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ON THE SPEED OF CONVERGENCE IN A BOUNDARY PROBLEM. I

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0. Introduction

Let $\xi_i, i = 1, \cdots, \infty$, be a sequence of identically distributed independent random variables with distribution function $F(x)$. We shall assume throughout that $\mathbf{D}\xi_1 = 1$. Let $a = \mathbf{E}\xi_1$ and $c_3 = \mathbf{E}|\xi_1 - a|^3$.

Let $g_i(t), t \geqq 0, i = 1, 2$, be functions satisfying the conditions

$$g_2(t) < g_1(t), \qquad g_2(0) < 0 < g_1(0),$$
$$|g_i(t + h) - g_i(t)| < Kh, \quad h > 0, \qquad i = 1, 2,$$

where K is some constant.

Set

$$S_{nk} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} (\xi_i - a).$$

Let

$$W_n = \mathbf{P}\left\{g_2\left(\frac{k}{n}\right) < S_{nk} < g_1\left(\frac{k}{n}\right), k = 1, \cdots, n\right\},$$
$$W = \mathbf{P}\{g_2(t) < \xi(t) - \xi(0) < g_1(t), 0 \leqq t \leqq 1\},$$

where $\xi(t)$ is a Brownian motion process. The main result of the paper is the following:

Theorem. *There exists an absolute constant L such that*

$$|W_n - W| < L\frac{c_3^2(K + 1)}{\sqrt{n}}.$$

But first we shall state a few words about results obtained earlier.

Yu. V. Prokhorov [2] obtained the estimate

$$W_n - W = O\left(\frac{\log^2 n}{n^{1/8}}\right)$$

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for a slightly more general situation (non-identically distributed summands having bounded absolute third moment).

A. V. Skorokhod [3] showed that

$$|W_n - W| < L\frac{\log n}{\sqrt{n}}$$

under the conditions of our theorem and the additional assumption $|\xi_i| < C < \infty$, where L is a constant depending just on $K, C, g_1(0)$ and $g_2(0)$.

The paper [4] is devoted to a review of results in the area of boundary problems.

We now introduce some notation. Let

$$S_n = \sum_{i=1}^{n} \xi_i, \qquad \bar{S}_n = \max_{1 \leq i \leq n} S_i, \qquad F_n(x) = \mathbf{P}(S_n < x),$$

$$\bar{F}_n(x) = \mathbf{P}(\bar{S}_n < x), \qquad \tilde{F}_n(x) = \mathbf{P}(\max[0, \bar{S}_n] < x)$$

and

$$f(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x), \qquad \varphi_n(t) = \int_{-\infty}^{\infty} e^{itx}\, d\tilde{F}_n(x).$$

Set

$$\Phi(t, z) = 1 + \sum_{n=1}^{\infty} \varphi_n(t) z^n, \qquad \Psi(t, z) = (1 - f(t)z)\Phi(t, z).$$

It is not hard to see that

$$\varphi_{n+1}(t) - f(t)\varphi_n(t) = \mathbf{P}(\bar{S}_{n+1} < 0) - \int_{-\infty}^{0-} e^{itx}\, d\bar{F}_{n+1}(x) \tag{0.1}$$

and hence,

$$\Psi(t, z) = 1 + \sum_{n=1}^{\infty} \bar{\varphi}_n(t) z^n, \tag{0.2}$$

where

$$\bar{\varphi}_n(t) = \mathbf{P}(\bar{S}_n < 0) - \int_{-\infty}^{0-} e^{itx}\, d\bar{F}_n(x).$$

Further,

$$1 - f(t)z = B^+(t, z)B^-(t, z),$$

where

$$B^+(t, z) = \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{0+}^{\infty} e^{itx}\, dF_n(x)\right\},$$

$$B^-(t, z) = \exp\left\{-\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-\infty}^{0+} e^{itx}\, dF_n(x)\right\}.$$

Clearly,

$$B^{\pm}(t, z) = 1 + \sum_{n=1}^{\infty} b_n^{\pm}(t)z^n,$$

where $b_n^+(t)\,(b_n^-(t))$ is the Fourier-Stieltjes transform on $(0, \infty)\,((-\infty, 0])$. Therefore,

$$\Psi(t, z) = B^-(t, z)g(z), \tag{0.3}$$

where

$$g(z) = 1 + \sum_{n=1}^{\infty} \mathbf{P}(S_n \leqq 0)z^n.$$

Because of (0.2), $\Psi(0, z) \equiv 1$. Hence,

$$g(z) = \frac{1}{B^-(0, z)}. \tag{0.4}$$

From (0.3) and (0.4), we obtain

$$\Psi(t, z) = \exp\left\{\sum_{n=1}^{\infty} \frac{z^n}{n} \int_{-\infty}^{0} (1 - e^{itx})\, dF_n(x)\right\}. \tag{0.5}$$

This incidentally leads right away to the well-known Spitzer identity [7]. Of course, it would be simpler to deduce (0.5) from that identity but we have not resisted the temptation of taking the opportunity to give a proof here of a relation equivalent to Spitzer's identity. A similar (unpublished) proof was found earlier by A. A. Borovkov.

Since

$$\Phi(t, z) = \frac{\Psi(t, z)}{1 - zf(t)},$$

it follows from (0.2) that

$$\varphi_n(t) = \sum_{k=0}^{n} f^k(t)\bar{\varphi}_{n-k}(t) \tag{0.6}$$

if we define $\bar{\varphi}_0(t) \equiv 1$. Using (0.1), we obtain

$$\int_{-\infty}^{\infty} e^{itx}\, d\bar{F}_n(x) = \varphi_n(t) + \int_{-\infty}^{0-} e^{itx}\, d\bar{F}_n(x) - \mathbf{P}(\bar{S}_n < 0) = f(t)\varphi_{n-1}(t). \tag{0.7}$$

From (0.6) and (0.7) it follows that

$$\int_{-\infty}^{\infty} e^{itx}\, d\bar{F}_n(x) = \sum_{k=1}^{n} f^k(t)\bar{\varphi}_{n-k}(t). \tag{0.8}$$

This relation will play an important role in the proof of Lemmas 5 and 6.

If $\mathbf{E}\xi_i = 0$, we shall sometimes use the notation $\bar{F}_n(x, c)$ for

$$\mathbf{P}\left(\max_{1 \leq k \leq n} (S_k + kc) < x\right).$$

Throughout the entire paper the symbol O will be used only when the corresponding constant is an absolute constant.

1. Estimates Related to the Distribution of the Maximum of Sums of Identically Distributed Independent Random Variables

Let

$$\psi_n(t) = \int_{-\infty}^{0} e^{itx}\, d\bar{F}_n(x),$$

$$\bar{a}_n = \int_{-\infty}^{0} x\, d\bar{F}_n(x), \qquad a_n^- = \int_{-\infty}^{0} x\, dF_n(x),$$

$$\bar{b}_n = \int_{-\infty}^{0} x^2\, d\bar{F}_n(x), \qquad b_n^- = \int_{-\infty}^{0} x^2\, dF_n(x),$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\, du, \qquad \Phi(x, m, \sigma) = \Phi\left(\frac{x - m}{\sigma}\right).$$

Lemma 1.

$$k\bar{a}_k = -\sqrt{\frac{k}{2\pi}}\, e^{-ka^2/2} + ak\Phi(-a\sqrt{k}) + O(c_3).$$

PROOF. By (0.5),

$$\Psi_t'(t, z) = -\sum_{k=1}^{\infty} \psi_k'(t) \frac{z^k}{k} \exp\left\{\sum_{k=1}^{\infty} \frac{z^k}{k} \int_{-\infty}^{0} (1 - e^{itx})\, dF_k(x)\right\}. \tag{1.1}$$

Hence

$$\Psi_t'(0, z) = -i \sum_{k=1}^{\infty} a_k^- \frac{z^k}{k}. \tag{1.2}$$

Clearly,

$$a_k^- = \frac{1}{\sqrt{2\pi k}} \int_{-\infty}^{0} x\, e^{-(x-ak)^2/k}\, dx + \int_{-\infty}^{0} x\, d(F_k(x) - \Phi(x, ka, k)). \tag{1.3}$$

Further,

$$F_k(x\sqrt{k} + ak) - \Phi(x) = O\left(\frac{c_3}{(1 + |x|^3)\sqrt{k}}\right) \tag{1.4}$$

(see [5], Theorem 1). Therefore,

$$\int_{-\infty}^{0} x\, d(F_k(x) - \Phi(x, ka, k)) = \int_{-\infty}^{0} (\Phi(x, ka, k) - F_k(x))\, dx$$

(1.5)
$$= \sqrt{k}\int_{-\infty}^{-a\sqrt{k}} (\Phi(x) - F_k(x\sqrt{k} + ak))\, dx = O(c_3).$$

Clearly,

$$\text{(1.6)}\qquad \frac{1}{\sqrt{2\pi k}}\int_{-\infty}^{0} x\, e^{-(x-ak)^2/2k}\, dx = -\sqrt{\frac{k}{2\pi}}\, e^{-ka^2/2} + ak\Phi(-a\sqrt{k}).$$

By virtue of (0.2), $\Psi_t'(0, z)$ is the generating function for $\bar{a}_n$. Therefore (1.2) leads to

$$\text{(1.7)}\qquad \bar{a}_k = \frac{a_k^-}{k}.$$

The assertion of the lemma easily results from (1.3) and (1.5)–(1.7).

Lemma 2. *If* $a = O(1)$, *then*

$$\bar{a}_k + \frac{1}{\sqrt{2\pi k}}\, e^{-ka^2/2} - a\Phi(-a\sqrt{k}) = r_{1k} + r_{2k},$$

where

$$\sum_{k=1}^{\infty} |r_{1k}| = O(c_3^2)\cdot \qquad r_{2k} = O\left(\frac{c_3|a|}{\sqrt{k}}\right).$$

Proof. Clearly,

$$\text{(1.8)}\qquad \int_{-\infty}^{\infty} e^{itx} x\, dF_k(x) = -ikf'(t)f^{k-1}(t).$$

On the other hand,

$$\int_{-\infty}^{\infty} e^{itx} x\, d(F_k(x) - \Phi(x, ka, k))$$

(1.9)
$$= it\int_{-\infty}^{\infty} e^{itx}\, d_x \int_{-\infty}^{x} y\, d_y(\Phi(y, ka, k) - F_k(y)).$$

Using the inversion formula and then integration by parts, we find from (1.8) and (1.9) that

$$\frac{a_k^-}{k} - \frac{1}{k}\int_{-\infty}^{0} x\, d\Phi(x, ka, k)$$

$$= \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T} (f'(t)f^{k-1}(t) + (t - ai)\, e^{iakt - kt^2/2})\frac{dt}{t}$$

(1.10)
$$= \frac{1}{2\pi}\int_{-\delta}^{\delta} (f'(t)f^{k-1}(t) + (t - ai)\, e^{iakt - kt^2/2})\frac{dt}{t} + \frac{1}{\pi k}\int_{\delta}^{\infty} \frac{\operatorname{Re} f^n(t)}{t^2}\, dt$$

$$- \frac{\operatorname{Re} f^k(\delta)}{k\delta} - \frac{1}{\pi k}\int_{\delta}^{\infty} \frac{\cos kat}{t^2}\, e^{-kt^2/2}\, dt + \frac{\cos ka\delta}{k\delta}\, e^{-k\delta^2/2}.$$

Let us first consider the integral

$$J_{1k}(\delta) = \int_{-\delta}^{\delta} \frac{f'(t)f^{k-1}(t) + (t - ai)\, e^{iakt - kt^2/2}}{t}\, dt.$$

First of all,

$$f'(t) = ai - t + it^2(R(t) + iI(t)),$$

where

$$R(t) = \frac{1}{t^2}\int_{-\infty}^{\infty} x(\cos tx - 1)\, dF(x),$$

$$I(t) = \frac{1}{t^2}\int_{-\infty}^{\infty} x(\sin tx - tx)\, dF(x).$$

Therefore,

$$\begin{aligned} J_{1k}(\delta) = &\int_{-\delta}^{\delta} e^{iat - t^2/2}\left(\exp\left\{i(k-1)at - \frac{(k-1)t^2}{2}\right\} - f^{k-1}(t)\right)\frac{t - ai}{t}\, dt \\ &+ \int_{-\delta}^{\delta} (e^{iat - t^2/2} - 1) f^{k-1}(t)\frac{t - ai}{t}\, dt \qquad (1.11) \\ &+ i\int_{-\delta}^{\delta} t f^{k-1}(t)(R(t) + iI(t))\, dt. \end{aligned}$$

Further,

$$\int_{-\delta}^{\delta} t f^{k}(t)(R(t) + iI(t))\, dt = d_{1k} + d_{2k}, \qquad (1.12)$$

where

$$d_{1k} = \int_{-\delta}^{\delta} t(f^k(t) - e^{ikat - kt^2/2})(R(t) + iI(t))\, dt + \int_{-\delta}^{\delta} tI(t)\, e^{iakt - kt^2/2}\, dt,$$

$$d_{2k} = \int_{-\delta}^{\delta} tR(t)\, e^{iat - kt^2/2}\, dt.$$

As we know (see, for example, [9], Sec. 40, Theorem 2),

$$|f^k(t)\, e^{-ikat} - e^{-kt^2/2}| < \tfrac{7}{6} c_3 k |t|^3\, e^{-kt^2/4} \qquad (1.13)$$

for $|t| \leqq 1/5c_3$. On the other hand,

$$|R(t)| + |I(t)| = O(c_3). \qquad (1.14)$$

Therefore,

$$\begin{aligned} &\int_{-\delta}^{\delta} t(f^k(t) - e^{iakt - kt^2/2})(R(t) + iI(t))\, dt \qquad (1.15) \\ &= O\left(c_3^2 k \int_0^{\delta} t^4\, e^{-kt^2/4}\, dt\right) = O\left(\frac{c_3^2}{k^{3/2}}\right) \end{aligned}$$

for $\delta < 1/5c_3$. Clearly,

$$\int_0^1 \frac{|I(t)|}{t}\, dt \leqq \int_{-\infty}^{\infty} |x|\, dF(x) \int_0^1 \frac{|\sin xt - xt|}{t^3}\, dt.$$

At the same time,

$$\int_0^1 \frac{|\sin xt - xt|}{t^3}\, dt = x^2 \int_0^x \frac{|\sin u - u|}{u^3}\, du < x^2 \int_0^\infty \frac{|\sin u - u|}{u^3}\, du.$$

Therefore,

$$\int_0^1 \frac{|I(t)|}{t}\, dt = O(c_3). \tag{1.16}$$

Using (1.16), we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} \left| \int_{-\delta}^{\delta} tI(t)\, e^{ikat - t^2/2}\, dt \right| &= O\left(\int_0^\delta \frac{t|I(t)|}{1 - e^{-t^2/2}}\, dt \right) \\ &= O\left(\int_0^\delta \frac{|I(t)|}{t}\, dt \right) = O(c_3), \qquad \delta \leqq 1. \end{aligned} \tag{1.17}$$

From (1.15) and (1.17) it follows that

$$\sum_{k=1}^{\infty} |d_{1k}| = O(c_3^2). \tag{1.18}$$

By virtue of (1.14) and the evenness of $R(t)$,

$$\begin{aligned} d_{2k} &= O\left(\left| \int_{-\delta}^{\delta} tR(t)\, e^{iakt - t^2/2}\, dt \right| \right) \\ &= O\left(c_3 |a| k \int_0^\delta t^2\, e^{-kt^2/2}\, dt \right) = O\left(\frac{c_3|a|}{\sqrt{k}} \right). \end{aligned} \tag{1.19}$$

Clearly,

$$f^k(t) - e^{iakt - kt^2/2} = (f(t) - e^{iat - t^2/2}) \sum_{m=0}^{k-1} f^{k-m}(t)\, e^{iamt - mt^2/2}. \tag{1.20}$$

Further,

$$f(t) - e^{iat - t^2/2} = t^4 R_1(t) - it^3 I_1(t) - e^{iat - t^2/2} + 1 + iat - \frac{t^2}{2}, \tag{1.21}$$

where

$$R_1(t) = \frac{1}{t^4} \int_{-\infty}^{\infty} \left(\cos tx - 1 + \frac{t^2x^2}{2} \right) dF(x),$$

$$I_1(t) = \frac{1}{t^3} \int_{-\infty}^{\infty} (\sin tx - tx)\, dF(x).$$

Therefore,

$$(f(t) - e^{iat - t^2/2})\, e^{iamt - mt^2/2} f^{k-m}(t)$$

$$(1.22)\qquad = (t^4 R_1(t) - it^3 I_1(t))\, e^{iakt - kt^2/2} + \left(1 + iat - \frac{t^2}{2} - e^{iat - t^2/2}\right) e^{iakt - kt^2/2}$$
$$+ (f(t) - e^{iat - t^2/2})(f^{k-m}(t) - e^{(k-m)(iat - t^2/2)})\, e^{iamt - mt^2/2}.$$

Clearly,

$$\int_0^1 |R_1(t)|\, dt \leqq \int_{-\infty}^{\infty} dF(x) \int_0^1 t^{-4} \left|\cos tx - 1 + \frac{t^2x^2}{2}\right| dt = O(c_3).$$

Hence

$$(1.23)\qquad \begin{aligned} &\sum_{k=0}^{\infty} k \int_{-\delta}^{\delta} t^4 |R_1(t)|\, e^{-kt^2/2}\, dt \\ &= O\left(\int_0^{\delta} R_1(t) t^4 (1 - e^{-t^2/2})^{-2}\, dt\right) = O(c_3), \qquad \delta \leqq 1. \end{aligned}$$

Further,

$$(1.24)\qquad I_1(t) = I_1(-t).$$

Thus

$$(1.25)\qquad \int_{-\delta}^{\delta} t^3 I_1(t)\, e^{iakt - kt^2/2}\, dt = O\left(|a| c_3 k \int_0^{\delta} t^4\, e^{-kt^2/2}\, dt\right) = O\left(\frac{|a| c_3}{k^{3/2}}\right).$$

Clearly,

$$e^{iat - t^2/2} - iat + \frac{t^2}{2} = O(|a| t^3 + t^4).$$

Therefore,

$$(1.26)\qquad \begin{aligned} &\int_{-\delta}^{\delta} \left(1 + iat - \frac{t^2}{2} - e^{iat - t^2/2}\right) e^{ikat - kt^2/2}\, dt \\ &= O\left(\int_0^{\delta} (|a| t^3 + t^4)\, e^{-kt^2/2}\, dt\right) = O\left(\frac{|a|}{k^2} + \frac{1}{k^{5/2}}\right). \end{aligned}$$

Finally by virtue of (1.13),

$$(1.27)\qquad \begin{aligned} &\sum_{m=0}^{k} \int_{-\delta}^{\delta} (f(t) - e^{iat - t^2/2})(f^{k-m}(t) - e^{i(k-m)at + (m-k)t^2/2})\, e^{iamt - mt^2/2}\, dt \\ &= O\left(c_3^2 \sum_{m=0}^{k} (k - m) \int_0^{\delta} t^6\, e^{-kt^2/4}\, dt\right) = O\left(\frac{c_3^2}{k^{3/2}}\right). \end{aligned}$$

From (1.20), (1.22), (1.23) and (1.25)–(1.27) it follows that

$$(1.28)\qquad \int_{-\delta}^{\delta} (e^{ikat - kt^2/2} - f^k(t))\, e^{iat - t^2/2}\, dt = c_{1k} + c_{2k},$$

and in addition,

$$\sum_{k=1}^{\infty} |c_{1k}| = O(c_3^2), \qquad c_{2k} = O\left(\frac{|a|c_3}{\sqrt{k}}\right).$$

By (1.13),

$$(1.29) \quad \int_{-\delta}^{\delta} e^{iat-t^2/2}(e^{iakt-kt^2/2} - f^k(t))\frac{dt}{t} = O\left(c_3 k \int_0^{\delta} t^2 e^{-kt^2/4}\,dt\right) = O\left(\frac{c_3}{\sqrt{k}}\right).$$

For $|t| \leqq 3/2c_3$,

$$\left| f(t)\,e^{-iat} - 1 + \frac{t^2}{2} \right| < \frac{t^2}{4}$$

and hence,

$$(1.30) \qquad |f(t)| \leqq e^{-t^2/4}.$$

Therefore,

$$(1.31) \int_{-\delta}^{\delta} (e^{iat-t^2/2} - 1)f^k(t)\,dt = O\left(\int_0^{\delta}\left(|a|t + \frac{t^2}{2}\right) e^{-kt^2/4}\,dt\right) = O\left(\frac{|a|}{k} + \frac{1}{k^{3/2}}\right).$$

In a similar way,

$$(1.32) \qquad \int_{-\delta}^{\delta} (e^{iat-t^2/2} - 1)\frac{f^k(t)}{t}\,dt = O\left(\frac{|a|}{\sqrt{k}} + \frac{1}{k}\right).$$

From (1.11), (1.12), (1.18), (1.19), (1.28), (1.29), (1.31) and (1.32) it follows that

$$(1.33) \qquad J_{1k}(\delta) = l_{1k} + l_{2k}, \qquad \delta < \frac{1}{5c_3},$$

where

$$\sum_{k=1}^{\infty} |l_{1k}| = O(c_3^2), \qquad l_{2k} = O\left(\frac{c_3|a|}{\sqrt{k}}\right).$$

We can now estimate the integral

$$J_{2k}(\delta) = \int_{\delta}^{\infty} \frac{\operatorname{Re} f^k(t)}{t^2}\,dt.$$

Clearly,

$$|J_{2k}(\delta)| < \int_{-\infty}^{\infty} dF_k(x)\left|\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt\right|.$$

For $x > 0$,

$$\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt = x\int_{\delta x}^{\infty} \frac{\cos t}{t^2}\,dt.$$

Further,

$$\int_{\delta x}^{\infty} \frac{\cos t}{t^2}\,dt = \int_{\delta x}^{k(\delta x)\pi} \frac{\cos t}{t^2}\,dt + \int_{k(\delta x)\pi}^{\infty} \frac{\cos t}{t^2}\,dt,$$

where

$$k(\delta x) = \left[\frac{\delta x}{\pi}\right] + 1.$$

It is not hard to see that

$$\left|\int_{k(\delta x)\pi}^{\infty} \frac{\cos t}{t^2}\,dt\right| < \int_{k(\delta x)\pi}^{(k(\delta x)+1)\pi} \frac{\cos t}{t^2}\,dt < \frac{\pi}{\delta^2 x^2}.$$

But this same estimate is also the value for

$$\int_{\delta x}^{k(\delta x)} \frac{\cos t}{t^2}\,dt.$$

Moreover, the integrals

$$\int_{k(\delta x)\pi}^{\infty} \frac{\cos t}{t^2}\,dt \quad \text{and} \quad \int_{\delta x}^{k(\delta x)\pi} \frac{\cos t}{t^2}\,dt$$

have opposite signs.

Therefore,

$$\left|\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt\right| < \frac{\pi}{|x|\delta^2}.$$

On the other hand,

$$\left|\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt\right| < \int_{\delta}^{\infty} \frac{dt}{t^2} = \frac{1}{\delta}.$$

Using these two estimates we obtain

$$|J_{2k}(\delta)| < \frac{1}{\delta}\int_{|x|<k^{1/4}c_3^{1/2}} dF_k(x) + \frac{\pi}{\delta^2}\int_{|x|\geqq k^{1/4}c_3^{1/2}} \frac{1}{|x|}\,dF_k(x). \tag{1.34}$$

Applying the well-known estimate of Esseen for the remainder in the central limit theorem (see, for example, [9], Sec. 40), we obtain

$$\begin{aligned}\int_{|x|<k^{1/4}c_3^{1/2}} dF_k(x) &= \Phi(c_3^{1/2}k^{-1/4} - a\sqrt{k}) \\ &\quad - \Phi(-c_3^{1/2}k^{-1/4} - a\sqrt{k}) + O\left(\frac{c_3}{\sqrt{k}}\right) \\ &= O\left(\frac{c_3^{1/2}}{k^{1/4}}\right) + O\left(\frac{c_3}{\sqrt{k}}\right).\end{aligned} \tag{1.35}$$

From (1.34) and (1.35) it is easy to deduce the estimate

$$(1.36)\qquad J_{2k}\left(\frac{1}{5c_3}\right) = O(c_3^{3/2}k^{-1/4}).$$

Finally, by virtue of (1.31),

$$(1.37)\qquad \sum_{k=1}^{\infty}\frac{1}{k}\left|f^k\left(\frac{1}{5c_3}\right)\right| = -\log\left(1-\left|f\left(\frac{1}{5c_3}\right)\right|\right) = O(\log c_3).$$

The assertion of the lemma follows from (1.10), (1.6), (1.33), (1.36) and (1.37).

Lemma 3. *If* $a = O(1)$, *then*

$$\bar{b}_k = O\left(c_3^2\left(\frac{1}{\sqrt{k}} + |a|\right)\right).$$

PROOF. First of all,

$$\Psi_t''(t,z) = \left[\left(\sum_{k=1}^{\infty}\psi_k'(t)\frac{z^k}{k}\right)^2 - \sum_{k=1}^{\infty}\psi_k''(t)\frac{z^k}{k}\right]\exp\left\{\sum_{k=1}^{\infty}\frac{z^k}{k}\int_{-\infty}^{0}(1-e^{itx})\,dF_k(x)\right\}.$$

Therefore,

$$(1.38)\qquad \Psi_t''(0,z) = \sum_{k=1}^{\infty}\frac{b_k^-}{k}z^k - \left(\sum_{k=1}^{\infty}\frac{a_k^-}{k}z^k\right)^2.$$

Clearly,

$$b_k^- = \frac{1}{\sqrt{2\pi k}}\int_{-\infty}^{0}x^2\,e^{-(x-ka)^2/2}\,dx + \int_{-\infty}^{0}x^2\,d[F_k(x) - \Phi(x,ka,k)].$$

Further,

$$\frac{1}{\sqrt{2\pi k}}\int_{-\infty}^{0}x^2\,e^{-(x-ka)^2/2k}\,dx = \frac{k}{\sqrt{2\pi}}\int_{-\infty}^{-\sqrt{k}a}(x+\sqrt{k}a)^2\,e^{-k^2/2}\,dx$$
$$= \frac{k}{\sqrt{2\pi}}\left(\int_{-\infty}^{-\sqrt{k}a}x^2\,e^{-x^2/2}\,dx - 2\sqrt{k}a\,e^{-ka^2/2} + ka^2\sqrt{2\pi}\Phi(-\sqrt{k}a)\right),$$

$$\int_{-\infty}^{-\sqrt{k}a}x^2\,e^{-x^2/2}\,dx = \sqrt{k}a\,e^{-ka^2/2} + \sqrt{2\pi}\Phi(-\sqrt{k}a).$$

Therefore,

$$\frac{1}{\sqrt{2\pi k}}\int_{-\infty}^{0}x^2\,e^{-(x-ak)^2/2k}\,dx = (k^2a^2+k)\Phi(-a\sqrt{k}) - \frac{ak^{3/2}}{\sqrt{2\pi}}e^{-ka^2/2}.$$

Using the estimate (1.14), we obtain

$$\frac{1}{2}\int_{-\infty}^{0} x^2\, d[F_k(x) - \Phi(x, ka, k)] = \int_{-\infty}^{0} x[\Phi(x, ak, k) - F_k(x)]\, dx$$
$$= \sqrt{k}\int_{-\infty}^{-a\sqrt{k}} (x\sqrt{k} + ak)[\Phi(x)$$
$$- F_k(x\sqrt{k} + ak)]\, dx$$
$$= O(c_3(\sqrt{k} + |a|k)).$$

Thus

$$(1.39)\quad b_k^- = (k^2a^2 + k)\Phi(-a\sqrt{k}) - \frac{ak^{3/2}}{\sqrt{2\pi}} e^{-ka^2/2} + O(c_3(\sqrt{k} + |a|k)).$$

By (1.7), the coefficient of z^k in the expansion of $(\sum_{j=1}^{\infty}(a_j^-/j)z^j)^2$ is $\sum_{j=1}^{k-1}\bar{a}_j\bar{a}_{k-j}$. Set

$$\bar{a}_k^0 = a\Phi(-a\sqrt{k}) - \frac{1}{\sqrt{2\pi k}} e^{-ka^2/2}, \quad r_k = \bar{a}_k - \bar{a}_k^0.$$

Then

$$(1.40)\qquad \sum_{j=1}^{k-1}\bar{a}_j\bar{a}_{k-j} = \sum_{j=1}^{k-1}\bar{a}_j^0\bar{a}_{k-j}^0 + 2\sum_{j=1}^{k-1}\bar{a}_j^0 r_{k-j} + \sum_{j=1}^{k-1} r_j r_{k-j}.$$

Clearly,

$$(1.41)\qquad \bar{a}_j^0 = O\left(\frac{1}{\sqrt{j}} + |a|\right).$$

At the same time, by Lemma 1,

$$(1.42)\qquad r_k = O\left(\frac{c_3}{k}\right).$$

Therefore,

$$\sum_{j=1}^{k/2}\bar{a}_j^0 r_{k-j} = O\left(\frac{c_3}{\sqrt{k}} + |a|c_3\right).$$

On the other hand, Lemma 2 and (1.41) imply that

$$\sum_{j=k/2}^{k-1}\bar{a}_j^0 r_{k-j} = O\left(c_3^2\left(\frac{1}{\sqrt{k}} + |a|\right)\right).$$

Thus

$$(1.43)\qquad \sum_{j=1}^{k-1}\bar{a}_j^0 r_{k-j} = O\left(c_3^2\left(\frac{1}{\sqrt{k}} + |a|\right)\right).$$

By (1.42),

$$\sum_{j=1}^{k-1} r_j r_{k-j} = O\left(\frac{c_3^2 \log k}{k}\right). \tag{1.44}$$

Let us now consider the sum

$$\sum_{j=1}^{k-1} \bar{a}_j^0 \bar{a}_{k-j}^0.$$

Applying Euler's summation formula with remainder (see, for example, [10], p. 297, (66)),[T-1] we obtain

$$\begin{aligned}\sum_{j=1}^{k-2} \bar{a}_j^0 \bar{a}_{k-j}^0 &= \int_1^{k-1} \varphi(t)\,dt + \varphi(k-1) - \varphi(1)\\ &\quad + O\left(\sum_{j=1}^{k-2} \int_0^1 \varphi'(j+1-t)\,dt\right),\end{aligned} \tag{1.45}$$

where

$$\varphi(t) = \varphi_1(t)\varphi_1(k-t), \qquad \varphi_1(t) = a\Phi(-a\sqrt{t}) - \frac{1}{\sqrt{2\pi t}} e^{-ta^2/2}.$$

Clearly,

$$\varphi_1'(t) = \frac{1}{2\sqrt{2\pi} t^{3/2}} e^{-ta^2/2}.$$

Hence,

$$\begin{aligned}&\int_1^{k-1} \varphi_1(t)\varphi_1'(k-t)\,dt\\ &\quad = O\left(e^{-ka^2/2} \int_1^{k-1} t^{-1/2}(k-t)^{-3/2}\,dt\right) + O\left(|a| \int_1^{k-1} t^{-3/2} e^{-ta^2/2}\,dt\right)\\ &\quad = O(k^{-1/2} e^{-ka^2/2} + |a|\, e^{-a^2/2}).\end{aligned}$$

This estimate is clearly valid also for

$$\int_1^{k-1} \varphi_1'(t)\varphi_1(k-t)\,dt.$$

Thus

$$\sum_{j=1}^{k-2} \int_0^1 \varphi'(j+1-t)\,dt = \int_1^{k-1} \varphi'(t)\,dt = O(k^{-1/2} e^{-ka^2/2} + |a|\, e^{-a^2/2}). \tag{1.46}$$

[T-1] Reference [10] was omitted in the listed references in the Russian original.

Clearly,

$$\varphi(kt) = a^2\Phi(-a\sqrt{kt})\Phi(-a\sqrt{k(1-t)}) - \frac{a}{\sqrt{2\pi kt}} e^{-ka^2t/2}\Phi(-a\sqrt{k(1-t)})$$
$$- \frac{a}{\sqrt{2\pi k(1-t)}} e^{-ka^2(1-t)/2}\Phi(-a\sqrt{kt}) + \frac{1}{2\pi k\sqrt{t(1-t)}} e^{-ka^2/2}.$$

Therefore,

$$\int_1^{k-1} \varphi(t)\,dt = k\int_{1/k}^{1-1/k} \varphi(kt)\,dt$$
(1.47)
$$= k\int_0^1 \varphi(kt)\,dt + O(k^{-1/2}e^{-ka^2/2} + |a| + a^2).$$

It is not hard to see that

(1.48)
$$k\int_0^1 \varphi(kt)\,dt = I_1(a\sqrt{k}),$$

where

$$I_1(x) = \frac{1}{2\pi}e^{-x^2/2}\int_0^1 t^{-1/2}(1-t)^{-1/2}\,dt$$
$$- \sqrt{\frac{2}{\pi}}x\, e^{-x^2/2}\int_0^1 \Phi(-x\sqrt{t})\, e^{tx^2/2}(1-t)^{-1/2}\,dt$$
$$+ x^2\int_0^1 \Phi(-x\sqrt{t})\Phi(-x\sqrt{1-t})\,dt.$$

From (1.40) and (1.43)–(1.48) it follows that

(1.49)
$$\sum_{j=1}^{k-1} \bar{a}_j\bar{a}_{k-j} = I_1(a\sqrt{k}) + O(c_3^2(k^{-1/2} + |a|)),$$

since $a^2 = O(|a|)$ for $a = O(1)$. Comparing (1.38), (1.39) and (1.49), we find

(1.50)
$$\bar{b}_k = -I_1(a\sqrt{k}) + I_2(a\sqrt{k}) + O(c_3^2(k^{-1/2} + |a|)),$$

where

$$I_2(x) = (x^2+1)\Phi(-x) - \frac{x}{\sqrt{2\pi}}e^{-x^2/2}.$$

We now show that

$$\lim_{k\to\infty} \bar{b}_k = 0$$

if $a = d/\sqrt{k}$ and $c_3^a = O(1)$, where d is a constant.

Indeed,

$$\bar{b}_k = \int_{-M}^{0} x^2\, d\bar{F}_k(x) + \int_{-\infty}^{-M} x^2\, d\bar{F}_k(x), \qquad M > 0.$$

Clearly, $\bar{F}_k(x) \leqq F(x)$. Therefore

$$\int_{-\infty}^{-M} x^2 \, d\bar{F}_k(x) \leqq \int_{-\infty}^{-M} x^2 \, dF(x).$$

On the other hand,

$$\int_{-M}^{0} x^2 \, d\bar{F}_k(x) < -M \int_{-\infty}^{0} x \, d\bar{F}_k(x) = -M\bar{a}_k.$$

From the last two estimates and Lemma 1 it follows that

$$\lim_{k\to\infty} \bar{b}_k = 0.$$

On the other hand,

$$\lim_{k\to\infty} \bar{b}_k = -I_1(d) + I_2(d).$$

Hence

$$-I_1(d) + I_2(d) = 0. \tag{1.51}$$

The assertion of the lemma follows from (1.50) and (1.51).

Lemma 4.

$$\bar{a}_n = O\left(\frac{1}{\sqrt{n}} + |a|\right).$$

PROOF. Clearly,

$$|a_n^-| \leqq \left(\int_{-\infty}^{\infty} x^2 \, dF_n(x)\right)^{1/2} = \sqrt{n}(1 + a^2 n)^{1/2}.$$

The assertion of the lemma now results from (1.7) and the inequality

$$(1 + y)^{1/2} < 1 + \sqrt{y}, \qquad y > 0.$$

Let

$$H(x) = \frac{3}{8\pi}\int_{-\infty}^{x} \left(\frac{\sin u/4}{u/4}\right)^4 du, \qquad h(t) = \int_{-\infty}^{\infty} e^{itx} \, dH(x).$$

Observe that $h(t) = 0$ for $|t| \geqq 1$.

Lemma 5.

$$\bar{F}_n(x + l) - \bar{F}_n(x) = O\left((c_3 l + c_3^2)\left(|a| + \frac{1}{\sqrt{n}}\right)\right), \qquad l > 0.$$

PROOF. Using (0.8), we easily show that

$$\begin{aligned} \Delta_n(x, l) &\equiv \int_{-\infty}^{\infty} [\bar{F}_n(x + l - y) - \bar{F}_n(x - y)] \, dH\left(\frac{y}{5c_3}\right) \\ &= \frac{1}{2\pi}\sum_{k=1}^{n}\int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-it(x+l)}}{it} \, h(5c_3 t) f^k(t)\bar{\varphi}_{n-k}(t) \, dt. \end{aligned} \tag{1.52}$$

Clearly,

$$\bar{\varphi}_n(t) = -it\bar{a}_n + O(\bar{b}_n t^2), \qquad n \geqq 1. \tag{1.53}$$

By means of (1.53) and (1.30), equation (1.52) can be rewritten in the following form:

$$\begin{aligned} \Delta_n(x, l) &= \frac{1}{2\pi} \sum_{k=1}^{n-1} \bar{a}_{n-k} \int_{|t| \geqq 1/5c_3} (e^{-it(x+l)} - e^{-itx}) f^k(t)\, dt \\ &+ \frac{1}{2\pi} \int_{|t| \leqq 1/5c_3} \frac{e^{-itx} - e^{-it(x-l)}}{it} f^n(t)\, dt \\ &+ O\left(l \sum_{k=1}^{n-1} \bar{b}_{n-k} \int_{|t| \leqq 1/5c_3} t^2 e^{-kt^2/4}\, dt \right). \end{aligned} \tag{1.54}$$

By virtue of (1.13),

$$\begin{aligned} &\int_{|t| \leqq 1/5c_3} (e^{-itx} - e^{-it(x+l)}) f^k(t) h(5c_3 t)\, dt \\ &= \int_{|t| \leqq 1/5c_3} (e^{-itx} - e^{-it(x+l)})\, e^{iakt - kt^2/2} h(5c_3 t)\, dt \\ &+ O\left(klc_3 \int_{|t| \leqq 1/5c_3} t^4 e^{-kt^2/4}\, dt \right). \end{aligned} \tag{1.55}$$

Clearly,

$$\int_0^\infty t^m e^{-kt^2/4}\, dt = O(k^{-(m+1)/2}). \tag{1.56}$$

Further,

$$\begin{aligned} I_k &\equiv \int_{-\infty}^{\infty} (e^{-itx} - e^{-it(x+l)})\, e^{-kt^2/2 + iakt} h(5c_3 t)\, dt \\ &= O\left(l \int_0^\infty t\, e^{-kt^2/2}\, dt \right) = O\left(\frac{l}{k} \right). \end{aligned} \tag{1.57}$$

On the other hand,

$$I_k = O\left(\int_0^\infty e^{-kt^2/2}\, dt \right) = O\left(\frac{1}{\sqrt{k}} \right). \tag{1.58}$$

Using the expansions

$$h(5c_3 t) = 1 + O(c_3^2 t^2)$$

and

$$e^{-itx} - e^{-it(x+l)} = itl + O(t^2 l^2),$$

we obtain

$$I_k = -l\int_{-\infty}^{\infty} t\,(\sin akt)\,e^{-kt^2/2}\,dt + O\left(c_3^2 l\int_0^{\infty} t^3\,e^{-kt^2/2}\,dt\right) + O\left(l^2\int_0^{\infty} t^2\,e^{-kt^2/2}\,dt\right).$$

Further,

$$\int_{-\infty}^{\infty} t(\sin akt)\,e^{-kt^2/2}\,dt = \sqrt{2\pi}ak^{-1/2}\,e^{-ka^2/2}.$$

Therefore, taking (1.56) into account, we obtain

$$I_k = O(|a|lk^{-1/2}\,e^{-ka^2/2} + c_3^2 lk^{-2} + l^2k^{-3/2}). \tag{1.59}$$

From (1.57)–(1.59) it follows that

$$\sum_{k=1}^{n-1} |I_k| = O\left(|a|l\sum_{k=1}^{n-1} k^{-1/2}\,e^{-ka^2/2}\right) + O\left(\sum_{k=1}^{[l^2]}\frac{1}{\sqrt{k}} + l^2\sum_{[l^2]}^{n/2}\frac{1}{k^{3/2}} + l\sum_{n/2}^{n}\frac{1}{k} + c_3^2 l\sum_{[l^2]}^{n/2}\frac{1}{k^2}\right)$$

since there is no loss of generality in assuming that $l < \sqrt{n/2}$. Further,

$$\sum_{k=1}^{n-1} k^{-1/2}\,e^{-ka^2/2} < \sqrt{n}\int_0^1 u^{-1/2}\,e^{-na^2u/2}\,du = \frac{1}{|a|}\int_0^{na^2}\frac{e^{-u/2}}{\sqrt{u}}\,du = \frac{2\sqrt{2}}{|a|}\int_0^{|a|\sqrt{n/2}} e^{-u^2}\,du.$$

Clearly,

$$\sum_{[l^2]}^{n/2}\frac{1}{k^2} = O(l^{-2}),\qquad \sum_{k=1}^{[l^2]}\frac{1}{\sqrt{k}} = O(l),$$

$$\sum_{[l^2]}^{n/2}\frac{1}{k^{3/2}} = O\left(\frac{1}{l}\right),\qquad \sum_{n/2}^{n}\frac{1}{k} = O(1).$$

Thus

$$\sum_{k=1}^{n-1} |I_k| = O(c_3^2 l^{-1} + l) = O(l) \tag{1.60}$$

since there is no loss of generality in assuming that $l \geqq c_3$. Further,

$$I_k = l\int_{-\infty}^{\infty} it\,e^{-kt^2/2+iakt}h(5c_3t)\,dt + O\left(l^2\int_0^{\infty} t^2\,e^{-kt^2/2}\,dt\right) = O\left((|a|kl + l^2)\int_0^{\infty} t^2\,e^{-kt^2/2}\,dt\right) = O\left(\frac{|a|l}{k^{1/2}}\right) + O\left(\frac{l^2}{k^{3/2}}\right),$$

since

$$\int_{-\infty}^{\infty} t\, e^{-kt^2/2} h(5c_3 t)\, dt = 0.$$

This estimate together with (1.57) and (1.58) implies that

$$\begin{aligned}\sum_{k=1}^{n-1} \frac{|I_k|}{\sqrt{n-k}} &= O\left(|a| l \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(n-k)}}\right) + O\left(\frac{1}{\sqrt{n}} \sum_{k=1}^{[l^2]} \frac{1}{\sqrt{k}}\right. \\ &\quad \left. + \frac{l^2}{\sqrt{n}} \sum_{[l^2]}^{n/2} \frac{1}{k^{3/2}} + l \sum_{n/2}^{n-1} \frac{1}{k(n-k)^{1/2}}\right) \\ &= O\left(|a| l + \frac{l}{n^{1/2}}\right).\end{aligned} \tag{1.61}$$

Applying Lemma 4 to (1.55) and (1.56), we obtain

$$\begin{aligned}&\sum_{k=1}^{n-1} \bar{a}_{n-k} \int_{-\infty}^{\infty} (e^{-itx} - e^{-it(x+l)}) f^k(t) h(5c_3 t)\, dt \\ &\quad = O\left(\sum_{k=1}^{n-1} |I_k| \left(|a| + \frac{1}{\sqrt{n-k}}\right)\right. \\ &\quad\quad + l c_3 \sum_{k=1}^{n/2} k \int_{|t| \leq 1/5c_3} t^4 e^{-kt^2/4}\, dt \left(\frac{1}{\sqrt{n}} + |a|\right) \\ &\quad\quad \left. + l c_3 \sum_{k=n/2}^{n} \left(\frac{1}{\sqrt{n-k}} + |a|\right) k^{-3/2}\right).\end{aligned} \tag{1.62}$$

Observe that

$$\sum_{k=n/2}^{n-1} k^{-3/2} = O(n^{-1/2})$$

and

$$\sum_{k=n/2}^{n-1} k^{-3/2}(n-k)^{-1/2} = O(n^{-1}).$$

In addition, there is no loss of generality in assuming that $c_3^2 < \sqrt{n}$. Therefore,

$$\sum_{k=n/2}^{n-1} k^{-3/2} = O\left(\frac{1}{c_3^2}\right), \qquad \sum_{k=n/2}^{n-1} k^{-3/2}(n-k)^{-1/2} = O\left(\frac{1}{c_3^2 \sqrt{n}}\right). \tag{1.63}$$

Further,

$$\begin{aligned}\sum_{k=1}^{n} k \int_{|t| \leq 1/5c_3} t^4 e^{-kt^2/4}\, dt &= O\left(\int_{|t| \leq 1/5c_3} \frac{t^4}{(1 - e^{-t^2/2})^2}\, dt\right) \\ &= O\left(\int_{|t| \leq 1/5c_3} dt\right) = O(1/c_3).\end{aligned}$$

Thus,

$$lc_3\left[\left(|a| + \frac{1}{\sqrt{n}}\right)\sum_{k=1}^{n/2} k \int_{|t|\leqq 1/5c_3} t^4\, e^{-kt^2/4}\, dt + \sum_{k=n/2}^{n-1}\left(\frac{1}{\sqrt{n-k}} + |a|\right)\frac{1}{k^{3/2}}\right]$$

(1.64)

$$= O\left(l\left(\frac{1}{\sqrt{n}} + |a|\right)\right).$$

Without loss of generality we may assume that $|a| \leqq 1$. Applying Lemma 3, we obtain

$$\sum_{k=0}^{n-1} \bar{b}_{n-k} \int_{|t|\leqq 1/5c_3} t^2\, e^{-kt^2/4}\, dt = O\left(c_3^2(|a| + n^{-1/2}) \sum_{k=0}^{n/2} \int_{|t|\leqq 1/5c_3} t^2\, e^{-kt^2/4}\, dt\right)$$

$$+ O\left(c_3^2\left(|a| \sum_{k=n/2}^{n} k^{-3/2} + \sum_{k=n/2}^{n} k^{-3/2}(n-k)^{-1/2}\right)\right).$$

Observe that

$$\sum_{k=0}^{n} \int_{|t|\leqq 1/5c_3} t^2\, e^{-kt^2/4}\, dt = O\left(\int_{|t|\leqq 1/5c_3} dt\right) = O(1/c_3).$$

Taking this estimate as well as estimate (1.63) into account, we obtain

$$l \sum_{k=0}^{n-1} \bar{b}_{n-k} \int_{|t|\leqq 1/5c_3} t^2\, e^{-kt^2/4}\, dt = O(c_3 l(|a| + n^{-1/2})). \tag{1.65}$$

Finally, by (1.30),

$$\int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-it(x+l)}}{it} f^n(t) h(5c_3 t)\, dt = O\left(l \int_0^{\infty} e^{-nt^2/4}\, dt\right) = O(l/\sqrt{n}).$$

The last estimate and (1.54), (1.62), (1.64) and (1.65) imply that

$$\Delta_n(x, l) = O(c_3 l(|a| + n^{-1/2})).$$

On the other hand,

$$\Delta_n(x, l) \geqq (\bar{F}_n(x + l - c_3) - \bar{F}_n(x + c_3)) \int_{-c_3}^{c_3} dH\left(\frac{y}{5c_3}\right).$$

Therefore,

$$\bar{F}_n(x + l) - \bar{F}_n(x) = O(\Delta_n(x, l + 2c_3)) = O\left((c_3 l + c_3^2)\left(|a| + \frac{1}{\sqrt{n}}\right)\right).$$

The lemma is proved.

Lemma 6.

$$\bar{F}_{n-m}(x) - \bar{F}_n(x) = O\left(\frac{m}{\sqrt{n}}\left(|a| + \frac{1}{\sqrt{n}}\right)\right) + O\left(c_3^2|a| + \frac{c_3^2}{\sqrt{n}}\right).$$

PROOF. Without loss of generality we may assume that $m < n/4$ and $|a| \leqq 1$. Taking (0.8) into account, it is not hard to see that

$$(1.66)\quad \begin{aligned}\Delta_{nm}(x) &\equiv \int_{-\infty}^{\infty} (\bar{F}_n(x-y) - \bar{F}_m(x-y))\, dH\left(\frac{y}{5c_3}\right) \\ &= -\int_{-\infty}^{\infty} \frac{e^{-itx}h(5c_3t)}{it}\left\{\sum_{k=1}^{n} f^k(t)\bar{\varphi}_{n-k}(t) - \sum_{k=1}^{n-m} f^k(t)\bar{\varphi}_{n-m-k}(t)\right\}.\end{aligned}$$

Clearly,

$$(1.67)\quad \begin{aligned}\sum_{k=1}^{n} f^k(t)\bar{\varphi}_{n-k}(t) - \sum_{k=1}^{n-m} f^k(t)\bar{\varphi}_{n-k-m}(t) &= \sum_{k=1}^{n/2-m} f^k(t)(\bar{\varphi}_{n-k}(t) - \bar{\varphi}_{n-m-k}(t)) \\ &+ \sum_{k=n/2+1}^{n} (f^k(t) - f^{k-m}(t))\bar{\varphi}_{n-k}(t) + \sum_{k=n/2-m+1}^{n/2} f^k(t)\bar{\varphi}_{n-k}(t).\end{aligned}$$

Further,

$$\begin{aligned}\bar{\varphi}_{k+j}(t) - \bar{\varphi}_j(t) &= \bar{F}_{k+j}(0) - \bar{F}_j(0) + \int_{-\infty}^{0-} e^{itx}\, d(\bar{F}_j(x) - \bar{F}_{k+j}(x)) \\ &= it\int_{-\infty}^{0-} (\bar{F}_{k+j}(x) - \bar{F}_j(x))\, e^{itx}\, dx.\end{aligned}$$

But

$$\bar{F}_{k+j}(x) \leqq \bar{F}_j(x).$$

Hence,

$$(1.68)\quad |\bar{\varphi}_{k+j}(t) - \bar{\varphi}_j(t)| < |t|\int_{-\infty}^{0-} (\bar{F}_j(x) - \bar{F}_{k+j}(x))\, dx = |t|(\bar{a}_{k+j} - \bar{a}_j).$$

By (1.13),

$$(1.69)\quad \begin{aligned}f^k(t) - f^{k-m}(t) &= e^{k(iat-t^2/2)} - e^{(k-m)(iat-t^2/2)} + O(c_3k|t|^3\, e^{-kt^2/4}) \\ &+ O(c_3(k-m)|t|^3\, e^{-(k-m)t^2/4}).\end{aligned}$$

Clearly,

$$(1.70)\quad \begin{aligned}e^{k(iat-t^2/2)} - e^{(k-m)(iat-t^2/2)} &= e^{(k-m)(iat-t^2/2)}(e^{-mt^2/2} - 1) \\ &+ e^{i(k-m)at-kt^2/2}(e^{iamt} - 1).\end{aligned}$$

Applying (1.30) and Lemma 4 we obtain

$$(1.71)\quad \begin{aligned}I_{km} &\equiv \int_{-\infty}^{\infty} \frac{f^k(t) - f^{k-m}(t)}{ti} e^{-itx}\bar{\varphi}_{n-k}(t)h(5c_3t)\, dt \\ &= \int_{|t| \leqq 1/5c_3} \frac{f^k(t) - f^{k-m}(t)}{it} e^{-itx}\bar{\varphi}_{n-k}(t)\, dt \\ &+ O\left(c_3^2\left(|a| + \frac{1}{\sqrt{n-k}}\right)\int_0^{\infty} t^2\, e^{-(k-m)t^2/4}\, dt\right).\end{aligned}$$

By (1.30), (1.53) and Lemma 3,

$$\int_{|t|\leqq 1/5c_3} \frac{f^k(t)-f^{k-m}(t)}{ti} e^{-itx}\overline{\varphi}_{n-k}(t)\,dt$$

(1.72)
$$= -\bar{a}_{n-k}\int_{|t|\leqq 1/5c_3} (f^k(t)-f^{k-m}(t))\,e^{-itx}\,dt + O\left(c_3^2\left(|a| + \frac{1}{\sqrt{n-k}}\right)\int_0^\infty t\,e^{-(k-m)t^2/4}\,dt\right).$$

From (1.69) and (1.70) it follows that

$$\int_{|t|\leqq 1/5c_3} (f^k(t)-f^{k-m}(t))\,e^{-itx}\,dt =$$

(1.73)
$$\int_{|t|\leqq 1/5c_3} e^{i(k-m)at-kt^2/2}(e^{iamt}-1)\,e^{-itx}\,dt + O\left(c_3(k-m)\int_0^\infty t^3\,e^{(m-k)t^2/4}\,dt + m\int_0^\infty t^2\,e^{-(k-m)t^2/2}\,dt\right).$$

It is not hard to see that

(1.74)
$$\int_{|t|\leqq 1/5c_3} e^{i(k-m)at-kt^2/2}(e^{iamt}-1)\,e^{-itx}\,dt = \int_{-\infty}^{\infty} (e^{k(iat-t^2/2)} - e^{(k-m)iat-kt^2/2})\,e^{-itx}\,dt + O\left(\frac{1}{\sqrt{k}}e^{-k/25c_3^2}\right).$$

Clearly,

(1.75)
$$\int_{-\infty}^{\infty} (e^{k(iat-t^2/2)} - e^{(k-m)iat-kt^2/2})\,e^{-itx}\,dt = \sqrt{\frac{2\pi}{k}}(e^{-(x-ka)^2/2k} - e^{-(x-(k-m)a)^2/2k}).$$

Further, setting $x - ka = A$ and $x - (k-m)a = B$ for brevity, we obtain

(1.76)
$$e^{-A^2/2k} - e^{-B^2/2k} = \int_{Ak^{-1/2}}^{Bk^{-1/2}} y\,e^{-y^2/2}\,dy = O(m|a|k^{-1/2}).$$

On the other hand,

(1.77)
$$\left|\int_{Bk^{-1/2}}^{Ak^{-1/2}} y\,e^{-y^2/2}\,dy\right| < \int_{Bk^{-1/2}}^{Ak^{-1/2}} |y|\,e^{-y^2/2}\,dy < 2\int_{Bn^{-1/2}}^{An^{-1/2}} |y|\,e^{-y^2/2}\,dy,$$

$$n \geqq k \geqq n/2,$$

since $e^{-ny^2/2k} < e^{-y^2/2}$.

Let

$$d_{km} = \int_{Bn^{-1/2}}^{An^{-1/2}} |y| \, e^{-y^2/2} \, dy.$$

It is easy to see that

$$\begin{aligned} \sum_{k=n/2}^{n} d_{km} &= \sum_{k=n/2}^{n} \sum_{j=k-m+1}^{k} d_{j1} = \sum_{k=n/2}^{n} \sum_{j=1-m}^{0} d_{(k+j)1} \\ &< m \sum_{i=-\infty}^{\infty} d_{i1} = m \int_{-\infty}^{\infty} |y| \, e^{-y^2/2} \, dy = 2m. \end{aligned} \tag{1.78}$$

Lemma 4 and (1.75)–(1.78) imply that

$$\begin{aligned} &\sum_{k=n/2}^{n-1} \bar{a}_{n-k} \int_{-\infty}^{\infty} (e^{k(iat-t^2/2)} - e^{(k-m)iat-kt^2/2}) \, e^{-itx} \, dt \\ &= O\left(|a| \sum_{k=n/2}^{n-1} \left(\frac{d_{km}}{\sqrt{k}} + \frac{m}{k\sqrt{n-k}}\right)\right) = O\left(\frac{m}{\sqrt{n}}|a|\right). \end{aligned} \tag{1.79}$$

Applying estimates (1.71)–(1.74), (1.56), (1.79) and Lemma 4, we obtain

$$\begin{aligned} \sum_{k=n/2+1}^{n-1} |I_{km}| &= O\left(\frac{m}{\sqrt{n}}|a| + \sum_{k=n/2+1}^{n-1} \left[\left(|a| + \frac{1}{\sqrt{n-k}}\right)\right.\right. \\ &\quad \times ((m + c_3^2)(k-m)^{-3/2} + k^{-1/2} e^{-k/25c_3^2}) \\ &\quad \left.\left. + (k-m)^{-1}(c_3^2|a| + c_3 + c_3^2(n-k)^{-1/2})\right]\right) \\ &= O\left(\left(c_3^2 + \frac{m}{\sqrt{n}}\right)\left(|a| + \frac{1}{\sqrt{n}}\right)\right), \end{aligned} \tag{1.80}$$

since

$$e^{-k/25c_3^2} = O\left(\frac{c_3^2}{k}\right),$$

and in addition, there is no less of generality in assuming $c_3^2 < \sqrt{n}$.

Observe that by (1.30),

$$f^n(t) - f^{n-m}(t) = O(m \, e^{-(n-m)t^2/4}(|at| + t^2)), \qquad t \leqq \frac{3}{2c_3}.$$

Therefore,

$$I_{nm} = O\left(m \int_0^{\infty} (|a|t + t^2) \, e^{-(n-m)t^2/4} \, dt\right) = O\left(\frac{m}{n}\left(|a| + \frac{1}{\sqrt{n}}\right)\right). \tag{1.81}$$

It is not hard to show that

$$\frac{1}{\sqrt{n-m}} - \frac{1}{\sqrt{n}} = O\left(\frac{m}{n^{3/2}}\right),$$

$$\Phi(-a\sqrt{n-m}) - \Phi(-a\sqrt{n}) = O\left(\frac{|a|m}{\sqrt{n}} e^{-a^2(n-m)/2}\right).$$

These estimates and Lemma 1 imply that

$$\bar{a}_n - \bar{a}_{n-m} = O\left(\frac{m}{n^{3/2}} + \frac{a^2 m}{\sqrt{n}} e^{-a^2(n-m)/2} + \frac{c_3}{n}\right). \tag{1.82}$$

Applying (1.30), (1.68), (1.71) and the estimate $x\,e^{-x^2} = O(1)$, we obtain

$$\begin{aligned} J_{km} &\equiv \int_{|t| \leqq 1/5c_3} |t^{-1} f^k(t)(\bar{\varphi}_{n-k}(t) - \bar{\varphi}_{n-m-k}(t))|\, dt \\ &= O\left(\left(\frac{m}{(n-k-m)^{3/2}} + \frac{m|a| + c_3}{n-k-m}\right) k^{-1/2}\right). \end{aligned}$$

Therefore,

$$\sum_{k=1}^{n/2-m} J_{km} = O\left(\frac{m}{\sqrt{n}}\left(|a| + \frac{1}{\sqrt{n}}\right) + \frac{c_3}{\sqrt{n}}\right). \tag{1.83}$$

From (1.30) and Lemma 4 it follows that

$$\begin{aligned} &\sum_{k=n/2-m}^{n/2} \int_{|t| \leqq 1/5c_3} |t^{-1} f^k(t) \bar{\varphi}_{n-k}(t)|\, dt \\ &\quad = O\left(\sum_{k=n/2-m}^{n/2} k^{-1/2} |\bar{a}_{n-k}|\right) = O\left(\frac{m}{\sqrt{n}}\left(|a| + \frac{1}{\sqrt{n}}\right)\right). \end{aligned} \tag{1.84}$$

From (1.66), (1.67), (1.80), (1.81), (1.83) and (1.84) we deduce

$$\Delta_{nm}(x) = O\left(\left(c_3^2 + \frac{m}{\sqrt{n}}\right)\left(|a| + \frac{1}{\sqrt{n}}\right)\right).$$

On the other hand,

$$\Delta_{nm}(x) \geqq (\bar{F}_n(x - c_3) - \bar{F}_{n-m}(x + c_3)) \int_{|x| < c_3} dH\left(\frac{x}{5c_3}\right).$$

By virtue of Lemma 5,

$$\bar{F}_n(x + 2c_3) - \bar{F}_n(x) = O\left(c_3^2\left(|a| + \frac{1}{\sqrt{n}}\right)\right).$$

The assertion of the lemma follows from the last three estimates.

Lemma 7.

$$\bar{F}_n(0) = O\left(\left(c_3^2\left(|a| + \frac{1}{\sqrt{n}}\right)\right)\right).$$

PROOF. Clearly,

$$\bar{F}_n(-1) \leqq -\int_{-\infty}^{-1} x\, d\bar{F}_n(x) \leqq -\bar{a}_n.$$

By Lemma 4,

$$\bar{a}_n = O\left(\frac{1}{\sqrt{n}} + |a|\right).$$

On the other hand, by Lemma 5,

$$\bar{F}_n(0) - \bar{F}_n(-1) = O\left(c_3^2\left(|a| + \frac{1}{\sqrt{n}}\right)\right).$$

These three estimates easily lead to the assertion of the lemma.

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