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SOME RENEWAL THEOREMS

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1. Introduction. Statement of Results

Let $F(x)$ be a distribution function such that $F(0) = 0$ and let

$$H(x, A) = \sum_{k=0}^{\infty} A^k F_k(x), \qquad A > 0,$$

where $F_k(x)$ is the k-fold convolution of $F(x)$, $k \geqq 1$, and $F_0(x)$ is the degenerate distribution function with jump at 0. Let

$$M_s = \int_{-\infty}^{\infty} x^s \, dF(x), \qquad \mu = M_1.$$

Throughout the following $\int_a^b g(x)\, dF(x)$ is to be interpreted as $\int_{a+}^{b+} g(x)\, dF(x)$.

Clearly, $H(x, A) < \infty$ for $A < 1$ and $x < \infty$. In addition, for this case $H(\infty, A) < \infty$. Let $g(z) = \int_{-\infty}^{\infty} e^{zx}\, dF(x)$. Clearly, when $g(\operatorname{Re} z) < 1/A$,

$$\int_{-\infty}^{\infty} e^{zx}\, dH(x, A) = \sum_{k=0}^{\infty} A^k g^k(z) = \frac{1}{1 - Ag(z)}. \tag{1.1}$$

If $\operatorname{Re} z < 0$ and $\operatorname{Im} z = 0$,

$$\sum_{k=0}^{\infty} A^k g^k(z) < \infty$$

for $1 \leqq A < 1/g(z)$.

On the other hand, for $\operatorname{Re} z < 0$ and $\operatorname{Im} z = 0$,

$$H(x, A) \leqq e^{-zx} \sum_{k=0}^{\infty} A^k g^k(z).$$

Thus,

$$H(x, A) < \infty$$

for $1 \leqq A < 1/g(-\infty) = 1/F(0+)$. Observe that $H(\infty, A) = \infty$ when $A \geqq 1$.

In renewal theory one usually studies the asymptotic behavior of $H(x, 1)$ or $H(x + l, 1) - H(x, 1)$ as $x \to \infty$. However the case where $A \neq 1$ is also of great interest. In particular, in branching processes with random particle life-time there arises a need for an asymptotic representation for $H(x + l, A) - H(x, A)$ as $x \to \infty$ with an estimate for the remainder term which is uniform in A. It should be noted that for fixed $A < 1$ and finite number of moments M_s the asymptotic behavior of $H(x + l, A) - H(x, A)$ will depend on the individual properties of $F(x)$. Results of a collective type may be obtained only when $A \uparrow 1$ sufficiently fast.

This paper studies the asymptotic behavior of $H(x + l, A) - H(x, A)$ under the assumption that $F(x)$ is a lattice distribution or has an absolutely continuous component. We shall say that $F(x)$ is a Δ-lattice distribution if it only increases by jumps at the points $k\Delta$, where k is a non-negative integer and the greatest common divisor of those k for which $F(k\Delta+) - F(k\Delta) > 0$ is equal to 1.

Consider a Δ-lattice $F(x)$ and set

$$f_k = F(k\Delta+) - F(k\Delta), \qquad f(z) = \sum_{k=0}^{\infty} f_k z^k, \qquad f_n(z) = \sum_{k=0}^{n} f_k z^k.$$

Let $\lambda(A)$ and $\lambda_n(A)$ be real non-negative roots of the equations $Af(z) = 1$ and $Af_n(z) = 1$, respectively.

Theorem 1. *If $F(x)$ is Δ-latticed and $M_s < \infty, s > 1$, then*

$$H(n\Delta+, A) - H(n\Delta, A) = \frac{\lambda_n^{-n-1}(A)}{Af'_n(\lambda_n(A))} + o\left(\frac{1}{n^{s-1}}\right) \tag{1.2}$$

uniformly for $A_n \leqq A \leqq 1$ such that

$$n^{1-s}\exp\{n\Delta(1 - A_n)/\mu\} = O(1) \tag{1.3}$$

and

$$H(n\Delta+, A) - H(n\Delta, A) = \frac{\lambda^{-n-1}(A)}{Af'(\lambda(A))} + o\left(\frac{1}{n^{s-1}}\right) \tag{1.4}$$

uniformly for $1 \leqq A \leqq \rho$, where $\rho < 1/f_0$ is such that $\lambda(A)$ is the only root of the equation $Af(z) = 1$ in the disc $|z| \leqq 1$ when $1 \leqq A \leqq \rho$.

Setting $A = 1$ in (1.4), we obtain

$$H(n\Delta+, 1) - H(n\Delta, 1) = \frac{\Delta}{\mu} + o\left(\frac{1}{n^{s-1}}\right). \tag{1.5}$$

This result is slightly weaker than A. O. Gel'fond's in [1], which in our notation can be stated as follows:

$$H(n\Delta+, 1) - H(n\Delta, 1) = \frac{\Delta}{\mu} + \frac{\Delta^2}{\mu^2}\sum_{k \geqq n+1} s_k + O\left(\frac{\log n}{n^s}\right),$$

where $s_k = \sum_{j \geqq k+1} f_j$, if $M_s < \infty$ for $s \geqq 1$.

A. A. Borovkov in [2] obtained an estimate such as (1.5) when $A = 1$, it is true, in the more general case where $F(0) > 0$. The function $H(x)$ in [2] is defined to be $\mathbf{E}\eta_x$, where η_x is the first passage time across the level x. Observe finally that condition (1.3) implies that $1 - A = O(\log n/n)$.

Let us now consider the non-lattice case. Let $\bar{F}_1(x)$ denote the absolutely continuous component of $F(x)$. Let

$$g_y(z) = \int_{-\infty}^{y-} e^{zx}\, dF(x).$$

Let $\Lambda_y(A)$ be a real root of the equation $Ag_y(z) = 1$ and let $\Lambda(A) = \Lambda_\infty(A)$.

Theorem 2. *If* $\bar{F}_1(\infty) > 0$ *and* $M_s < \infty$ *for* $s > 1$, *then, for any* $L > 0$,

$$H(y, A) - H(y - l, A) = \frac{e^{-\Lambda_y(A)(y-l)} - e^{-\Lambda_y(A)y}}{Ag_y'(\Lambda_y(A))\Lambda_y(A)} + o\left(\frac{1}{y^{s-1}}\right), \quad y > 0, \tag{1.6}$$

uniformly for $0 \leqq l \leqq L$ *and* $A_y \leqq A \leqq 1$ *for which*

$$y^{1-s}\exp\{y(1 - A_y)\mu\} = O(1)$$

and

$$H(y, A) - H(y - l, A) = \frac{e^{-\Lambda(A)(y-l)} - e^{-\Lambda(A)y}}{Ag'(\Lambda(A))\Lambda(A)} + o\left(\frac{1}{y^{s-1}}\right), \quad y > 0, \tag{1.7}$$

uniformly for $0 \leqq l \leqq L$ *and* $1 \leqq A \leqq R$, *where* $R < \min\{F(0+)^{-1}, (1 - \bar{F}_1(\infty)/2)^{-1}\}$ *is such that* $\Lambda(A)$ *is the only root of the equation* $Ag(z) = 1$ *in the half-plane* $\operatorname{Im} z \leqq 0$ *when* $1 \leqq A \leqq R$.

Corollary.[1] *If* $\bar{F}_1(\infty) > 0$ *and* $M_s < \infty$ *for* $s > 1$, *then*

$$H(y, 1) - H(y - l, 1) = \frac{l}{\mu} + o\left(\frac{1}{y^{s-1}}\right) \tag{1.8}$$

uniformly for l *in any finite interval* $0 \leqq l \leqq L$.

To prove this, one merely has to set $A = 1$ in (1.7) and observe that $\Lambda(1) = 0$.

We point out that, generally speaking, Theorem 2 ceases to be true even for $A = 1$ if one assumes $F(x)$ not to be Δ-latticed, and at the same time omits the requirement $\bar{F}_1(\infty) > 0$.

Indeed, suppose $M_3 < \infty$ and (1.8) holds. Then

$$\Delta_l(x) \equiv H(x + l, 1) - H(x, 1) - l/\mu = o(x^{-2})$$

and it is therefore absolutely integrable. Thus

$$\lim_{z \to it, \operatorname{Im} z < 0} \int_{-\infty}^{\infty} \Delta_l(x)\, e^{zx}\, dx = \int_{-\infty}^{\infty} e^{itx}\Delta_l(x)\, dx,$$

[1] This result was also proved in [7] under the conditions $s \geqq 2$ and $\mu > 0$ but without the restriction that $F(0) = 0$.

and

$$\left|\int_{-\infty}^{\infty} e^{itx}\Delta_l(x)\,dx\right| < \infty \tag{1.9}$$

uniformly in t.

On the other hand,

$$\lim_{z\to it, \operatorname{Im} z<0} \int_{-\infty}^{\infty} (H(x+l,1) - H(x,1))\, e^{zx}\,dx = \frac{1-e^{-itl}}{it(1-g(it))},$$

and hence

$$\varphi_l(t) \equiv \int_{-\infty}^{\infty} e^{itx}\Delta_l(x)\,dx = \frac{1-e^{-itl}}{it(1-g(it))} + \frac{l}{\mu i t}.$$

Now let $g(z) = e^{z\alpha}\psi(z)$, where $\psi(z)$ is the generating function of a Δ-lattice distribution with $\Delta = 1$ and α is a positive irrational for which there exists a sequence of integers k_j such that $\lim_{j\to\infty} k_j\{(2k_j+1)\alpha\} = 0$. Clearly

$$\lim_{j\to\infty} k_j(g(2\pi(2k_j+1)i) - 1) = 0.$$

On the other hand, $1 - \exp\{-2\pi(2k_j+1)li\} \equiv 2$ when $l = \frac{1}{2}$. Therefore,

$$\lim_{j\to\infty} |\varphi_{1/2}(2\pi(2k_j+1))| = \infty,$$

and this contradicts (1.9).

Let $g_i(z) = \int_{0-}^{\infty} e^{zx}\,d\bar{F}_i(x)$, $i = 1, 2$, where $\bar{F}_2(x) = F(x) - \bar{F}_1(x)$.

Theorem 3. *Let* $\bar{F}_1(\infty) > 0$, $\int_{0-}^{\infty} e^{hx}\,dF(x) < \infty$ *for* $0 \leqq h < h_0$ *and let* $\Lambda(A)$ *be the only zero of* $Ag(z) - 1$ *in the half-plane* $\operatorname{Re} z \leqq h_1 < h_0$, *where* $h_1 > 0$ *and* $g_2(h_1) < 1$, *when* $1/g(h_1) < A \leqq 1$. *Then, for any* $\varepsilon > 0$ *and* $L > 0$,

$$H(x,A) - H(x-l,A) = \frac{e^{-\Lambda(A)(x-l)} - e^{-\Lambda(A)x}}{Ag'(\Lambda(A))\Lambda(A)} + O(e^{-h_1 x}) \tag{1.10}$$

uniformly for $1/g(h_1) + \varepsilon \leqq A \leqq 1$ *and* $0 < l \leqq L$.

A similar result is derived in [3], wherein it is assumed that $\bar{F}_2(x) \equiv 0$ and $F'(x)$ is integrable to some power of $p > 1$.

The proof of each of the three theorems is based on the same method. We shall therefore give a detailed proof of Theorem 1 only; in proving Theorems 2 and 3 we shall concentrate our attention on what is new as compared to the lattice case while referring the reader to corresponding places in the proof of Theorem 1 at the first opportunity.

2. Proof of Theorem 1

Without loss of generality we may assume that $\Delta = 1$. In the following we shall denote by $C_n(f(z))$ the coefficient of z^n in the Taylor series of a function $f(z)$ which is analytic in some neighborhood of 0. Let $u_n = H(n+, A)$

$-H(n, A)$. Clearly,

$$u_n = C_n(1/(1 - Af(z))).$$

On the other hand, when $k \leqq n$

$$C_k(1/(1 - Af(z))) = C_k(1/(1 - Af_n(z))).$$

Thus

$$u_k = C_k(1/(1 - Af_n(z))), \qquad k \leqq n. \tag{2.1}$$

Let $\mu_n(z) = f'_n(z)$ and $h_n = [(s - 1)\log n + c]/n$, where c is a positive constant to be chosen later.

Let $h > 0$. It is not hard to show that for any distribution function we have

$$\int_{1/h}^{y} e^{hx}\, dF(x) < e\left(1 - F\left(\frac{1}{h}\right)\right) + h\int_{1/h}^{y} (1 - F(x))\, e^{hx}\, dx, \qquad y > \frac{1}{h}. \tag{2.2}$$

It is easy to see that

$$\int_{1/h}^{y} (1 - F(x))\, e^{hx}\, dx < L_s \frac{e^{hy}}{hy^s}\int_{1/h}^{\infty} x^s\, dF(x), \tag{2.3}$$

where $L_s = 1 + s(s + 1)^{s+1} e^{-s}$ (see [5], p. 217). Further,

$$1 - F\left(\frac{1}{h}\right) < \left(\frac{s}{ey}\right)^s e^{hy}\int_{1/h}^{\infty} x^s\, dF(x). \tag{2.4}$$

From (2.2)–(2.4) follows

$$\int_{1/h}^{y} e^{hx}\, dF(x) < \bar{L}_s \frac{e^{hy}}{y^s}\int_{1/h}^{\infty} x^s\, dF(x), \qquad y > \frac{1}{h}, \tag{2.5}$$

where $\bar{L}_s$ is some constant.

Let us show that

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty}\sup_{z\in U_n(\varepsilon)} |\mu_n(z) - \mu| = 0, \tag{2.6}$$

where $U_n(\varepsilon) = \{z : 1 \leqq |z| \leqq e^{h_n}, |\arg z| \leqq \varepsilon\}$. Indeed,

$$\begin{aligned} |\mu_n(z) - \mu| \leqq &\int_0^{1/h_n} |z^{x-1} - 1|x\, dF(x) + \int_{1/h_n}^{n} |z|^{x-1} x\, dF(x) \\ &+ \int_{1/h_n}^{\infty} x\, dF(x). \end{aligned} \tag{2.7}$$

But $|z|^x < e^{h_n x}$ for $z \in U_n(\varepsilon)$. Therefore by (2.5),

$$\sup_{z\in U_n(\varepsilon)}\int_{1/h_n}^{n} |z|^x x\, dF(x) = O\left(\frac{e^{h_n n}}{n^{s-1}}\int_{1/h_n}^{\infty} x^s\, dF(x)\right). \tag{2.8}$$

On the other hand,

$$|z^{x-1} - 1| \leqq x|z - 1|\, |z|^x \tag{2.9}$$

when $|z| \geqq 1$ and $x \geqq 1$. Further,

$$(2.10) \qquad |z - 1| < |x - \exp\{i \arg z\}| + |1 - \exp\{i \arg z\}|.$$

Using estimates (2.9) and (2.10), we obtain

$$(2.11) \qquad \int_0^M |z^{x-1} - 1| x \, dF(x) < e((e^{h_n} - 1) + \varepsilon) M^2, \qquad z \in U_n(\varepsilon).$$

At the same time,

$$\int_M^{1/h_n} |z^{x-1} - 1| x \, dF(x) = O\left(\int_M^\infty x \, dF(x)\right).$$

Setting $M = (h_n + \varepsilon)^{-1/3}$, we have

$$(2.12) \quad \int_0^{1/h_n} |z^{x-1} - 1| x \, dF(x) = O((h_n + \varepsilon)^{1/3}) + O\left(\int_{(h_n+\varepsilon)^{-1/3}}^\infty x \, dF(x)\right), \qquad z \in U_n(\varepsilon).$$

Formulas (2.7), (2.8) and (2.12) easily imply (2.6).

Let us estimate the difference $\lambda_n(A) - \lambda(A)$ for $A \geqq 1$. First of all,

$$(2.13) \qquad \lambda_n(A) \geqq \lambda(A),$$

since $f(z) \geqq f_n(z)$ for non-negative z.

Clearly, $\lambda(A) \leqq 1$ when $A \geqq 1$ and

$$(2.14) \qquad \int_{0-}^n (\lambda_n^x(A) - \lambda^x(A)) \, dF(x) = \int_n^\infty \lambda^x(A) \, dF(x).$$

In this case, $\lambda_n^x(A) - \lambda^x(A) \geqq x(\lambda_n(A) - \lambda(A))\lambda^x(A)$. Therefore,

$$(2.15) \qquad (\lambda_n(A) - \lambda(A)) \int_{0-}^n x \lambda^x(A) \, dF(x) \leqq \int_n^\infty \lambda^x(A) \, dF(x).$$

Hence

$$(2.16) \qquad \lambda_n(A) - \lambda(A) = o\left(\frac{\lambda^n(A)}{n^s}\right)$$

uniformly for $1 \leqq A \leqq \rho$.

Indeed, there exists an $\omega > 0$ such that $\int_{0-}^\omega x \, dF(x) > \mu/2$. Therefore, for sufficiently large n and $A \leqq \rho$,

$$(2.17) \qquad \int_{0-}^n x \lambda^x(A) \, dF(x) > \lambda^\omega(\rho) \int_{0-}^\omega x \, dF(x) > \mu \lambda^\omega(\rho)/2,$$

since $\lambda(A)$ is non-increasing with increasing A. From (2.15) and (2.17) results (2.16).

If $A < 1$, $\lambda(A)$ is generally speaking not well-defined. In that case, we estimate the difference $\lambda_n(A) - 1$. It is not hard to see that

$$\int_{0-}^n (\lambda_n^x(A) - 1) \, dF(x) = \int_n^\infty dF(x) + \frac{1 - A}{A}.$$

Further,

$$\lambda_n^x(A) - 1 \geqq (\lambda_n(A) - 1)x.$$

Therefore,

$$(2.18) \qquad (\lambda_n(A) - 1)\int_{0-}^{n} x\, dF(x) \leqq \int_n^\infty dF(x) + \frac{1-A}{A}.$$

Condition (1.3) implies the existence of a constant c_1 such that

$$\frac{n(1-A)}{\mu} - (s-1)\log n < c_1,$$

i.e.,

$$(2.19) \qquad \frac{1-A}{\mu} < \frac{(s-1)\log n + c_1}{n}.$$

Then by virtue of (2.19),

$$(2.20) \qquad 1/A = 1 + O(1-A) = 1 + O(n^{-1}\log n).$$

From (2.18)–(2.20) we conclude that

$$(\lambda_n(A) - 1)\left(\mu + o\left(\frac{1}{n^{s-1}}\right)\right) \leqq \frac{\mu(s-1)}{n}\log n + \frac{\mu c_1}{n} + O\left(\frac{\log^2 n}{n^2}\right) + o\left(\frac{1}{n^s}\right).$$

Thus

$$(2.21) \qquad \lambda_n(A) - 1 < \frac{(s-1)\log n + c_1}{n} + o\left(\frac{1}{n}\right),$$

and therefore, for sufficiently large n, $\lambda_n(A)$ will lie inside a circle of radius e^{h_n} if we set $c = 2c_1$.

Let us now show that, for sufficiently large n, $\lambda_n(A)$ is the only root of the equation $Af_n(z) = 1$ in the disc $|z| \leqq e^{h_n}$ for all $A \leqq 1$ satisfying condition (1.3) and $1 \leqq A \leqq \rho$.

First of all, because of (2.6) and the equicontinuity of $f'_n(z)$ in the unit circle we have

$$(2.22) \qquad Af_n(z) - 1 = A\mu_n(\lambda_n(A))(z - \lambda_n(A)) + o(z - \lambda_n(A))$$

uniformly in n and in the admissible values of A. Therefore, there exists an ε_0 such that $Af_n - 1$ has no other zeros in the disc $|z - \lambda_n(A)| < \varepsilon_0$ apart from $\lambda_n(A)$ for all $A \in \mathfrak{U}_n$, where $\mathfrak{U}_n$ is the set of all admissible values of A for given n.

Clearly,

$$|\mu_n(z)| \leqq \int_0^{1/h_n} x|z|^{x-1}\, dF(x) + \int_{1/h_n}^{n} x|z|^{x-1}\, dF(x).$$

But

$$\int_0^{1/h_n} x|z|^{x-1}\, dF(x) < e\mu$$

for $|z| \leqq e^{h_n}$. Using in addition the estimate (2.8), we conclude that

$$\sup_{n,|z| \leqq e^{h_n}} |\mu_n(z)| < \infty. \tag{2.23}$$

Therefore,

$$\lim_{n\to\infty} \sup_{\substack{1 \leqq r \leqq e^{h_n} \\ 0 \leqq \varphi \leqq 2\pi}} |f_n(re^{i\varphi}) - f_n(e^{i\varphi})| = 0. \tag{2.24}$$

On the other hand,

$$\lim_{n\to\infty} \sup_{0 \leqq \varphi \leqq 2\pi} |f_n(e^{i\varphi}) - f(e^{i\varphi})| = 0. \tag{2.25}$$

But, for any $\varepsilon > 0$,

$$m(\varepsilon) = \inf_{1 \leqq A \leqq \rho} \inf_{\varepsilon \leqq |\varphi| \leqq \pi} |Af(e^{i\varphi}) - 1| > 0, \tag{2.26}$$

since otherwise there would exist $1 \leqq A_0 \leqq \rho$ and $\varepsilon \leqq \varphi_0 \leqq \pi$ such that $A_0 f(e^{i\varphi_0}) = 1$ and this would contradict the hypothesis of the theorem.

From (2.24)–(2.26) it follows that, for sufficiently large n,

$$\inf_{\varepsilon \leqq |\varphi| \leqq \pi} |Af_n(r\, e^{i\varphi}) - 1| > m(\varepsilon)/2 \tag{2.27}$$

for all $1 \leqq r \leqq e^{h_n}$ and $A \in \mathfrak{U}_n$.

On the basis of all of the above, we can assert that if $Af_n(z) - 1$ has a zero $\tilde{\lambda}_n(A)$ in the disc $|z| \leqq e^{h_n}$ differing from $\lambda_n(A)$, then $\tilde{\lambda}_n(A)$ will lie outside the region $\{z : 1 \leqq z \leqq e^{h_n}, |\arg z| \geqq \varepsilon\}$ for n sufficiently large.

Observe that

$$Af_n(z) - 1 \geqq A\mu_n(\lambda_n(A))(z - \lambda_n(A))$$

for real $z > \lambda_n(A)$. This inequality plus the equicontinuity of $f_n(z)$ in the disc $|z| \leqq e^{h_n}$ (which is assured by (2.23)) imply the existence of a positive ε_1 such that $|Af_n(z) - 1| > \varepsilon_1$, $A \in \mathfrak{U}_n$, provided $z \in U_n(\varepsilon_1)$ and $\lambda_n(A) \leqq 1 - \varepsilon_0/2$. Setting $\varepsilon = \varepsilon_1$ in (2.27), we arrive at the conclusion that, for n sufficiently large, $\tilde{\lambda}_n(A)$ cannot lie in the annulus $1 \leqq |z| \leqq e^{h_n}$ if $\lambda_n(A) \leqq 1 - \varepsilon_0/2$.

But if $\lambda_n(A) \geqq 1 - \varepsilon_0/2$, there exists an ε_2 such that $\tilde{\lambda}_n(A)$ does not lie in the region $1 - \varepsilon_0/2 \leqq |z| \leqq e^{h_n}$, $|\arg z| < \varepsilon_2$, and therefore not in the annulus $1 \leqq |z| \leqq e^{h_n}$.

Observe that $|\tilde{\lambda}_n(A)| > |\lambda_n(A)|$. Therefore, when $A \leqq 1$, $\tilde{\lambda}_n(A)$ cannot lie in the disc $|z| \leqq e^{h_n}$.

Consider now the case $A \geqq 1$. Let

$$m = \inf_{1 \leqq A \leqq \rho} \inf_{z \in V(A)} |1 - Af(z)|,$$

where $V(A) = \{z : |z| \leqq 1, |z - \lambda(A)| \geqq \varepsilon_0/2\}$. It is not hard to see that m is positive since otherwise there would exist an A_0, $1 \leqq A_0 \leqq \rho$, such that $A_0 f(z_0) = 1$, where $z_0 \neq \lambda(A)$ and $|z_0| \leqq 1$. For sufficiently large n and $|z| \leqq 1$, $|f_n(z) - f(z)| < m/2\rho$ and hence $|\tilde{\lambda}_n(A) - \lambda(A)| < \varepsilon_0/2$ for all $1 \leqq A \leqq \rho$. By (2.16), $\lambda_n(A) - \lambda(A) < \varepsilon_0/2$ for n sufficiently large. On the other hand, $\tilde{\lambda}_n(A)$

$\notin \{z:|z - \lambda_n(A)| < \varepsilon_0\}$. Therefore, $|\tilde{\lambda}_n(A) - \lambda(A)| > \varepsilon_0/2$. Thus, $\tilde{\lambda}_n(A) \notin \{z:|z| \leqq e^{h_n}\}$ also when $1 \leqq A \leqq \rho$.

In what follows, we shall assume that the c occurring in the definition of h_n has been chosen so that $\lambda_n(A)$ lies in the disc $|z| < e^{h_n - 1/n}$ for n sufficiently large.

Let γ_n be a circle of radius $r_n = e^{h_n}$. It is not hard to see that

$$u_k = \frac{\lambda_n^{-k-1}(A)}{Af'_n(\lambda_n(A))} + \frac{1}{2\pi i}\int_{\gamma_n} \frac{z^{-k-1}}{1 - Af_n(z)} dz, \qquad k \leqq n, \tag{2.28}$$

for n sufficiently large. Clearly,

$$\frac{1}{2\pi i}\int_{\gamma_n} \frac{z^{-k-1}\,dz}{1 - Af_n(z)} = \frac{1}{2\pi r_n^k}\int_{-\pi}^{\pi} \frac{e^{-ikt}\,dt}{1 - Af_n(r_n e^{it})}. \tag{2.29}$$

By (2.6),

$$f_n(r_n) - f_n(\lambda_n(A)) - \mu(r_n - \lambda_n(A)) = o(r_n - \lambda_n(A)) \tag{2.30}$$

uniformly for $A \in \mathfrak{U}_n$. Set

$$f_{n1}(z) = \int_{0-}^{1/h_n} z^x\,dF(x), \qquad f_{n2}(z) = \int_{1/h_n}^{n} z^x\,dF(x),$$

$$\varphi_n(z) = A(f_{n1}(z) - f_{n1}(r_n) - f'_{n1}(r_n)(z - r_n)),$$

$$\psi_n(z) = 1 - Af_{n1}(r_n) - Af'_{n1}(r_n)(z - r_n).$$

The following identity holds:

$$\frac{1}{1 - Af_n(z)} - \frac{1}{\psi_n(z)} = \frac{\varphi_n(z) + Af_{n2}(z)}{(1 - Af_n(z))\psi_n(z)}. \tag{2.31}$$

Let $\gamma_n(\varepsilon) = \gamma_n \cap U_n(\varepsilon)$ and let $\bar{\gamma}_n(\varepsilon)$ be the complement of $\gamma_n(\varepsilon)$ with respect to γ_n. By (2.31),

$$\int_{\gamma_n} \frac{z^{-k-1}\,dz}{1 - Af_n(z)} = \sum_{j=1}^{4} I_j(n, \varepsilon, k), \tag{2.32}$$

where

$$I_1 = \int_{\gamma_n} \psi_n^{-1}(z) z^{-k-1}\,dz, \qquad I_2 = \int_{\gamma_n(\varepsilon)} \frac{\varphi_n(z) + Af_{n2}(z)}{\psi_n(z)(1 - Af_n(z))z^{k+1}} dz,$$

$$I_3 = -\int_{\bar{\gamma}_n(\varepsilon)} \psi_n^{-1}(z) z^{-k-1}\,dz, \qquad I_4 = \int_{\bar{\gamma}_n(\varepsilon)} \frac{z^{-k-1}}{1 - Af_n(z)} dz.$$

For $|b| < c^2|a|$,

$$\int_{|z|=c^2} \frac{z^{-n}\,dz}{az + b} = 0, \qquad n > 0.$$

By virtue of (2.8) and (2.30), $1 - Af_{n1}(r_n) < 0$ for sufficiently large n. On the other hand, $1 - Af_{n1}(r_n) + Af'_{n1}(r_n)r_n \geqq 1 - Af_0$. Taking into account all of the above, one can easily see that $I_1 = 0$ for sufficiently large n.

Let us now consider I_2. We first estimate the variation of $\varphi_n(z)/(1 - Af_n(z))\psi_n(z)$ on $\gamma_n(\varepsilon)$, defining this to be the sum of the variations of the real and imaginary parts. Some simple computations lead to the estimate

$$\operatorname*{var}_{z \in \gamma_n(\varepsilon)} \frac{\varphi_n(z)}{(1 - Af_n(z))\psi_n(z)} \leqq \sqrt{2} \int_{\gamma_n(\varepsilon)} \left(\left| \frac{\psi_n'(z)\varphi_n(z)}{(1 - Af_n(z))\psi_n^2(z)} \right| + \left| \frac{Af_n'(z)\varphi_n(z)}{(1 - Af_n(z))^2\psi_n(z)} \right| + \left| \frac{\varphi_n'(z)}{(1 - Af_n(z))\psi_n(z)} \right| \right) dl, \tag{2.33}$$

where dl is differential of arc along $\gamma_n(\varepsilon)$. Indeed, let $\omega_n(z) = \varphi_n(z)/(1 - Af_n(z))\psi_n(z)$. Clearly,

$$\left(\left| \frac{d}{dl} \operatorname{Re} \omega_n(z) \right| + \left| \frac{d}{dl} \operatorname{Im} \omega_n(z) \right| \right)^2 \leqq 2\left(\left| \frac{d}{dl} \operatorname{Re} \omega_n(z) \right|^2 + \left| \frac{d}{dl} \operatorname{Im} \omega_n(z) \right|^2 \right) = 2\left| \frac{d}{dz} \omega_n(z) \right|^2 .$$

On the other hand,

$$\operatorname*{var}_{z \in \gamma_n(\varepsilon)} \omega_n(z) \leqq \int_{\gamma_n(\varepsilon)} \left(\left| \frac{d}{dl} \operatorname{Re} \omega_n(z) \right| + \left| \frac{d}{dl} \operatorname{Im} \omega_n(z) \right| \right) dl.$$

These two inequalities easily imply (2.33).

Now

$$|1 - Af_n(z)|^2 = |1 - Af_n(r_n)|^2 + A^2|f_n(r_n) - f_n(z)|^2 + 2A(1 - Af_n(r_n)) \operatorname{Re} (f_n(r_n) - f_n(z)).$$

By (2.6) there exist an ε_1 and n_1 such that, for $n > n_1$,

$$|f_n(z) - f_n(r_n) - \mu(z - r_n)| < \frac{\mu}{16}|z - r_n|$$

for $z \in U_n(\varepsilon_1)$. On the other hand, for ε sufficiently small, $|\operatorname{Re}(z - r_n)| < |z - r_n|^2$ for $z \in \gamma_n(\varepsilon)$. Therefore, there exists an ε_2 such that, for $z \in \gamma_n(\varepsilon_2)$, $n > n_1$,

$$|1 - Af_n(z)|^2 > |1 - Af_n(r_n)|^2 + A^2|f_n(r_n) - f_n(z)| - \frac{A\mu}{4}|z - r_n|\,|1 - Af_n(r_n)|.$$

But, for ε sufficiently small,

$$|f_n(z) - f_n(r_n)|^2 > \frac{3\mu^2}{4}|z - r_n|^2, \qquad z \in \gamma_n(\varepsilon).$$

Hence, there exist an n_0 and ε_0 such that, for $z \in \gamma_n(\varepsilon_0)$, $n > n_0$,

$$|1 - Af_n(z)| > \frac{1}{2}|1 - Af_n(r_n)| + \frac{A\mu}{4}|z - r_n|. \tag{2.34}$$

In what follows we shall assume that n_0 and ε_0 have been chosen so that, for

$z \in \gamma_n(\varepsilon_0)$, $n > n_0$, the following inequality holds:

$$|\psi_n(z)| > \frac{1}{2}|1 - Af_{n1}(r_n)| + \frac{A\mu}{4}|z - r_n|, \tag{2.35}$$

the proof of which is similar to that of inequality (2.34).

In consequence of (2.5),

$$f_{n2}(r_n) = o(n^{-1}). \tag{2.36}$$

For sufficiently large n,

$$r_n - \lambda_n(A) > \tfrac{3}{4}n \tag{2.37}$$

and therefore by (2.30),

$$Af_n(r_n) - 1 > A\mu/2n. \tag{2.38}$$

In addition, (2.30), (2.37) and (2.38) imply that

$$Af_{n1}(r_n) - 1 > A\mu/2n \tag{2.39}$$

for n sufficiently large. Now

$$|f''_{n1}(z)| < eM_2, \quad s \geqq 2, \quad |f''_{n1}(z)| < eM_s n^{2-s}, \quad 1 < s < 2. \tag{2.40}$$

Therefore,

$$\varphi_n(z) = O(|z - r_n|^2 n^{\max(0,2-s)}), \tag{2.41}$$

$$\varphi'_n(z) = O(|z - r_n| n^{\max(0,2-s)}). \tag{2.42}$$

From (2.34)–(2.36), (2.38) and (2.39) we conclude that

$$\int_{|t|<\varepsilon_0} \frac{|f_{n2}(r_n e^{it})|\, dt}{\psi_n(r_n e^{it})(1 - Af_n(r_n e^{it}))} = o\left(\frac{1}{n}\int_0^{\varepsilon_0} \frac{dt}{(n^{-1} + t)^2}\right) = o(1) \tag{2.43}$$

uniformly for $A \in \mathfrak{U}_n$. Similarly, on this occasion taking (2.41) and (2.42) into consideration, we obtain, on setting $\nu(s) = \max(-1, 1 - s)$,

$$\begin{aligned}
&n^{-1}\int_{\gamma_n(\varepsilon_0)} \left|\frac{\psi'_n(z)\varphi_n(z)}{(1 - Af_n(z))\psi_n^2(z)}\right| dl = O\left(n^{\nu(s)}\int_0^{\varepsilon_0} \frac{t^2\, dt}{(n^{-1} + t)^3}\right) = O(n^{\nu(s)} \log n),\\
&n^{-1}\int_{\gamma_n(\varepsilon_0)} \frac{|f'_n(z)\varphi_n(z)|\, dl}{|\psi_n^2(z)(1 - Af_n(z))|} = O\left(n^{\nu(s)}\int_0^{\varepsilon_0} \frac{t^2\, dt}{(n^{-1} + t)^3}\right) = O(n^{\nu(s)} \log n),\\
&n^{-1}\int_{\gamma_n(\varepsilon_0)} \frac{|\varphi'_n(z)|\, dl}{|(1 - Af_n(z))\psi_n(z)|} = O\left(n^{\nu(s)}\int_0^{\varepsilon_0} \frac{t\, dt}{(n^{-1} + t)^2}\right) = O(n^{\nu(s)} \log n).
\end{aligned} \tag{2.44}$$

From (2.33) and (2.44) we deduce using the familiar estimate for Fourier coefficients that

$$\int_{|t|<\varepsilon_0} \frac{\varphi_n(r_n e^{it})\, e^{-int}\, dt}{(1 - Af_n(r_n e^{it}))\psi_n(r_n e^{it})} = O(n^{\nu(s)} \log n) = o(1). \tag{2.45}$$

In a similar way we can derive the estimates

$$(2.46)\qquad \begin{aligned} \int_{\varepsilon_0 \leqq |t| \leqq \pi} \frac{e^{-int}\,dt}{1 - Af_n(r_n e^{it})} &= O\left(\frac{1}{n}\right), \\ \int_{\varepsilon_0 \leqq |t| \leqq \pi} \frac{e^{-int}\,dt}{\psi_n(r_n e^{it})} &= O\left(\frac{1}{n}\right). \end{aligned}$$

From (2.28), (2.29), (2.32), (2.43), (2.45), (2.46) and the relation $I_1 = 0$, it follows that (1.2) holds uniformly for $A \in \mathfrak{U}_n$. To prove the second part of the theorem, we make use of the inequality

$$(2.47)\qquad \begin{aligned} |f'_n(\lambda_n(A)) - f'(\lambda(A))| &< \int_0^n |\lambda_n^x(A) - \lambda^x(A)| x\,dF(x) \\ &+ \int_n^\infty x\lambda^x(A)\,dF(x) < (\lambda_n^{n-1}(A) + 1)(\lambda_n(A) - \lambda(A)) \int_0^n x^2\,dF(x) \\ &+ \lambda^n(A) \int_n^\infty x\,dF(x). \end{aligned}$$

By virtue of (2.16),

$$(2.48)\qquad \lambda_n^n(A) = O(1)$$

when $1 \leqq A \leqq \rho$. Further,

$$(2.49)\qquad \int_0^n x^2\,dF(x) = o(n).$$

Using (2.16) again, we conclude from (2.47)–(2.49) that

$$(2.50)\qquad f'_n(\lambda_n(A)) - f'(\lambda(A)) = o(\lambda^n(A)/n^{s-1}).$$

On the other hand, by (2.16)

$$(2.51)\qquad \lambda_n^{-n}(A) - \lambda^{-n}(A) = o(1/n^{s-1}).$$

Equation (1.4) easily follows from (1.2) and the estimates (2.50) and (2.51). The proof of Theorem 1 is now complete.

3. Proof of Theorems 2 and 3

Let $F^{(y)}(x) = F(x)$ for $x \leqq y$ and $F^{(y)}(x) = F(y)$ for $x > y$. Let

$$H_y(x, A) = \sum_{k=0}^{\infty} A^k F_k^{(y)}(x),$$

where $F_k^{(y)}(x)$ stands for the k-fold convolution of $F^{(y)}(x)$. For $x \leqq y$ and $l > 0$,

$$(3.1)\qquad H(x, A) - H(x - l, A) = H_y(x, A) - H_y(x - l, A).$$

Set

$$F_{1y}(x) = \int_{A_x \cap (-\infty, y)} d\bar{F}_1(u),$$

where $A_x = \{u: \bar{F}'_1(u) < \bar{L} < \infty,\ u \leqq \min(x, M)\}$, $M < \infty$ and $\bar{L}$ is chosen in such a way that $F_{1M}(\infty) > \frac{1}{2}\bar{F}_1(\infty)$. Further, let $F_{2y}(x) = F^{(y)}(x) - F_{1y}(x)$ and $g_{iy}(z) = \int_{0-}^{\infty} e^{zx}\, dF_{iy}(x)$, $i = 1, 2$. The following identity is easy to verify:

$$(3.2)\qquad \frac{1}{1 - Ag_y(z)} = \frac{Ag_{1y}(z)}{(1 - Ag_y(z))(1 - Ag_{2y}(z))} + \frac{1}{1 - Ag_{2y}(z)}.$$

If y and $y - l$ are points of continuity of $H_y(u, A)$, then for $\sigma = \operatorname{Re} z < \Lambda_y(A)$,

$$(3.3)\qquad H_y(y, A) - H_y(y - l, A) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma - iT}^{\sigma + iT} \frac{e^{-z(y-l)} - e^{-zy}}{z(1 - Ag_y(z))}\, dz.$$

Indeed, it is not hard to see that

$$\int_{0-}^{\infty} (H_y(x, A) - H_y(x - l, A))\, e^{zx}\, dx = e^{zl} \int_{-\infty}^{\infty} (H_y(x + l, A)$$

$$- H_y(x, A))\, e^{zx}\, dx = \frac{e^{zl} - 1}{z(1 - Ag_y(z))}, \qquad \operatorname{Re} z < \Lambda_y(A).$$

To arrive at (3.3), one merely has to apply the formula for the inverse Laplace transform (see, for example, [6], Theorem 7.6a).

It is not hard to see that a point of continuity of $H(x, A)$ is at the same time a point of continuity of $H_y(x, A)$ if $x \leqq y$. Therefore, (3.3) holds if $y - l$ and y are points of continuity of $H(u, A)$.

For y sufficiently large, $\Lambda_y(A) < [(s - 1)\log y + c_1]/y$, where c_1 is some positive constant. This is proved in exactly the same way as the corresponding assertion for $\lambda_n(A)$ in the lattice case (see the proof of Theorem 1).

In addition, for y sufficiently large, $\Lambda_y(A)$ is the only root of the equation $Ag_y(z) = 1$ in the half-plane $\operatorname{Re} z \leqq [(s - 1)\log y + c]/y$ for all admissible values of A, where c is a constant greater than c_1.

To prove this statement, one has to use the fact that, for any $\varepsilon > 0$,

$$g_{1y}(\sigma + it) = \int_{0-}^{M} e^{(\sigma + it)x}\, dF_{1M}(x) \to 0, \qquad y > M,$$

as $t \to \infty$ uniformly for $\sigma \leqq \varepsilon$. Since $g_y(z) = g_{1y}(z) + g_{2y}(z)$ and

$$(3.4)\qquad |g_{2y}(z)| \leqq \left| \int_{0}^{\infty} e^{zx}\, dF_{2\infty}(x) \right| < \gamma = 1 - \tfrac{1}{2}\bar{F}_1(\infty)$$

for $\operatorname{Re} z \leqq 0$, one can find a δ_0, Y_0 and K_0 such that

$$\inf_{\substack{\operatorname{Re} z \leqq \sigma(y) \\ |\operatorname{Im} z| > K_0}} |1 - Ag_y(z)| > \delta_0, \qquad \sigma(y) = [(s - 1)\log y + c]/y,$$

for all $y > Y_0$ and $A \leqq \gamma^{-1}$. In going from $\operatorname{Re} z \leqq 0$ to $\operatorname{Re} z \leqq \sigma(y)$, one has to make use of (2.5).

On the other hand, there exists a negative K_1 such that

$$\inf_{y}\ \inf_{A < \gamma^{-1}}\ \inf_{\operatorname{Re} z < K_1} |1 - Ag_y(z)| > 0.$$

Thus the problem reduces to establishing the uniqueness of $\Lambda_y(A)$ in the rectangle $K_1 \leqq \operatorname{Re} z \leqq [(s-1)\log y + c]/y$, $|\operatorname{Im} z| \leqq K_0$. This fact can be proved in exactly the same way as the uniqueness of $\lambda_n(A)$ in the disc $|z| \leqq e^{h_n}$. The reduction to a rectangle is necessary because in proving uniqueness one has to make use of compactness.

In consequence of (3.4) and (2.5), $|1 - Ag_{2y}(z)| > \delta_0 > 0$ when $A \leqq \gamma^{-1}$ and $\operatorname{Re} z \leqq [(s-1)\log y + c]/y$ providing y is sufficiently large. Hence it follows particularly that $1 - Ag_{2y}(z)$ has no zeros for this range of values of A and z.

Applying the residue theorem and (3.2), we obtain from (3.3),

$$
\begin{aligned}
H_y(y, A) - H_y(y-l, A) = e^{-\Lambda_y(A)y}\, & \frac{e^{\Lambda_y(A)l} - 1}{Ag'(\Lambda_y(A))\Lambda_y(A)} \\
& + \frac{A}{2\pi i}\int_{\Gamma_y} \frac{(e^{-z(y-l)} - e^{-zy})g_{1y}(z)}{(1 - Ag_y(z))(1 - Ag_{2y}(z))z}\, dz \\
& + \frac{1}{2\pi}\lim_{T\to\infty}\int_{-T}^{T} \frac{e^{-it(y-l)} - e^{-ity}}{(1 - Ag_{2y}(it))it}\, dt,
\end{aligned} \tag{3.5}
$$

where Γ_y is the line $\operatorname{Re} z = [(s-1)\log y + c]/y$.

Let $\Gamma_y(\varepsilon) = \{z : z \in \Gamma_y, |\operatorname{Im} z| < \varepsilon\}$ and $\bar{\Gamma}_y(\varepsilon) = \Gamma_y - \Gamma_y(\varepsilon)$. Clearly,

$$
\int_{|t| \geqq \varepsilon} \left|\frac{g_{1y}(\sigma + it)}{t}\right|^2 dt \leqq \frac{1}{2}\int_{|t|\geqq\varepsilon} |g_{1y}(\sigma + it)|^2\, dt + \frac{1}{\varepsilon}. \tag{3.6}
$$

Further,

$$
\int_{|t|\geqq\varepsilon} |g_{1y}(\sigma + it)|^2\, dt \leqq \int_{-\infty}^{\infty} F'^2_{1y}(x)\, e^{2\sigma x}\, dx \leqq \bar{L}\, e^{2\sigma M}. \tag{3.7}
$$

Using (3.6) and (3.7) as well as analogous estimates for $g'_{1y}(z)$, one can easily show that

$$
\begin{aligned}
&\left|\int_{\bar{\Gamma}_y(\varepsilon)} \frac{(e^{-zy} - e^{-z(y-l)})g_{1y}(z)}{(1 - Ag_y(z))(1 - Ag_{2y}(z))z}\, dz\right| \\
&\qquad < \frac{K}{y^s}\left|\int_{\bar{\Gamma}_y(\varepsilon)} \left|\frac{d}{dz}\frac{g_{1y}(z)(e^{zl} - 1)}{(1 - Ag_y(z))(1 - Ag_{2y}(z))z}\right| dz\right| < \frac{C(\varepsilon, L)}{y^s}
\end{aligned} \tag{3.8}
$$

uniformly for $0 \leqq l \leqq L$. Here K is an absolute constant. In proving (3.8), one also has to use the fact that $g'_y(\sigma) < Q < \infty$ for $\sigma \leqq [(s-1)\log y + c]/y$ (cf. (2.23)). On the other hand, reasoning in an analogous way as in the proof of (2.45), one can show that for ε sufficiently small

$$
\int_{\Gamma_y(\varepsilon)} \frac{(e^{-z(y-l)} - e^{-zy})g_{1y}(z)\, dz}{(1 - Ag_y(z))(1 - Ag_{2y}(z))z} = o\left(\frac{1}{y^{s-1}}\right) \tag{3.9}
$$

uniformly in l ranging over any finite interval $0 \leqq l \leqq L$.

It remains to estimate

$$I = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{-it(y-l)} - e^{-ity}}{(1 - Ag_{2y}(it))it}\, dt.$$

It is not hard to see that when $y - l > 0$,

$$I = \sum_{k=1}^{\infty} A^k(F_{2,k}^{(y)}(y) - F_{2,k}^{(y)}(y - l)), \tag{3.10}$$

where $F_{2,k}^{(y)}(x)$ stands for the k-fold convolution of $F_{2y}(x)$. Clearly,

$$F_{2,k}^{(y)}(y) - F_{2,k}^{(y)}(y - l) \leqq (y - l)^{-s} \int_{-\infty}^{\infty} u^s\, dF_{2,k}(u). \tag{3.11}$$

Set $\bar{F}_2^{(y)}(x) = F_{2y}(x)/F_{2y}(\infty)$. Let $\bar{F}_{2,k}^{(y)}(x)$ denote the k-fold convolution of $\bar{F}_2^{(y)}(x)$. Using this notation, we have

$$\begin{aligned} \int_{-\infty}^{\infty} x^s\, dF_{2,k}^{(y)}(x) &= F_{2y}^k(\infty) \int_{-\infty}^{\infty} x^s\, d\bar{F}_{2,k}^{(y)}(x) \\ &< k^s F_{2y}^k(\infty) \int_{-\infty}^{\infty} x^s d\bar{F}_2^{(y)}(x) < k^s M_s F_{2y}^{k-1}(\infty). \end{aligned} \tag{3.12}$$

From (3.10)–(3.12) we obtain the estimate

$$I \leqq \sum_{k=1}^{\infty} \frac{k^s M_s}{(y - l)^s} A^k \delta^{k-1}, \tag{3.13}$$

where $\delta = F_{2\infty}(\infty)$.

The asymptotic representation (1.6) in the case where y and $y - l$ are points of continuity of $H(u, A)$ is now easily obtained from (3.1) and (3.5) by using the estimates (3.8), (3.9) and (3.13). If one of the points y and $y - l$ is a discontinuity of $H(u, A)$, we choose a sequence $y_n \uparrow y$ so that each y_n and $y_n - l$ is a point of continuity of $H(u, A)$. It is not hard to see that

$$\lim_{y_n \uparrow y} g_{y_n}(z) = g_y(z), \qquad \lim_{y_n \uparrow y} g'_{y_n}(z) = g'_y(z).$$

Hence,

$$\lim_{y_n \uparrow y} \Lambda_{y_n}(A) = \Lambda_y(A) \quad \text{and} \quad \lim_{y_n \uparrow y} g'_{y_n}(\Lambda_{y_n}(A)) = g'_y(\Lambda_y(A)).$$

On the other hand,

$$\lim_{y_n \uparrow y} (H(y_n, A) - H(y_n - l, A)) = H(y, A) - H(y - l, A).$$

From all of the above, we conclude that the asymptotic representation (1.6) holds for any pair y and $y - l$. The passage from (1.6) to (1.7) is accomplished in the same way as the passage from (1.2) to (1.4) in the proof of Theorem 1.

We now proceed to prove Theorem 3. First of all,

$$\overline{\lim_{|t|\to\infty}} |g(\sigma + it)| \leqq g_2(\sigma),$$

and, to any h and $\varepsilon > 0$, there exists a positive K such that $|g(\sigma + it)| < g_2(\sigma) + \varepsilon$ for all $0 \leqq \sigma \leqq h$ and $|t| < K$.

Hence there exists a K_0 such that

$$\inf_{|t| > K_0} |Ag(\sigma + it) - 1| > \delta > 0 \tag{3.14}$$

for all $0 \leqq \sigma \leqq h_1$ and $A \leqq 1$, where δ depends on K_0.

On the other hand,

$$\inf_{|t| < K_0} |Ag(h_1 + it) - 1| > \delta_1 > 0 \tag{3.15}$$

for all $g^{-1}(h_1) + \varepsilon \leqq A \leqq 1$ since otherwise there would be an A_0, $g^{-1}(h_1) + \varepsilon \leqq A_0 \leqq 1$, and $t_0 \neq 0$ such that $A_0 g(h_1 + it_0) = 1$. And this would contradict the hypothesis of the theorem.

Choose now M and $\bar{L}$, occurring in the definition of $F_{1y}(x)$, so that $g_{2M}(h_1) < 1$. Then, when $A \leqq 1$,

$$\inf_{\mathrm{Re}\, z \leqq h_1} |Ag_{2M}(z) - 1| > \delta_2 > 0. \tag{3.16}$$

Hence it follows in particular that $Ag_{2M}(z) - 1$ has no zeroes in the half-plane $\mathrm{Re}\, z \leqq h_1$.

It is not hard to see that

$$H(x, A) - H(x - l, A) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\sigma - iT}^{\sigma + iT} \frac{e^{-z(x-l)} - e^{-zx}}{z(1 - Ag(z))} dz, \qquad A \leqq 1,$$

if $\sigma < 0$ and $x - l$ and x are points of continuity of $H(u, A)$.

Applying the residue theorem and (3.2), we have

$$\begin{aligned} H(x, A) - H(x - l, A) &= \frac{e^{-\Lambda(A)x}(e^{\Lambda(A)l} - 1)}{Ag'(\Lambda(A))\Lambda(A)} \\ &\quad + \frac{A}{2\pi i} \int_{\mathrm{Re}\, z = h_1} \frac{(e^{-z(x-l)} - e^{-zx}) g_{1M}(z)\, dz}{(1 - Ag_M(z))(1 - Ag_{2M}(z))z} \\ &\quad + \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-it(x-l)} - e^{-itx}}{(1 - Ag_{2M}(it))it} dt. \end{aligned} \tag{3.17}$$

By virtue of (3.6), (3.7) and (3.14)–(3.16),

$$\int_{\mathrm{Re}\, z = h_1} \frac{(e^{-z(x-l)} - e^{-zx}) g_{1M}(z)\, dz}{(1 - Ag_M(z))(1 - Ag_{2M}(z))z} = O(e^{-h_1 x}) \tag{3.18}$$

uniformly for A in $g^{-1}(h_1) + \varepsilon \leqq A \leqq 1$.

Further (cf. (3.10))

$$\begin{aligned} \frac{1}{2\pi} \lim_{T \to \infty} \int_{-T}^{T} \frac{e^{-it(x-l)} - e^{-itx}}{it(1 - Ag_{2M}(it))} dt &= \sum_{k=1}^{\infty} A^k (F_{2,k}^{(M)}(x) - F_{2,k}^{(M)}(x - l)) \leqq e^{h_1(l-x)} \\ &\times \sum_{k=1}^{\infty} A^k \int_{-\infty}^{\infty} e^{h_1 y}\, dF_{2,k}^{(M)}(y) = e^{h_1(l-x)} \sum_{k=1}^{\infty} A^k g_{2M}^k(h_1), \qquad x - l > 0. \end{aligned} \tag{3.19}$$

The assertion of Theorem 3 in the case where x and $x - l$ are both points of continuity of $H(u, A)$ follows from (3.17)–(3.19). The passage to arbitrary x and $x - l$ is accomplished just as in the proof of Theorem 2.

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