)$, where $w(x)$ satisfies equation (7) for $-\infty < x < \infty$ and coincides with $P^* * G(x)$ on $(0, 1)$.

Lemma 1. *For* $x > 1$, $\bar{w}(x) \leqq 0$.

PROOF. If $x > 1$, then by (2), (3) and (5),

$$\left(\lambda^2\frac{d^2}{dx^2} + m\frac{d}{dx}\right)P^* * G(x) = -F(1 - x) - \Phi(x).$$

Clearly,

$$-\Phi(x) = \int_0^1 P(y)\,dF(y - x) \leqq F(1 - x).$$

Thus, for $x > 1$,

$$\left(\lambda^2\frac{d^2}{dx^2} + m\frac{d}{dx}\right)P^* * G(x) \leqq 0.$$

Hence

$$\left(\lambda^2\frac{d^2}{dx^2} + m\frac{d}{dx}\right)\bar{w}(x) \leqq 0, \qquad x > 1.$$

Moreover, $\bar{w}(1) = 0$ and $\bar{w}'(1) = 0$. This implies

$$\lambda^2\bar{w}'(x) + m\bar{w}(x) \leqq 0, \qquad x > 1,$$

and therefore

$$\frac{\bar{w}(x)}{\bar{w}(y)} \leqq e^{m(y-x)/\lambda^2}, \qquad x \geqq y.$$

Thus the function $\bar{w}(x)\,e^{mx/\lambda^2}$ is non-increasing and this means that $\bar{w}(x) \leqq 0$ for $x > 1$. The lemma is proved.

Let us now consider values of $x < 0$. Let $w_1(x)$ denote the solution of equation (7) satisfying the initial conditions

$$w_1(0) = P * G(0), \qquad w_1'(0) = \frac{d}{dx} P * G(x)\bigg|_{x=0}.$$

Lemma 2. *For $x < 0$, $P * G(x) \geqq w_1(x)$.*

PROOF. Let $\bar{w}_1(x) = P * G(x) - w_1(x)$. From (2) and (3) it follows that

$$\left(\lambda^2 \frac{d^2}{dx^2} + m\frac{d}{dx}\right) P * G(x) = -\Phi(x) \geqq 0$$

for $x < 0$. Hence

$$\left(\lambda^2 \frac{d^2}{dx^2} + m\frac{d}{dx}\right) \bar{w}_1(x) \geqq 0.$$

This implies that

$$\lambda^2 \frac{d}{dx} \bar{w}_1(x) = m\bar{w}_1(x) \leqq 0,$$

since $w_1(0) = 0$ and $\bar{w}_1'(0) = 0$. Therefore,

$$\frac{\bar{w}_1(y)}{\bar{w}_1(x)} \leqq e^{m(x-y)/\lambda^2}, \qquad x < y.$$

Thus, the function $\bar{w}_1(x)\, e^{mx/\lambda^2}$ is non-increasing. Thus it follows that $\bar{w}_1(x) \geqq 0$ for $x < 0$ and this is equivalent to the assertion of the lemma.

Lemma 3. *For $x \leqq 0$,*

$$|w(x) - w_1(x)| \leqq \frac{c_3}{2\lambda^2}\left(\frac{\lambda^2}{m}(e^{-mx/\lambda^2} - 1) + 1\right).$$

PROOF. Let $w_2(x) = w(x) - w_1(x)$. By (6),

$$w_2(0) = G(-1), \qquad w_2'(0) = G'(-1).$$

Further,

$$g(t) = \frac{1}{m - \lambda^2 it} \frac{1 - f(-t)}{it}.$$

Clearly,

$$S^{-1}\left(\frac{1}{m - \lambda^2 it}\right) = \begin{cases} \lambda^{-2}\, e^{-mx/\lambda^2}, & x \geqq 0, \\ 0, & x < 0. \end{cases} \tag{9}$$

Therefore, for $x \leqq 0$,

$$G'(x) = e^{-mx/\lambda^2}\lambda^{-2} \int_{-\infty}^{x} e^{my/\lambda^2}\bar{F}(y)\, dy.$$

From this we obtain

$$0 \leqq G'(x) \leqq \frac{c_3}{2\lambda^2 |x|^2}, \qquad 0 \leqq G(x) \leqq \frac{c_3}{2\lambda^2 |x|}.$$

Thus,

$$0 \leqq w_2(0) \leqq \frac{c_3}{2\lambda^2}, \qquad 0 \leqq w_2'(0) \leqq \frac{c_3}{2\lambda^2}.$$

On the other hand,

$$w_2(x) = \frac{\lambda^2}{m}(1 - e^{-mx/\lambda^2}) w_2'(0) + w_2(0).$$

Hence,

$$|w_2(x)| \leqq \frac{c_3}{2\lambda^2}\left(\frac{\lambda^2}{m}(e^{-mx/\lambda^2} - 1) + 1\right), \qquad x \leqq 0,$$

as required.

Lemma 4.

$$|w'(x)| \leqq 2\left(\frac{m}{\lambda^2} + 1\right) e^{-mx/\lambda^2}.$$

PROOF. Observe first of all that

$$w(x) = \alpha\, e^{-mx/\lambda^2} + \beta, \tag{10}$$

where α and β are constants. Clearly,

$$0 \leqq P^*(x) \leqq 1. \tag{11}$$

From (8) and (11) it follows that

$$|w(x)| \leqq \operatorname*{Var}_{-\infty<x<\infty} G(x). \tag{12}$$

Let us now estimate $\operatorname{Var}_{-\infty<x<\infty} G(x)$. Clearly,

$$g(t) = \frac{1 - f(-t) - imt}{imt + \lambda^2 t^2} + \frac{m}{m - i\lambda^2 t}.$$

Further,

$$\frac{1}{imt + \lambda^2 t^2} = \frac{1}{\lambda^2 t^2} - \frac{m}{\lambda^2 t^2 (m - i\lambda^2 t)}.$$

Therefore,

$$g(t) = f_2(t)\left(1 - \frac{m}{m - i\lambda^2 t}\right) + \frac{m}{m - i\lambda^2 t}. \tag{13}$$

It is not hard to see that

$$S^{-1}(f_2(t)) = \begin{cases} -\lambda^{-2}m + \int_{-\infty}^{x} F_1(u)\,du, & x > 0, \\ \int_{-\infty}^{x} F_1(u)\,du, & x \leqq 0, \end{cases} \tag{14}$$

where

$$F_1(x) = \begin{cases} \frac{1}{\lambda^2}(\bar{F}(x) - 1), & x > 0, \\ \frac{1}{\lambda^2}\bar{F}(x), & x \leqq 0. \end{cases}$$

Clearly,

$$\frac{m}{\lambda^2} \leqq \int_{-\infty} F_1(u)\,du, \qquad x > 0.$$

Hence

$$S^{-1}(f_2(t)) \geqq 0. \tag{15}$$

This implies that

$$\operatorname*{Var}_{-\infty<x<\infty} S_1^{-1}(f_2(t)) = f_2(0) = 1. \tag{16}$$

By (9), we have

$$\operatorname*{Var}_{-\infty<x<\infty} S_1^{-1}\left(\frac{m}{m - \lambda^2 it}\right) = 1. \tag{17}$$

From (12), (13), (16) and (17) it follows that

(18) $$|w(x)| \leqq 2, \qquad 0 \leqq x \leqq 1.$$

Consequently,

$$|\alpha(e^{-m/\lambda^2} - 1)| = |w(1) - w(0)| \leqq 2.$$

From this we obtain

$$|w'(x)| < \frac{m}{\lambda^2}|\alpha|\, e^{-mx/\lambda^2} \leqq \frac{2m}{\lambda^2}(1 - e^{-m/\lambda^2})^{-1} e^{-mx/\lambda^2} \leqq 2\left(\frac{m}{\lambda^2} + 1\right) e^{-mx/\lambda^2},$$

as required.

Let $F_2(x) = S_1^{-1}(f_2(t))$.

Lemma 5.

$$\operatorname*{Var}_{-\infty<x<\infty} [G(x) - F_2(x)] < \frac{c_3 m}{3\lambda^4}.$$

Proof. Using the representation (13), we can easily obtain

(19) $$g(t) = f_2(t) + f_3(t)\frac{mit}{m - i\lambda^2 t},$$

where $f_3(t) = [1 - f_2(t)]/it$. It is easy to see that

$$S^{-1}(f_3(t)) = \begin{cases} F_2(x) - 1, & x > 0, \\ F_2(x), & x \leqq 0. \end{cases}$$

From this representation, using (14), we obtain

(20) $$\operatorname*{Var}_{-\infty<x<\infty} S_1^{-1}(f_3(t)) = \int_{-\infty}^{\infty} |x|\, dF_2(x) \leqq \frac{c_3}{6\lambda^2}.$$

On the other hand,

$$\frac{mit}{m - i\lambda^2 t} = \frac{m}{\lambda^2} - \frac{m^2}{\lambda^2(m - \lambda^2 it)}.$$

By virtue of (9) this implies

(21) $$\operatorname*{Var}_{-\infty<x<\infty} S_1^{-1}\left(\frac{mit}{m - i\lambda^2 t}\right) \leqq \frac{2m}{\lambda^2}.$$

From (19) and (21), we can deduce the assertion of the lemma without any difficulty.

We now proceed to the final step in the proof. Without loss of generality, we may assume that $c_3/\lambda^2 < 1$ and $c_3 m/\lambda^4 < 1$. Let $c_3(1 + m/\lambda^2)/\lambda^2$ be denoted by γ.

By Lemma 5,

(22) $$|P^* * G(x) - P^* * F_2(x)| < \frac{c_3 m}{3\lambda^4}, \qquad |P * G(x) - P * F_2(x)| < \frac{c_3 m}{3\lambda^4}.$$

Let

$$W(x) = \begin{cases} w(0), & x \leqq 0, \\ w(x), & 0 < x < 1, \\ w(1), & x \geqq 1. \end{cases}$$

Clearly,

$$W(x) = W * F_2(x) + \int_{-\infty}^{\infty} (W(x) - W(x - y))\, dF_2(y),$$

$$\left|\int_{-\infty}^{\infty} (W(x) - W(x - y))\, dF_2(y)\right| < \max_{0 \leqq x \leqq 1} |w'(x)| \int_{-\infty}^{\infty} |y|\, dF_2(y).$$

Using (20) and Lemma 4, we obtain

(23) $$|W(x) - W * F_2(x)| < \frac{\gamma}{3}.$$

By (22),

$$(24)\quad \begin{aligned} P^* * G(x) &\geqq P^* * F_2(x) - \frac{mc_3}{3\lambda^4} \geqq P^* * F_2(1) - \frac{mc_3}{3\lambda^4} \\ &\geqq P^* * G(1) - \frac{2}{3}\frac{mc_3}{\lambda^4} = w(1) - \frac{2}{3}\frac{mc_3}{\lambda^4} = W(x) - \frac{2}{3}\frac{mc_3}{\lambda^4} \end{aligned}$$

for $x \geqq 1$. Applying Lemmas 1 and 4, we have

$$(25)\quad P^* * G(x) \leqq w(x) \leqq w(1) + 2\left(\frac{m}{\lambda^2} + 1\right)(x - 1), \qquad x \geqq 1.$$

From (24) and (25) it follows that

$$(26)\quad |P^* * G(x) - W(x)| < 2\gamma$$

for $1 \leqq x \leqq c_3/\lambda^2 + 1$. Consider now values of $x < 0$. By (22),

$$(27)\quad P * F_2(x) \geqq P * G(x) - \frac{1}{3}\frac{mc_3}{\lambda^4}.$$

On the other hand, by Lemma 3,

$$(28)\quad w_1(x) > w(x) - \frac{c_3}{2\lambda^2}\left(\frac{c_3}{\lambda^2}e + 1\right) > w(x) - \frac{c_3}{2\lambda^2}(e + 1)$$

for $-c_3/\lambda^2 \leqq x \leqq 0$. Using Lemma 4, we conclude that

$$(29)\quad w(x) \geqq w(0) + 2e\left(\frac{m}{\lambda^2} + 1\right)x \geqq W(x) - 2e\gamma, \qquad -\frac{c_3}{\lambda^2} \leqq x \leqq 0.$$

From (27)–(29) and Lemma 2 it follows that, for $-c_3/\lambda^2 \leqq x \leqq 0$,

$$(30)\quad P * F_3(x) \geqq W(x) - 8\gamma.$$

Using (20), we find

$$P^* * F_2(0) = P * F_2(0) + F_2(-1) \leqq P * F_2(0) + \frac{mc_3}{6\lambda^4}.$$

By (22) and Lemma 3,

$$P * F_2(0) \leqq P * G(0) + \frac{mc_3}{3\lambda^4} = w_1(0) + \frac{mc_3}{3\lambda^4} \leqq w(0) + \gamma/2.$$

Thus, for $x \leqq 0$,

$$(31)\quad P^* * F_2(x) \leqq P^* * F_2(0) \leqq W(x) + 2\gamma/3.$$

From (30) and (31) it follows that, for $-c_3/\lambda^2 \leqq x \leqq 0$,

$$(32)\quad |P^* * F_2(x) - W(x)| < 8\gamma.$$

In turn, (8), (23), (26) and (32) lead to the estimate

$$(33)\quad |(P^* - W) * F_2(x)| < \tfrac{25}{3}\gamma, \qquad -\frac{c_3}{\lambda^2} \leqq x \leqq 1 + \frac{c_3}{\lambda^2}.$$

Let $\Delta = \sup_{0 \leqq x \leqq 1} |P^*(x) - W(x)|$. There obviously exists an x_0, $0 \leqq x_0 \leqq 1$, such that $\Delta = |P^*(x_0) - W(x_0)|$. Suppose that $P^*(x_0) - W(x_0) = \Delta$. Clearly,

$$(34)\quad (P^* - W) * F_2\left(x_0 + \frac{c_3}{\lambda^2}\right) = \int_{-\infty}^{\infty} (W(y) - P^*(y))\, dF_2\left(x_0 + \frac{c_3}{\lambda^2}y\right).$$

By virtue of Lemma 4,

$$P^*(x) - W(x) \geqq \Delta + 2\gamma(x_0 - x)$$

for $x > x_0$. Therefore,

$$(35)\qquad \int_{x_0}^{\infty}(W(y)-P^*(y))\,dF_2\left(x_0+\frac{c_3}{\lambda^2}-y\right) > \Delta F_2\left(\frac{c_3}{\lambda^2}\right) -2\gamma\int_{x_0}^{\infty}(x_0-y)\,dF_2\left(x_0+\frac{c_3}{\lambda^2}-y\right).$$

Further,

$$(36)\qquad F_2\left(\frac{c_3}{\lambda^2}\right) > \tfrac{5}{6}.$$

It is not hard to see that

$$(37)\qquad \int_{x_0}^{\infty}(x_0-y)\,dF_2\left(x_0+\frac{c_3}{\lambda^2}-y\right) \leqq \frac{c_3}{\lambda^2}+\int_{-\infty}^{\infty}|x|\,dF_2(x) \leqq \frac{7}{6}\frac{c_3}{\lambda^2}.$$

From (35)–(37) it follows that

$$(38)\qquad \int_{x_0}^{\infty}(W(y)-P^*(y))\,dF_2\left(x_0+\frac{c_3}{\lambda^2}-y\right) > \tfrac{5}{6}\Delta-\tfrac{7}{6}\gamma.$$

Finally, by virtue of (36),

$$(39)\qquad \left|\int_{-\infty}^{x_0}(P^*(y)-W(y))\,dF_2\left(x_0+\frac{c_3}{\lambda^2}-y\right)\right| < \frac{\Delta}{6}.$$

From (34), (38) and (39) we conclude that

$$(40)\qquad (P^*-W)*F_2\left(x_0+\frac{c_3}{\lambda^2}\right) > \tfrac{2}{3}\Delta-\tfrac{7}{6}\gamma.$$

Comparing (33) and (40), we obtain

$$(41)\qquad \Delta < 15\gamma.$$

From (41) it follows in particular that

$$|w(0)| < 15\gamma, \qquad |1-w(1)| < 15\gamma.$$

This yields

$$(42)\qquad \sup_{0\leqq x\leqq 1}|u(x)-w(x)| \leqq \max[|w(0)|,|1-w(1)|] < 15\gamma.$$

From (41) and (42) we obtain

$$\sup_{0\leqq x\leqq 1}|P(x)-u(x)| < 30\gamma.$$

Thus the theorem is proved under the assumption that $P^*(x_0)-W(x_0)=\Delta$.
The case $P^*(x_0)-W(x_0)=-\Delta$ is argued in exactly the same way.

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