ESTIMATING THE RATE OF CONVERGENCE FOR THE DISTRIBUTION OF THE MAXIMUM SUMS OF INDEPENDENT RANDOM QUANTITIES

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Let ξ_n ($n = 1, \infty$) be a sequence of independent identically distributed random quantities with $M\xi_1 = 0$ and $D\xi_1 = \sigma^2 < \infty$. We set

$$c_3 = M|\xi_1|^3, \quad S_n = \sum_{i=1}^{n} \xi_i, \quad \bar{S}_n = \max_{1 \leq i \leq n} S_i, \quad F(x) = P(\xi_1 < x),$$

$$F_n(x) = P(S_n < x), \quad \bar{F}_n(x) = P(\bar{S}_n < x), \quad \tilde{F}_n(x) = P(\max[0, S_n] < x).$$

As shown by Erdős and Katz [1]

$$\lim_{n\to\infty} \bar{F}_n(\sigma\sqrt{n}\,x) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-u^2/2}\,du, \quad x \geq 0.$$

Chung [3] obtained the estimate $c_3 < \infty$ for the rate of convergence $\bar{F}_n(\sigma\sqrt{n}\,x)$ given the condition $O(\ln n / n^{1/26})$.

The problem of estimating the rate of convergence of $\bar{F}_n(\sigma\sqrt{n}\,x)$ is tied up with the so-called boundary-value problems for the sums of independent random quantities. Yu. V. Prokhorov [10] obtained the estimate $O(\ln^2 n / n^{1/8})$ in a problem with two curved boundaries, assuming that $c_3 < \infty$. A. V. Shorokhod refined the estimate in this problem to $O(\ln n / n^{1/2})$, but at the price of a very great restriction on $|\xi_1| < c < \infty$.

A detailed survey of results applying to this group of problems can be found in [5].

This study is devoted to a proof of the following theorem.

THEOREM. There exists an absolute constant L such that

$$\left| \bar{F}_n(x\sigma\sqrt{n}) - \left(\frac{2}{\pi}\right)^{1/2} \int_0^x e^{-u^2/2}\,du \right| < \frac{Lc_3^2}{\sigma^6\sqrt{n}} \min\left[\ln n, \frac{1+x^2}{x^2}\right]; \quad x \geq 0.$$

Note that it is much easier to obtain an estimate $O(\ln n / n^{1/2})$ uniform with respect to x. A discussion is given at the end of this article after the proof of the theorem.

Proof. We shall consider the notation which shall be used below. First, the symbol O will be required only when the corresponding constant does not depend on the distribution $F(x)$. The term $S(G(x))$ will represent the operator converting the bounded-variation function $G(x)$ into the Fourier-Stiltjes transform $\int_{-\infty}^{\infty} e^{itx}\,dG(x)$; the term $S_1(g(x))$ will represent the operator converting the absolute integrable function $g(x)$ into the Fourier-Lebesgue transform $\int_{-\infty}^{\infty} e^{itx} g(x)\,dx$. The symbols S^{-1} and S_1^{-1} will represent the inverse operators of, respectively, S and S_1.

We set $f(t) = S(F)$, $\varphi_n(t) = S(\tilde{F}_n)$. Let $\psi_n(t)$ be the characteristic function of $[0, S_n]$. As is known (see [3]),

$$\sum_{n=0}^{\infty} \varphi_n(t) z^n = \exp\left\{ \sum_{n=1}^{\infty} \frac{\psi_n(t)}{n} z^n \right\}, \quad \varphi_0(t) \equiv 1. \tag{1}$$

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We set

$$\Phi(t,z)=\sum_{n=0}^{\infty}\varphi_n(t)z^n,\quad \Psi(t,z)=(1-f(t)z)\sum_{n=0}^{\infty}\varphi_n(t)z^n.$$

Clearly,

$$F_n(x)=\begin{cases}F_n(x), & x>0;\\ 0, & x\leqslant 0.\end{cases}$$

Therefore,

$$F_{n+1}(x)=\int_{-\infty}^{x}F_n(x-y)\,dF(y)=\int_{-\infty}^{\infty}F_n(x-y)\,dF(y),$$

whence

$$S(F_{n+1})=\varphi_n(t)f(t).$$

It is not difficult to see that

$$\varphi_n(t)=P_+S(F_n)+P(\bar{S}_n<0),$$

where P_+ is the projector (in the space of the Fourier-Stiltjes transforms) of the bounded-variation functions converting

$$g(t)=\int_{-\infty}^{\infty}e^{itx}\,dG(x)\ \text{into}\ P_+g(t)=\int_{0-}^{\infty}e^{itx}\,dG(x).$$

Thus,

$$\varphi_{n+1}(t)-f(t)\varphi_n(t)=P(\bar{S}_{n+1}<0)-\int_{-\infty}^{0-}e^{itx}\,dF_{n+1}(x)$$

and, consequently,

$$\Psi(t,z)=1+\sum_{n=1}^{\infty}\bar{\varphi}_n(t)z^n, \tag{2}$$

where

$$\bar{\varphi}_n(t)=P(\bar{S}_n<0)-\int_{-\infty}^{0-}e^{itx}\,dF_n(x).$$

We set

$$\bar{a}_n=\int_{-\infty}^{0}x\,dF_n(x),\quad a_n^+=\int_{0}^{\infty}x\,dF_n(x),$$

$$\bar{b}_n=\int_{-\infty}^{0}x^2\,dF_n(x),\quad b_n^+=\int_{0}^{\infty}x^2\,dF_n(x).$$

Without loss of generality we shall henceforth assume that $\sigma^2=1$.

LEMMA 1. $\bar{a}_n=\bar{a}_n^+/n$, $n>0$.

Proof. From (2), the generating function for $\bar{a}_n$ is $i\Psi_t'(0,z)$. Clearly,

$$\Psi_t'(t,z)=\left((1-zf(t))\sum_{n=1}^{\infty}\frac{\Psi_n'(t)}{n}z^n-f'(t)z\right)\Phi(t,z). \tag{3}$$

Therefore,

$$\Psi_t'(0,z)=i(1-z)\Phi(0,z)\sum_{n=1}^{\infty}\frac{a_n^+}{n}z^n=i\sum_{n=1}^{\infty}\frac{a_n^+}{n}z^n. \tag{4}$$

The statement of the lemma now easily follows.

LEMMA 2. $a_n^+ = \sqrt{\frac{n}{2\pi}} + O(c_3), \quad b_n^+ = \frac{n}{2} + O(c_3\sqrt{n})$.

Proof. Clearly,

$$a_n^+ = \int_0^\infty (1 - F_n(x))\,dx, \quad b_n^+ = 2\int_0^\infty (1 - F_n(x))\,x\,dx.$$

Moreover,

$$|F_n(x\sqrt{n}) - \Phi(x)| < \frac{Lc_3}{(1 + |x|^3)\sqrt{n}}, \tag{5}$$

where L is an absolute constant, and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2}\,du$ (see [8], Theorem 1). Therefore,

$$a_n^+ = \sqrt{n} \int_0^\infty (1 - \Phi(x))\,dx + O(c_3) = \sqrt{\frac{n}{2\pi}} + O(c_3), \tag{6}$$

$$b_n^+ = 2n \int_0^\infty (1 - \Phi(x))\,x\,dx + O(c_3\sqrt{n}) = \frac{n}{2} + O(c_3\sqrt{n}), \tag{7}$$

which we were required to prove.

LEMMA 3.* $\sum_{n=1}^{\infty} \left| \frac{a_n^+}{n} - \frac{1}{\sqrt{2\pi n}} \right| = O(c_3^2)$.

Proof. Clearly,

$$\int_{-\infty}^{\infty} e^{itx} x\,dF_n(x) = -inf'(t) f^{n-1}(t).$$

On the other hand,

$$\int_{-\infty}^{\infty} e^{itx} x\,dF_n(x) = it \int_{-\infty}^{\infty} e^{itx}\,dx \int_x^\infty y\,dF_n(y).$$

Using the inversion formula and integrating by parts we obtain

$$\frac{1}{n}\int_0^\infty x\,dF_n(x) = -\frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{f'(t) f^{n-1}(t)}{t}\,dt$$

$$= -\frac{1}{2\pi}\int_{-\delta}^{\delta} \frac{f'(t) f^{n-1}(t)}{t}\,dt - \frac{1}{\pi n}\int_{\delta}^{\infty} \frac{\operatorname{Re} f^n(r)}{t^2}\,dt + \frac{\operatorname{Re} f^n(\delta)}{n\delta}, \quad \delta > 0. \tag{8}$$

Clearly,

$$\frac{1}{\sqrt{2\pi n}} = \frac{1}{n} \int_0^\infty x\,d\Phi(x, n), \tag{9}$$

where

$$\Phi(x, n) = \frac{1}{\sqrt{2\pi n}} \int_{-\infty}^{x} e^{-u^2/2n}\,du.$$

*In this article by Rosen [4] convergence of the series $M\xi_1 = 0$, $D\xi_1 < \infty$ was proved for the condition $\sum_{n=1}^{\infty}$ $1/n\,|P(S_n < 0) - {}^1/_2|$. The method involved in proving Lemma 3 is a modification of the Rosen method.

It follows from (8) and (9) that

$$\frac{a_n^+}{n}-\frac{1}{\sqrt{2\pi n}}=-\frac{1}{2\pi}\int_{-\delta}^{\delta}\frac{te^{-nt^2/2}+f'(t)f^{n-1}(t)}{t}\,dt$$

$$-\frac{1}{\pi n}\int_{\delta}^{\infty}\frac{\operatorname{Re}f^n(t)}{t^2}\,dt+\frac{\operatorname{Re}f^n(\delta)}{n\delta}+\frac{1}{\pi n}\int_{\delta}^{\infty}\frac{e^{-nt^2/2}}{t^2}\,dt-\frac{e^{-n\delta^2/2}}{n\delta}. \tag{10}$$

To begin with, we use the integral

$$J_{1n}(\delta)=\int_{-\delta}^{\delta}\frac{te^{-nt^2/2}+f'(t)f^{n-1}(t)}{t}\,dt.$$

First,

$$f'(t)+te^{-t^2/2}=it^2(R(t)+iI(t))+t(e^{-t^2/2}-1), \tag{11}$$

where

$$R(t)=\frac{1}{t^2}\int_{-\infty}^{\infty}x\cos tx\,dF(x),\quad I(t)=\frac{1}{t^2}\int_{-\infty}^{\infty}x(\sin tx-tx)\,dF(x).$$

Therefore,

$$J_{1n}=\int_{-\delta}^{\delta}e^{-t^2/2}(e^{-(n-1)t^2/2}-f^{n-1}(t))\,dt+\int_{-\delta}^{\delta}(e^{-t^2/2}-1)f^{n-1}(t)\,dt+i\int_{-\delta}^{\delta}tf^{n-1}(t)(R(t)+iI(t))\,dt. \tag{12}$$

Furthermore,

$$\int_{-\delta}^{\delta}tf^n(t)(R(t)+iI(t))\,dt=\int_{-\delta}^{\delta}t(f^n(t)-e^{-nt^2/2})(R(t)+iI(t))\,dt+i\int_{-\delta}^{\delta}tI(t)e^{-nt^2/2}\,dt. \tag{13}$$

As is known (see, for example, [9], §40, Theorem 2),

$$|f^n(t)-e^{-nt^2/2}|<\frac{7}{6}c_3n|t|^3e^{-nt^2/4},\ n>0, \tag{14}$$

for $|t|\le 1/5c_3$. Clearly,

$$R(t)+iI(t)=O(c_3). \tag{15}$$

Therefore,

$$\int_{-\delta}^{\delta}t(f^n(t)-e^{-nt^2/2})(R(t)+iI(t))\,dt=O(c_3^2n\int_{0}^{\delta}t^4e^{-nt^2/4}\,dt)=O\Big(\frac{c_3^2}{n^{3/2}}\Big) \tag{16}$$

is uniform for $\delta\le 1/5c_3$. Clearly,

$$\int_{0}^{1}\frac{|I(t)|}{t}\,dt\leqslant\int_{-\infty}^{\infty}|x|\,dF(x)\int_{0}^{1}\frac{|\sin xt-xt|}{t^3}\,dt.$$

On the other hand,

$$\int_{0}^{x}\frac{|\sin xt-xt|}{t^3}\,dt=x^2\int_{0}^{x}\frac{|\sin u-u|}{u^3}\,du<x^2\int_{0}^{\infty}\frac{|\sin u-u|}{u^3}\,du.$$

Therefore,

$$\int_{0}^{1}\frac{|I(t)|}{t}\,dt=O(c_3). \tag{17}$$

Using (17), we have

$$\sum_{n=1}^{\infty}\Big|\int_{-\delta}^{\delta}tI(t)e^{-nt^2/2}dt\Big|=O\Big(\int_{0}^{\delta}\frac{t|I(t)|}{1-e^{-t^2/2}}dt\Big)=O\Big(\int_{0}^{\delta}\frac{|I(t)|}{t}dt\Big)=O(c_3),\quad \delta\leqslant 1. \tag{18}$$

Clearly,

$$f^n(t) - e^{-nt^2/2} = (f(t) - e^{-t^2/2}) \sum_{k=0}^{n-1} e^{-kt^2/2} f^{n-k}(t). \tag{19}$$

Moreover,

$$f(t) - e^{-t^2/2} = \frac{\beta_3}{6}(it)^3 + t^4(R_1(t) + iI_1(t)) + \left(1 - \frac{t^2}{2} - e^{-t^2/2}\right),$$

where

$$R_1(t) = \frac{1}{t^4} \int_{-\infty}^{\infty} \left(\cos tx - 1 + \frac{t^2x^2}{2}\right) dF(x),$$

$$I_1(t) = \frac{1}{t^4} \int_{-\infty}^{\infty} \left(\sin tx - tx + \frac{t^3x^3}{6}\right) dF(x),$$

$$\beta_3 = \int_{-\infty}^{\infty} x^3 dF(x).$$

Therefore

$$(f(t) - e^{-t^2/2}) e^{-kt^2/2} f^{n-k}(t) = \frac{\beta_3}{6}(it)^3 e^{-nt^2/2} + t^4(R_1(t) + iI_1(t)) e^{-nt^2/2}$$

$$+ (f(t) - e^{-t^2/2})(f^{n-k}(t) - e^{-(n-k)t^2/2}) e^{-kt^2/2} + \left(1 - \frac{t^2}{2} - e^{-t^2/2}\right) e^{-nt^2/2}. \tag{20}$$

Clearly,

$$\int_0^1 |R_1(t)| dt \leqslant \int_{-\infty}^{\infty} dF(x) \int_0^1 \frac{|\cos tx - 1 - t^2x^2/2|}{t^4} dt = O(c_3).$$

Therefore,

$$\sum_{n=1}^{\infty} n \int_{-\delta}^{\delta} t^4 |R_1(t)| e^{-nt^2/4} dt = 0\left(\int_0^{\delta} \frac{|R_1(t)| t^4 dt}{(1 - e^{-t^2/2})^2}\right) = O(c_3) \quad \delta \leqslant 1. \tag{21}$$

Moreover,

$$\int_{-\delta}^{\delta} t^3 e^{-nt^2/2} dt = 0 \tag{22}$$

and

$$\int_{-\delta}^{\delta} t^4 I_1(t) e^{-nt^2/2} dt = 0, \tag{23}$$

since $I_1(t) = -I_1(-t)$.

From (14),

$$(f(t) - e^{-t^2/2})(f^{n-k}(t) - e^{-\frac{n-k}{2}t^2}) e^{-kt^2/2} = O(c_3^2 (n-k) |t|^6 e^{-nt^2/4}),$$
$$|t| \leqslant 1/5c_3.$$

We now have

$$\sum_{k=0}^{n} \int_{-\delta}^{\delta} |(f(t) - e^{-t^2/2})(f^{n-k}(t) - e^{-\frac{n-k}{2}t^2})| \cdot e^{-\frac{k+1}{2}t^2} dt = O\left(c_3^2 \sum_{k=1}^{n} (n-k) n^{-5/2}\right) = O\left(\frac{c_3^2}{n^{1/2}}\right), \quad \delta < \frac{1}{5c_3}. \tag{24}$$

It follows from (19)-(24) that

$$\sum_{n=1}^{\infty} \left| \int_{-\delta}^{\delta} (f^n(t) - e^{-nt^2/2}) e^{-t^2/2} dt \right| = O(c_3^2). \tag{25}$$

Finally

$$\int_{-\delta}^{\delta} (e^{-t^2/2}-1)f^n(t)\,dt = O\Big(\int_{-\delta}^{\delta} t^2 e^{-nt^2/4}\,dt\Big) = O(n^{-3/2}) \tag{26}$$

is uniform with respect to $\delta \leq 3/2c_3$, since for $|t| \leq 3/2c_3$ we have

$$|f(t) - 1 + t^2/2| \leqslant t^2/4.$$

and, therefore,

$$|f(t)| < e^{-t^2/4}, \quad |t| \leqslant 3/2c_3. \tag{27}$$

From (12), (13), (16), (18), (25), and (26) we see that

$$\sum_{n=1}^{\infty} |J_{1n}(\delta)| = O(c_3^2) \tag{28}$$

is uniform for $\delta < 1/5c_3$.

We now evaluate the integral

$$J_{2n}(\delta) = \int_{\delta}^{\infty} \frac{\operatorname{Re} f^n(t)}{t^2}\,dt.$$

Clearly,

$$|J_{2n}(\delta)| \leqslant \int_{-\infty}^{\infty} dF_n(x) \,\Big|\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt\Big|.$$

For $x > 0$,

$$\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt = x\int_{\delta x}^{\infty} \frac{\cos t}{t^2}\,dt.$$

Moreover,

$$\int_{\delta x}^{\infty} \frac{\cos t}{t^2}\,dt = \int_{\delta x}^{k(\delta x)\pi} \frac{\cos t}{t^2}\,dt + \int_{k(\delta x)\pi}^{\infty} \frac{\cos t}{t^2}\,dt,$$

where $k(\delta x) = [\delta x/\pi] + 1$.

It is not difficult to see that

$$\Big|\int_{k(\delta x)\pi}^{\infty} \frac{\cos t}{t^2}\,dt\Big| < \int_{k(\delta x)\pi}^{(k(\delta x)+1)\pi} \frac{\cos t}{t^2}\,dt < \frac{\pi}{\delta^2 x^2}.$$

This estimate is also valid for $\int_{\delta x}^{k(\delta x)\pi} \frac{\cos t}{t^2}\,dt$. Moreover, $\int_{k(\delta x)\pi}^{\infty} \frac{\cos t}{t^2}\,dt$ and $\int_{\delta x}^{k(\delta x)\pi} \frac{\cos t}{t^2}\,dt$ have different signs. Therefore,

$$\Big|\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt\Big| < \frac{\pi}{|x|\delta^2}.$$

On the other hand,

$$\Big|\int_{\delta}^{\infty} \frac{\cos xt}{t^2}\,dt\Big| < \int_{\delta}^{\infty} \frac{dt}{t^2} = \frac{1}{\delta}.$$

Using these two estimates we obtain

$$|J_{2n}(\delta)| < \frac{\pi}{\delta} \int_{|x|<n^{1/4}c_3^{1/3}} dF_n(x) + \frac{1}{\delta^2} \int_{|x|>n^{1/4}c_3^{1/3}} \frac{1}{|x|}\,dF_n(x). \tag{29}$$

Now applying the familiar Esseen estimate for the remaining term in the central limit theorem (see, for example, [9], §40), we have

$$\int\limits_{|x|<n^{1/4}c_3^{1/4}} dF_n(x) = \Phi(c_3^{1/4}n^{-1/4}) - \Phi(-c_3^{1/4}n^{-1/4}) + O(c_3/\sqrt{n}) = O(c_3^{1/4}/n^{1/4}) + O(c_3/\sqrt{n}). \tag{30}$$

From (29) and (30), it is easy to obtain the estimate

$$J_{2n}(1/5c_3) = O(c_3^{1/4}n^{-1/4}). \tag{31}$$

Finally, from (27),

$$\sum_{n=1}^{\infty} \frac{1}{n}\left|f^n\left(\frac{1}{5c_3}\right)\right| = -\ln\left(1 - \left|f\left(\frac{1}{5c_3}\right)\right|\right) = O(\ln c_3). \tag{32}$$

The statement of the lemma now follows from (10), (28), (31), and (32).

LEMMA 4. $\bar{b}_n = O(c_3^2/\sqrt{n})$.

Proof. Using (4), it is not difficult to show

$$\Psi_t''(0,z) = \left[\left(\left(\sum_{n=1}^{\infty}\frac{\psi_n'(0)}{n}z^n\right)^2 + \sum_{n=1}^{\infty}\frac{\psi_n''(0)}{n}z^n\right)\cdot(1-z)\right]\Phi(0,z) = -\left(\sum_{n=1}^{\infty}\bar{a}_n z^n\right)^2 - \sum_{n=1}^{\infty}\frac{b_n^+}{n}z^n + \frac{z}{1-z}. \tag{33}$$

In view of Lemmas 1-3,

$$\sum_{n=1}^{\infty}\bar{a}_n z^n = -\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}} + \sum_{n=1}^{\infty} r_n z^n, \tag{34}$$

where $\sum\limits_{n=1}^{\infty} |r_n| = O(c_3^2)$ and $r_n = O(c_3/n)$.

Employing the asymptotic expansion of the remaining term in the local limit theorem of De Moivre-Laplace (see [9], §51, Theorem 1), we have

$$\frac{1\cdot 3\cdot \ldots \cdot(2n-1)}{2^n n!} = \frac{(2n)!}{2^{2n}(n!)^2} = \frac{1}{\sqrt{\pi n}} + O\left(\frac{1}{n^{3/2}}\right).$$

Therefore,

$$\frac{1}{\sqrt{2\pi}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}} = \frac{1}{\sqrt{2}}(1-z)^{-1/2} + \sum_{n=0}^{\infty}\rho_{1n}z^n, \tag{35}$$

where $\rho_{1n} = O(n^{-3/2})$. From Lemma 2, we have

$$\sum_{n=1}^{\infty}\frac{b_n^+}{n}z^n = \frac{1}{2}(1-z)^{-1} + \sum_{n=0}^{\infty}\rho_{2n}z^n, \tag{36}$$

where $\rho_{2n} = O(c_3/\sqrt{n})$. Moreover,

$$\sum_{k=1}^{n-1}\frac{|r_k|}{\sqrt{n-k}} = \sum_{k=1}^{n/2}\frac{|r_k|}{\sqrt{n-k}} + \sum_{k=n/2+1}^{n-1}\frac{|r_k|}{\sqrt{n-k}} = O\left(\frac{c_3^2}{\sqrt{n}}\right) + O\left(\frac{c_3}{\sqrt{n}}\right) = O\left(\frac{c_3^2}{\sqrt{n}}\right). \tag{37}$$

$$\sum_{k=1}^{n-1}|r_k||r_{n-k}| = O\left(\frac{c_3^2}{\sqrt{n}}\right).$$

$$\left(\sum_{n=1}^{\infty}\bar{a}_n z^n\right)^2 = \frac{1}{2\pi}\left(\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}}\right)^2 + \sum_{n=0}^{\infty}\rho_{3n}z^n, \tag{38}$$

where $\rho_{3n} = O(c_3^2/\sqrt{n})$. From (35), we have

$$\frac{1}{2\pi}\left(\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n}}\right)^2 = \frac{1}{2}(1-z)^{-1} + \sum_{n=0}^{\infty}\rho_{4n}z^n, \tag{39}$$

where $\rho_{4n} = O(1/\sqrt{n})$. The statement of the lemma easily follows from (33), (36), (38), and (39).

Let

$$h(t)=\begin{cases}0, & |t|>1;\\ 2(1-|t|)^3, & 1/2\leqslant|t|\leqslant 1;\\ 1-6t^2+6|t|^3, & 0\leqslant|t|\leqslant 1/2.\end{cases}$$

It is not difficult to verify that

$$S^{-1}(h(t))=\frac{3}{8\pi}\int_{-\infty}^{x}\left(\frac{\sin u/4}{u/4}\right)^4 du.$$

We set

$$g_n(t)=(f^n(t)-e^{-nt^2/2})h(5c_3t),\quad G_n(x)=S^{-1}(g_n(t)).$$

LEMMA 5. $G_n'(x)=O\left(c_3\min\left(\frac{1}{n},\ \frac{1}{x^2}\left(1+\frac{c_3^2}{n}\right)\right)\right)$.

Proof. Clearly,

$$\int_{-\infty}^{\infty}e^{itx}x^2G_n'(x)\,dx=-g_n''(t). \tag{40}$$

Moreover,

$$g_n''(t)=\sum_{k=0}^{2}C_2^k\frac{d^k}{dt^k}(f^n(t)-e^{-nt^2/2})\frac{d^{2-k}}{dt^{2-k}}h(5c_3t). \tag{41}$$

Clearly,

$$\frac{d}{dt}(f^n(t)-e^{-nt^2/2})=n(f'(t)f^{n-1}(t)+te^{-nt^2/2}), \tag{42}$$

$$\frac{d^2}{dt^2}(f^n(t)-e^{-nt^2/2})=n[f''(t)f^{n-1}(t)+e^{-nt^2/2}+(n-1)f'^2(t)f^{n-2}(t)-nt^2e^{-nt^2/2}]. \tag{43}$$

It follows from (42), (11), (14), (15), and (27) that

$$\frac{d}{dt}(f^n(t)-e^{-nt^2/2})=O((n^2t^4+nt^2c_3)e^{-nt^2/4}) \tag{44}$$

for $|t|\leq 1/5c_3$. Note that

$$f''(t)=-1+O(c_3t). \tag{45}$$

Therefore,

$$f''(t)f^{n-1}(t)+e^{-nt^2/2}=(f''(t)+e^{-t^2/2})f^{n-1}(t)$$
$$+e^{-nt^2/2}-e^{-t^2/2}f^{n-1}(t)=O((nc_3|t|^3+c_3|t|)\cdot e^{-nt^2/4}),\quad t\leqslant\frac{1}{5c_3}. \tag{46}$$

Similarly, we obtain

$$f'^2f^{n-2}(t)-t^2e^{-nt^2/2}=(f'^2(t)+t^2e^{-t^2})f^{n-2}(t)+t^2e^{-t^2}(f^{n-2}(t)$$
$$-e^{-\frac{n-2}{2}t^2})=O(c_3(n|t|^5+|t|^3)e^{-nt^2/4}),\quad |t|\leqslant 1/5c_3. \tag{47}$$

From (43), (46), and (47) it follows that

$$\frac{d^2}{dt^2}(f^n(t)-e^{-nt^2/2})=O(c_3(n^3|t|^5+n^2|t|^3+n|t|)e^{-nt^2/4}),\quad |t|\leqslant 1/5c_3. \tag{48}$$

From (40) we have

$$x^2G_n'(x)=-\frac{1}{2\pi}\int_{|t|\leqslant 1/5c_3}g_n''(t)e^{-itx}\,dt. \tag{49}$$

Using estimates (14), (44), and (48), equations (49) and (41) yield

$$x^2 G_n'(x) = O\left(c_3 \int_0^\infty (n^3t^5 + n^2t^3 + nt + c_3^2(n^2t^5 + nt^3))e^{-nt^2/4}\,dt\right) = O\left(c_3\left(1 + \frac{c_3^2}{n}\right)\right). \tag{50}$$

Moreover,

$$G_n'(x) = \frac{1}{2\pi} \int_{|t| \leq 1/3c_3} g_n(t) e^{-itx}\,dt.$$

Now, in view of (14), it is easy to obtain the estimate

$$G_n'(x) = O\left(c_3 n \int_0^\infty t^3 e^{-nt^2/4}\,dt\right) = O\left(\frac{c_3}{n}\right). \tag{51}$$

The statement of the lemma follows from (50) and (51).

We shall now directly prove the theorem.

From the identity $\Phi(t, z) = \Psi(t, z)/(1 - f(t)z)$ it follows that

$$\varphi_n(t) = f^n(t) + \sum_{k=0}^{n-1} f^k(t)\varphi_{n-k}(t). \tag{52}$$

We write (52) as

$$\varphi_n(t) = f^n(t) + \sum_{k=0}^{n-1} e^{-kt^2/2}\varphi_{n-k}(t) + \sum_{k=1}^{n-1} (f^k(t) - e^{-kt^2/2})\varphi_{n-k}(t). \tag{53}$$

Clearly,

$$\varphi_k(t) = -it\bar{a}_k + \tilde{\varphi}_k(t)t^2, \tag{54}$$

where

$$\tilde{\varphi}_k(t)t^2 = \int_{-\infty}^{0-} (e^{itx} - 1 - itx)\,d\bar{F}_k(x).$$

It is not difficult to see that

$$ite^{-kt^2/2} = S_1\left(\frac{x}{\sqrt{2\pi}\,k^{3/2}} e^{-x^2/2k}\right). \tag{55}$$

Let

$$\Phi_n(x) = S^{-1}\left(\frac{i}{2\pi} \sum_{k=1}^{n-1} te^{-kt^2/2}/\sqrt{n-k}\right).$$

Note that

$$\int_x^\infty ue^{-u^2/2k}\,du = ke^{-x^2/2k}. \tag{56}$$

From (55) and (56) we obtain

$$\Phi_n(\infty) - \Phi_n(x\sqrt{n}) = \frac{1}{2\pi} \sum_{k=1}^{n-1} \frac{e^{-nx^2/2k}}{\sqrt{k(n-k)}}. \tag{57}$$

It is not difficult to show that

$$\frac{e^{-nx^2/2k}}{\sqrt{k(n-k)}} = \int_{k/n}^{(k+1)/n} u^{-1/2}(1-u)^{-1/2} e^{-x^2/2u}\,du + O\left(\frac{1}{\sqrt{k(n-k)}}\left(\frac{1}{k} + \frac{1}{n-k} + \frac{nx^2}{k^2}\right)e^{-\frac{nx^2}{2(k+1)}}\right). \tag{58}$$

In addition,

$$\int_0^1 u^{-1/2}(1-u)^{-1/2} e^{-x^2/2u}\,du = \int_1^\infty u^{-1}(u-1)^{-1/2} e^{-x^2/2u}\,du = 2^{1/4}|x|^{-1/2}\Gamma\left(\frac{1}{2}\right)e^{-x^2/4}W_{-1/4,1/4}\left(\frac{x^2}{2}\right)$$

(see [12], p. 333, formula 4). Moreover,

$$\sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}\,du = 1 - 2^{1/4}(\pi x)^{-1/2}W_{-1/4,\,1/4}\left(\frac{x^2}{2}\right)e^{-x^2/4}$$

(see [12], p. 1077, 9.236, formula 1). Thus,

$$\int_0^1 u^{-1/2}(1-u)^{-1/2}e^{-x^2/2u}\,du = \pi\left(1-\sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2/2}\,du\right) = 2\pi(1-\Phi(x)). \tag{59}$$

It follows from (56)-(59) that

$$\Phi_n(\infty)-\Phi_n(x\sqrt{n}) = 1-\Phi(x)+O\left(\sum_{k=1}^{n-1}\frac{1}{\sqrt{k(n-k)}}\left(\frac{1}{k}+\frac{1}{n-k}\right)\right) = 1-\Phi(x)+O\left(\frac{1}{\sqrt{n}}\right) \tag{60}$$

Let

$$R_n(x) = S^{-1}\left(\sum_{k=1}^{n-1} ite^{-kt^2/2}r_{n-k}\right),$$

where $r_k = -\bar{a}_k - 1/\sqrt{\pi k}$.

In view of Lemmas 1-3 and relationships (55) and (56), we have

$$R_n(x\sqrt{n}) = O\left(c_3\sum_{k=1}^{n/2}\frac{1}{\sqrt{k(n-k)}}\right)+O\left(\sum_{k=n/2+1}^{n-1}\frac{r_{n-k}}{\sqrt{k}}\right) = O\left(\frac{c_3^2}{\sqrt{n}}\right). \tag{61}$$

We set

$$\Omega_{1n}(x) = S^{-1}\left(\sum_{k=1}^{n-1} t^2\tilde{\varphi}_{n-k}(t)e^{-kt^2/2}\right).$$

Clearly,

$$t^2e^{-kt^2/2} = S_1\left(-\frac{d^2}{dx^2}\sqrt{\frac{1}{2\pi k}}e^{-x^2/2k}\right).$$

Therefore,

$$\frac{t^2}{n}e^{-kt^2/2} = S_1^{-1}\left(\frac{1}{k}\sqrt{\frac{n}{2\pi k}}\left(1-\frac{n}{k}x^2\right)e^{-nx^2/2k}\right). \tag{62}$$

Note that

$$\frac{n^{3/2}}{k^{5/2}}\int_x^\infty u^2e^{-nu^2/2k}\,du = O\left(\frac{1}{x^2n}\right), \quad x>0;$$

$$\frac{n^{1/2}}{k^{3/2}}\int_x^\infty e^{-nu^2/2k}\,du = O\left(\frac{1}{x^2n}\right), \quad x>0. \tag{63}$$

Clearly,

$$\operatorname*{Var}_{-\infty<x<\infty} S^{-1}(\tilde{\varphi}_k(t)) = \frac{b_k}{2}. \tag{64}$$

Using Lemma 4 and (62)-(64), we obtain

$$\Omega_{1n}(\infty)-\Omega_{1n}(x\sqrt{n}) = O\left(\frac{1}{x^2n}\sum_{k=1}^{n} b_{n-k}\right) = O\left(\frac{c_3^2}{x^2\sqrt{n}}\right), \quad x>0. \tag{65}$$

Moreover,

$$\frac{n^{3/2}}{k^{5/2}}\int_0^\infty u^2e^{-nu^2/2k}\,du = \frac{\sqrt{2\pi}}{k}, \quad \frac{n^{1/2}}{k^{3/2}}\int_0^\infty e^{-nu^2/2k}\,du = \frac{\sqrt{2\pi}}{k}.$$

Consequently,

$$\Omega_{1n}(\infty)-\Omega_{1n}(x\sqrt{n})=O\Big(\sum_{k=1}^{n-k}\frac{\delta_{n-k}}{k}\Big)=O\Big(c_3^2\sum_{k=1}^{n-1}\frac{1}{k\sqrt{n-k}}\Big)=O\Big(\frac{c_3^3\ln n}{\sqrt{n}}\Big),\quad x\geqslant 0. \tag{66}$$

We set

$$\Omega_{2n}(x)=S^{-1}\Big(\sum_{k=1}^{n-1}(f^k(t)-e^{-kt^2/2})h(5c_3t)\varphi_{n-k}(t)\Big).$$

Clearly,

$$S^{-1}((f^k(t)-e^{-kt^2/2})h(5c_3t)\varphi_{n-k}(t))=G_k\cdot S^{-1}(\varphi_{n-k}(t))=G_k(x)F_{n-k}(0)-\int_{-\infty}^{0-}G_k(x-u)\,dF_{n-k}(u)=\int_{-\infty}^{0-}\Delta_u G_k(x)\,dF_{n-k}(u),$$

where $\Delta_u G_k(x)=G_k(x)-G_k(x-u)$. Therefore,

$$\Omega_{2n}(x)=\sum_{k=1}^{n-1}\int_{-\infty}^{0-}\Delta_u G_k(x)\,dF_{n-k}(u). \tag{67}$$

In view of Lemma 5, for $u\leq 0$, $x\geq 0$ we have

$$\Delta_u G_k(x\sqrt{n})=O\Big(c_3\frac{u}{\sqrt{n}}\min\Big[\frac{1}{x^2}\Big(\frac{1}{\sqrt{n}}+\frac{1}{k}\Big),\ \frac{\sqrt{n}}{k}\Big]\Big) \tag{68}$$

since we can assume $c_3^2<\sqrt{n}$ without loss of generality.

Lemma 1 and the inequality $M|\xi_1|\leq(M\xi_1^2)^{1/2}$ lead to the estimate

$$\bar{a}_n=O(1/\sqrt{n}). \tag{69}$$

From (67)-(69) it follows that

$$\Omega_{2n}(x\sqrt{n})=O\Big(\frac{c_3}{\sqrt{n}}\sum_{k=1}^{n-1}|a_{n-k}|\min\Big[\frac{1}{x^2}\Big(\frac{1}{\sqrt{n}}+\frac{1}{k}\Big),\frac{\sqrt{n}}{k}\Big]\Big)=O\Big(\frac{c_3}{\sqrt{n}}\min\Big[\frac{1}{x^2},\ \ln n\Big]\Big),\quad x\geqslant 0. \tag{70}$$

Let

$$\Omega_{3n}(x)=S^{-1}\Big(\sum_{k=1}^{n-1}(f^k(t)-e^{-kt^2/2})(1-h(5c_3t))\varphi_{n-k}(t)\Big)+S^{-1}(\varphi_n(t)),$$

$$\bar{\Omega}_{3n}(x)=\Omega_{3n}(x)+S^{-1}(f^n(t)(1-h(5c_3t))).$$

We set

$$q_n(t)=(f^n(t)-e^{-nt^2/2})(1-h(5c_3t)),\quad n>0.$$

Clearly,

$$\varphi_{n-k}(t)q_k(t)=f^k(t)\varphi_{n-k}(t)-(f^k(t)h{-}5c_3t)+e^{-kt^2/2}(1-h(5c_3t))\varphi_{n-k}(t). \tag{71}$$

We set

$$w_n(t)=\sum_{k=1}^{n-1}f^k(t)h(5c_3t)\varphi_{n-k}(t),\qquad W_n(x)=S^{-1}(w_n(t)).$$

<u>LEMMA 6.</u> $\sup|W_n'(x)|=O(c_3/\sqrt{n})$.

<u>Proof.</u> First,

$$W_n'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}w_n(t)e^{-itx}\,dt. \tag{72}$$

Clearly,

$$\varphi_n(t) = -\bar{a}_n it + O(\delta_n t^2). \tag{73}$$

Furthermore,

$$\int_{-\infty}^{\infty} f^k(t)\, th(5c_3t)e^{-itx}\,dt = \int_{|t|\leq 1/5c_3} (f^k(t) - e^{-kt^2/2})\, th(5c_3t)e^{-itx}\,dt - i\int_{-\infty}^{\infty} e^{-kt^2/2}\, th(5c_3t)\sin tx\,dt. \tag{74}$$

In view of (14),

$$\int_{|t|\leq 1/5c_3} (f^k(t) - e^{-kt^2/2})\, th(5c_3t)e^{-itx}\,dt = O\left(c_3k\int_0^{\infty} t^4e^{-kt^2/4}\,dt\right) = O(c_3k^{-3/2}). \tag{75}$$

Using the estimate $h(t) - 1 = O(t^2)$, we have

$$\int_{-\infty}^{\infty} e^{-kt^2/2}\, th(5c_3t)\sin tx\,dx = \int_{-\infty}^{\infty} e^{-kt^2/2}\, t\sin tx\,dt + O\left(c_3^2\int_0^{\infty} e^{-kt^2/2}t^3\,dt\right) = -\sqrt{2\pi}\,\frac{x}{k^{3/2}}e^{-x^2/2k} + O\left(\frac{c_3^2}{k^2}\right). \tag{76}$$

From (27) we have

$$\int_{|t|\leq 1/5c_3} |f^k(t)|\,t^2\,dt = O(k^{-3/2}). \tag{77}$$

It follows from (73)-(77), (69) and Lemma 4 that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f^k(t)h(5c_3t)\varphi_{n-k}(t)|\,dt = O\left(\left(\frac{|x|e^{-x^2/2k}}{k^{3/2}\sqrt{2\pi}} + c_3^2k^{-3/2}\right)(n-k)^{-1/2}\right). \tag{78}$$

Moreover,

$$\int_{|t|\leq 1/5c_3} |f^k(t)\, th(5c_3t)|\,dt = O\left(\int_0^{1/5c_3} t\,dt\right) = O\left(\frac{1}{c_3^2}\right).$$

From (69),

$$\varphi_n(t) = O(|\bar{a}_n t|) = O(|t|\,/\,\sqrt{n}). \tag{79}$$

From the last two estimates we have

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f^k(t)h(5c_3t)\varphi_{n-k}(t)|\,dt = O(c_3^{-2}(n-k)^{-1/2}). \tag{80}$$

Using estimates (78) and (80) and the inequality $\min(a_1 + a_2,\ a_3) \leq a_1 + \min(a_2,\ a_3)$, $a_i > 0$ we obtain

$$W_n'(x) = O\left(\sum_{k=1}^{n-1} |x|k^{-3/2}(n-k)^{-1/2}e^{-x^2/2k}\right) + O\left(\sum_{k=1}^{n-1} \min[c_3^2k^{-3/2},\ c_3^{-2}](n-k)^{-1/2}\right). \tag{81}$$

We first estimate

$$W_1 = |x|\sum_{k=1}^{n-1} k^{-3/2}(n-k)^{-1/2}e^{-x^2/2k}.$$

The function $u^{-3/2}(n-u)^{-1/2}e^{-x^2/2u}$ increases monotonically in the intervals $(0, u_{1n})$ and (u_{2n}, n), and decreases monotonely in the interval (u_{1n}, u_{2n}) where $u_{1n} < u_{2n}$ are roots of the equation $4u^2 - (3n + x^2)u + x^2n = 0$. Therefore

$$\sum_{\substack{1\leq k\leq \max(1,\, u_{1n}-1)\\ u_{2n}\leq k\leq n-1}} k^{-3/2}(n-k)^{-1/2}e^{-x^2/2k} < \left(\int_1^{\max(1,\, u_{1n})} + \int_{u_{2n}}^{n}\right)\frac{e^{-x^2/2u}du}{u^{3/2}(n-u)^{1/2}}$$

and

$$\sum_{v_n\leq k\leq \bar{u}_{2n}} k^{-3/2}(n-k)^{-1/2}e^{-x^2/2k} < \int_{v_n}^{\bar{u}_{2n}} u^{-3/2}(n-u)^{-1/2}e^{-x^2/2u}du +$$

$$+ \bar{v}_n^{-3/2}(n-\bar{v}_n)^{-1/2}e^{-x^2/2\bar{v}_n},\quad v_n = \max[1,\ u_{1n}],\ \bar{v}_n = \min[v_n,\ n-1],$$
$$\bar{u}_{2n} = \min[u_{2n},\ n-1].$$

Thus,

$$W_1 \leqslant |x| \int\limits_1^n u^{-3/2}(n-u)^{-1/2} e^{-x^2/2u}\, du + 1/\sqrt{n-1},$$

since $|x| e^{-x^2/2u} < \sqrt{2u}/e$ and $\bar{v}_n^{-1}(n-\bar{v}_n)^{-1/2} < (n-1)^{-1/2}$. Clearly,

$$\int\limits_1^n u^{-3/2}(n-u)^{-1/2} e^{-x^2/2u}\, du = \frac{1}{n} \int\limits_{1/n}^1 \frac{e^{-x^2/2un}\, du}{u^{3/2}(1-u)^{1/2}}.$$

From (59),

$$x \int\limits_{1/n}^1 u^{-3/2}(1-u)^{-1/2} e^{-x^2/2u}\, du = \sqrt{2\pi}\, e^{-x^2/2} + \frac{d}{dx} \int\limits_0^{1/n} u^{-1/2}(1-u)^{-1/2} e^{-x^2/2u}\, du.$$

However, for $x \geq 0$,

$$\frac{d}{dx} \int\limits_0^{1/n} u^{-1/2}(1-u)^{-1/2} e^{-x^2/2u}\, du \leqslant 0.$$

This means that

$$\sqrt{n}\, |x| \int\limits_1^n u^{-3/2}(n-u)^{-1/2} e^{-x^2/2u}\, du \leqslant \sqrt{2\pi}\, e^{-x^2/2n}. \tag{83}$$

It follows from (82) and (83) that

$$W_1 < 4/\sqrt{n}. \tag{84}$$

We now estimate

$$W_2 = \sum_{k=1}^{n-1} \min\,[c_3^2 k^{-3/2},\, c_3^{-2}](n-k)^{-1/2}.$$

Without loss of generality we can assume that $c_3 < \sqrt{n/2}$. Then

$$c_3^2 \sum_{k=n/2}^{n-1} k^{-3/2}(n-k)^{-1/2} = O(c_3^2/n) = O(c_3/\sqrt{n}),$$

$$c_3^{-2} \sum_{k=0}^{c_3^2} (n-k)^{-1/2} = O(1/\sqrt{n}),$$

$$c_3^2 \sum_{k=c_3^2}^{n/2} k^{-3/2}(n-k)^{-1/2} = O\Big(\frac{c_3^2}{\sqrt{n}} \sum_{k=c_3^2}^{n/2} k^{-3/2}\Big) = O(c_3/\sqrt{n}).$$

Thus,

$$W_2 = O(c_3/\sqrt{n}). \tag{85}$$

The statement of the lemma follows from (81), (84) and (85).

We set

$$v_n(t) = \sum_{k=1}^{n-1} e^{-kt^2/2} \varphi_{n-k}(t)\,(1-h(5c_3 t)), \qquad V_n(x) = S^{-1}(v_n(t)).$$

LEMMA 7. $\sup\limits_x |V_n'(x)| = O(c_3/\sqrt{n})$.

Proof. Clearly,

$$V_n'(x) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} v_n(t) e^{-itx}\, dt.$$

Using (79) we obtain

$$\int_{-\infty}^{\infty} e^{-kt^2/2}(1-h(5c_3t))\,|\varphi_{n-k}(t)|\,dt = O\left(c_3^2|\bar{a}_{n-k}|\int_0^{\infty} e^{-kt^2/2}t^3\,dt\right) = O(c_3^2k^{-2}(n-k)^{-1/2}). \tag{86}$$

On the other hand,

$$\int_{-\infty}^{\infty} e^{-kt^2/2}(1-h(5c_3t))\,|\varphi_{n-k}(t)|\,dt = O\left(|\bar{a}_{n-k}|\int_0^{\infty} e^{-kt^2/2}t\,dt\right) = O\left(\frac{1}{k\sqrt{n-k}}\right). \tag{87}$$

From estimates (86) and (87) we obtain an estimate for $V'_n(x)$:

$$V_n'(x) = O\left(\sum_{k=1}^{n-1}\min(c_3^2k^{-2},\,k^{-1})\,(n-k)^{-1/2}\right). \tag{88}$$

Since without loss of generality we can assume that $c_3 < n/2$, then

$$c_3^2\sum_{k=n/2}^{n-1}\frac{1}{k^2(n-k)^{1/2}} = O\left(\frac{c_3^2}{n^{3/2}}\right) = O\left(\frac{c_3}{\sqrt{n}}\right)$$

$$c_3^2\sum_{k=c_3}^{n/2}\frac{1}{k^2(n-k)^{1/2}} = O\left(\frac{c_3^2}{\sqrt{n}}\sum_{k=c_3}^{n/2}k^{-2}\right) = O\left(\frac{c_3}{\sqrt{n}}\right), \tag{89}$$

$$\sum_{n=1}\frac{1}{k(n-k)^{1/2}} = O\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{c_3}\frac{1}{k}\right) = O\left(\frac{\ln c_3}{\sqrt{n}}\right) = O\left(\frac{c_3}{\sqrt{n}}\right).$$

The statement of the lemma follows from (88) and (89).

It is not difficult to see that

$$\bar{\Omega}_{3n}(x) = F_n(x) + S^{-1}\left(\sum_{k=0}^{n-1}f^k(t)\varphi_{n-k}(t) - W_n(x) - V_n(x) - S^{-1}(f^n(t)h(5c_3t))\right).$$

From (52) we have

$$F_n(x) + S^{-1}\left(\sum_{k=0}^{n-1}f^k(t)\varphi_{n-k}(t)\right) = F_n(x).$$

In view of (27),

$$\sup_x\left|\frac{d}{dx}S^{-1}(f^n(t)h(5c_3t))\right| = O\left(\int_0^{\infty}e^{-nt^2/4}\,dt\right) = O\left(\frac{1}{\sqrt{n}}\right). \tag{90}$$

Setting

$$Q_n(x) = W_n(x) + V_n(x) + S^{-1}(f^n(t)h(5c_3t)),$$

we have

$$\bar{\Omega}_{3n}(x) = F_n(x) - Q_n(x).$$

It follows from Lemmas 6 and 7 and estimate (90) that

$$\sup_x|Q_n'(x)| = O(c_3/\sqrt{n}).$$

Clearly, $Q_n(x)$ has bounded variation, and, moreover, $Q_n(-\infty) = 0$.

By employing a modification of the Esseen Theorem (see [7], Theorem 2) we obtain

$$\sup_x|\bar{\Omega}_{3n}(x)| = O\left(\frac{c_3^2}{\sqrt{n}}\right) + O\left(\int_{|t|\le 1/5c_3}\left|\frac{\omega_{3n}(t)}{t}\right|dt\right), \tag{91}$$

where $\omega_{3n}(t) = S(\bar{\Omega}_{3n})$. Clearly,

$$|\omega_{3n}(t)| \leq \sum_{k=0}^{n-1} |q_k(t)\varphi_{n-k}(t)| + |f^n(t)(1-h(5c_3t))|, \quad q_0(t) \equiv 1, \tag{92}$$

From (14) and (79) we have

$$\int_{|t|\leq 1/5c_3} \left|\frac{\varphi_n(t)}{t}\right| dt = O\left(\frac{1}{c_3\sqrt{n}}\right),$$

$$\int_{|t|\leq 1/5c_3} \left|\frac{\varphi_{n-k}(t)q_k(t)}{t}\right| dt = O\left(\frac{c_3^3k}{\sqrt{n-k}} \int_{|t|\leq 1/5c_3} |t|^5 e^{-kt^2/4}dt\right) = O\left(\frac{1}{\sqrt{n-k}}\min\left[\frac{c_3^3}{k^2},\ kc_3^{-3}\right]\right). \tag{93}$$

Moreover,

$$\int_{|t|\leq 1/5c_3} \left|\frac{f^n(t)(1-h(5c_3t))}{t}\right| dt = O\left(c_3^2\int_0^\infty te^{-nt^2/4}dt\right) = O\left(\frac{c_3^2}{n}\right). \tag{94}$$

From (92)-(94) and estimate (69) we obtain

$$\int_{|t|\leq 1/5c_3} \left|\frac{\omega_{3n}(t)}{t}\right| dt = O\left(\sum_{k=1}^{n-1}\min\left[\frac{c_3^3}{k^2},\ kc_3^{-3}\right](n-k)^{-1/2}\right) + O\left(\frac{1}{c_3\sqrt{n}}\right) + O\left(\frac{c_3^2}{n}\right). \tag{95}$$

Without loss of generality we can assume $c_3^2 < n/2$. Therefore,

$$c_3^{-3}\sum_{k=1}^{c_3^2}\frac{1}{\sqrt{n-k}} = O\left(\frac{c_3^{-1}}{\sqrt{n}}\right),$$

$$c_3^3\sum_{k=c_3^2}^{n/2}\frac{1}{k^2\sqrt{n-k}} = O\left(\frac{c_3^3}{\sqrt{n}}\sum_{k=c_3^2}^{n/2}\frac{1}{k^2}\right) = O\left(\frac{c_3}{\sqrt{n}}\right),$$

$$c_3^3\sum_{k=n/2}^{n-1} 1/k^2\sqrt{n-k} = O(c_3^3/n^{3/2}) = O(c_3/\sqrt{n}). \tag{96}$$

From (95) and (96) it follows that

$$\int_{|t|\leq 1/5c_3} \left|\frac{\omega_{3n}(t)}{t}\right| dt = O\left(\frac{c_3}{\sqrt{n}}\right).$$

Substituting this estimate into (91) we obtain

$$\sup_x |\overline{\Omega}_{3n}(x)| = O(c_3^2/\sqrt{n}). \tag{97}$$

Now employing Theorem 2 of [7] and estimates (90) and (94) it is not difficult to show that

$$\sup_x |S^{-1}(f^n(t)(1-h(5c_3t)))| = O(c_3/\sqrt{n} + c_3^2/n). \tag{98}$$

It follows from estimates (97) and (98) that

$$\sup_x |\Omega_{3n}(x)| = O(c_3^2/\sqrt{n}). \tag{99}$$

Using identity (53) it is easy to obtain the representation

$$\mathbf{F}_n(x) = F_n(x) + \Phi_n(x) + R_n(x) + \sum_{i=1}^{3}\Omega_{in}(x), \quad x \geq 0.$$

To complete the proof it is now sufficient to use estimates (60), (65), (66), (70), and (99) and the familiar Berri-Esseen estimate for the difference $F_n(x) - \Phi(x/\sqrt{n})$ (see, for example, [9], §40, Theorem 1).

In conclusion we shall show that the estimate $O(c_3^2 \ln n/\sqrt{n})$ is derived in a much simpler manner.

First note that from (84) we have

$$\sup_x |\Phi_n'(x)| = O(1/\sqrt{n}). \tag{100}$$

Moreover,

$$s_n(t) \equiv S(F_n(x) - \Phi_n(x) - \Phi(x/\sqrt{n})) = \sum_{k=1}^{n-1} (itr_{n-k} + t^2\varphi_{n-k}(t)e^{-kt^2/2}$$

$$+\varphi_n(t)+\sum_{k=1}^{n-1}\left(f_k(t)-e^{-kt^2/2}\right)\varphi_{n-k}(t)+f^n(t)-e^{-nt^2/2}. \tag{101}$$

From (101), identity (54), and estimates (14), and (79) we have

$$\int\limits_{|t|\leq 1/4c_3}\left|\frac{s_n(t)}{t}\right|dt=O\left(\sum_{k=1}^{n-1}\left(|r_{n-k}|k^{-1/2}+\delta_{n-k}k^{-1}+c_3|\bar{a}_{n-k}|k^{-1}\right)+\frac{c_3}{\sqrt{n}}\right). \tag{102}$$

However,

$$\sum_{k=1}^{n-1}|r_{n-k}|k^{-1/2}=O\left(\frac{c_3^2}{\sqrt{n}}\right) \tag{103}$$

(see (61)). Moreover, in view of Lemma 4,

$$\sum_{k=1}^{n-1}\delta_{n-k}k^{-1}=O\left(\frac{c_3^2\ln n}{\sqrt{n}}\right). \tag{104}$$

Finally, from (69)

$$\sum_{k=1}^{n-1}k^{-1}|\bar{a}_{n-k}|=O\left(\frac{\ln n}{\sqrt{n}}\right). \tag{105}$$

It follows from (102)-(105) that

$$\int\limits_{|t|\leq 1/4c_3}\left|\frac{s_n(t)}{t}\right|dt=O\left(\frac{c_3^2\ln n}{\sqrt{n}}\right).$$

Using the above modification of the Esseen theorem

$$F_n(x)-\Phi_n(x)-\Phi(x/\sqrt{n})=O\left(\frac{c_3^2\ln n}{\sqrt{n}}\right).$$

It now remains to use equality (60).

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