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SOME LIMIT THEOREMS FOR LARGE DEVIATIONS

S. V. NAGAEV

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1. Introduction. Formulation of results

Let $\xi_1, \xi_2, \cdots, \xi_n, \cdots$ be a sequence of identically distributed independent random variables with distribution function $F(x)$, $\mathbf{E}\xi_i = 0$ and $\mathbf{D}\xi_i = 1$, and let $F_n(x)$ be the distribution function of $\sum_{k=1}^{n} \xi_k$.

Of great importance is the study of the asymptotic behavior of $1-F_n(x)$ and $F_n(-x)$ as $n \to \infty$ and $x/\sqrt{n} \to \infty$. A highly distinctive feature of this behavior is its dependence on both the rate of increase of $x/\sqrt{n}$ and rate of decrease of $1-F(x)$ $(F(-x))$.

The laws existing here can be described qualitatively as follows.

If $x/\sqrt{n}$ does not increase very fast, then $1-F_n(x)$ is approximated by $1-\Phi(x/\sqrt{n})$ ([1], [2], [3]) or $\{1-\Phi(x/\sqrt{n})\}\times\exp\{(x^3/n^2)\lambda^{[s]}(x/n)\}$, where $\Phi(u)$ is the normal distribution and $\lambda^{[s]}(u)$ is a segment of the so-called Cramér series consisting of its first s terms, the integer s depending on the rate of decrease of $1-F(x)$, [1], [3], [4].

If $1-F(x)$ decreases so fast that $\int_0^\infty e^{hx}\,dF(x) < \infty$ for all $h > 0$, then, under very broad assumptions concerning the decrease of $1-F(x)$,

$$1-F_n(x) \sim \frac{1}{H\left(\frac{x}{n}\right)}\sqrt{\frac{H'\left(\frac{x}{n}\right)}{2\pi n}}\exp\left\{-n\int_0^{x/n} H(u)\,du\right\},$$

where $x/n \to \infty$ and $H(u)$ is a certain function determined by $F(x)$, [5].

But if $\int_0^\infty e^{hx}dF(x) = \infty$ for all $h > 0$ and $1-F(x)$ decreases sufficiently, then

$$1-F_n(x) \sim n(1-F(x)) \tag{1.1}$$

for $x > \varphi(F, n)$, where $\varphi(F, n)$ is a monotone increasing function of n (depending on F), [6].

As to an upper estimate for $1-F_n(x)$, it can be obtained under very general assumptions; namely, in this paper we proved the following

Theorem 1. *If* $c_m = \mathbf{E}|\xi_i|^m < \infty$, $m > 2$, *then for* x *and* y *positive,*

$$(1.2)\qquad 1-F_n(x) > n(1-F(y)) + \exp\left\{2n\left[\frac{m\log y-\log(nc_m K_m)}{y}\right]^2+1\right\}\left[\frac{nc_m K_m}{y_m}\right]^{x/y},$$

where

$$K_m = 1+(m+1)^{m+2}e^{-m}.$$

An analogous assertion holds for $F(-x)$.

We now state two corollaries to Theorem 1.

Corollary 1. *If* $c_m < \infty$, $m > 2$, *then for* $x > k(c_m K_m)^{1/m}\sqrt{n}\log n$, $n \geqq 3$ *and* $k \geqq 1$,

$$1-F_n(x) < n\left(1-F\left(\frac{x}{k}\right)\right) + \exp\left\{2k^2m^2\left(\frac{1}{e}+\frac{1}{2K_m^{1/m}}\right)^2+1\right\}\left[\frac{nc_m k^m K_m}{x^m}\right]^k.$$

Setting $y = x/2$ in (1.2) if $n^{m/2-1}/K_m c_m \geqq e$ and $y = x$ if $n^{m/2-1}/K_m c_m < e$ but $x^m > c_m n K_m$ (the case $x^m < c_m n K_m$ is trivial), we obtain

Corollary 2. *If* $c_m < \infty, m > 2$, *then*

$$1-F_n(x) < \frac{B_m c_m n}{x^m} \tag{1.3}$$

for

$$x > 4\sqrt{n \max\left[\log\frac{n^{m/2-1}}{K_m c_m}, 0\right]},$$

where B_m *is an absolute constant depending only on m.*

The estimate (1.3) is a generalization of the inequality $1-F_n(x) < n/x^2$.

In addition, an estimate is derived in the paper for $F_n(x)-\Phi(x/\sqrt{n})$ which is optimum in the sense of dependence on x.

Theorem 3. *If* $c_3 < \infty$, *then there exisis an absolute constant L such that*

$$|F_n(x\sqrt{n})-\Phi(x)| < \frac{Lc_3}{\sqrt{n}(1+|x|^3)}. \tag{1.4}$$

It follows immediately from (1.1) that the power of $|x|$ in (1.4) cannot be replaced by a higher one.

The methods applied in the proof of Theorems 1 and 2 permit us to sharpen the known results of Yu. V. Linnik [2] and V. V. Petrov [4].

Let $g(x)$ be a continuous function with a monotone decreasing continuous derivative which satisfies the conditions

$$0 < g'(x) < \frac{\alpha g(x)}{x}, \quad \alpha < 1, \quad x > B(g), \tag{1.5}$$

and

$$g(x) > \rho(x)\log x, \tag{1.6}$$

where $\rho(x)$ is a function which approaches infinity in an arbitrarily slow manner as $x \to \infty$.

Let $\Lambda(n)$ be a solution of the equation $x^2 = ng(x)$.

Theorem 3. *The condition*

$$\mathbf{E}\exp\{g(|\xi_1|)\} < \infty \tag{1.7}$$

is sufficient in order that

$$(1.8)\quad \begin{aligned} \frac{F_n(-x)}{\Phi\left(-\dfrac{x}{\sqrt{n}}\right)} &= \exp\left\{-\frac{x^3}{n^2}\lambda^{[\alpha/(1-\alpha)]}\left(-\frac{x}{n}\right)\right\}(1+o(1)),\\ \frac{1-F_n(x)}{1-\Phi\left(\dfrac{x}{\sqrt{n}}\right)} &= \exp\left\{\frac{x^3}{n^2}\lambda^{[\alpha|(1-\alpha)]}\left(\frac{x}{n}\right)\right\}(1+o(1)),\end{aligned}$$

for $0 \leqq x \leqq \Lambda(n)$, *where* $\lambda^{[\alpha/(1-\alpha)]}(u)$ *is the segment of the first* $\alpha/(1-\alpha)$ *terms of the Cramér series* (*for* $\alpha < \frac{1}{2}$, $\lambda^{[\alpha/(1-\alpha)]}(u) \equiv 0$), *and necessary in order that* (1.8) *hold for* $0 \leqq x \leqq 2\Lambda(n)$.

It is not hard to see that the class of functions $g(x)$ satisfying (1.5) with $\alpha < \frac{1}{2}$ and (1.6) contains the classes I and II introduced by Yu. V. Linnik [2].

Theorem 4. *The condition*

$$(1.9)\qquad \mathbf{E}|\xi_1|^m < \infty$$

is sufficient in order that

$$(1.10)\qquad \frac{F_n(-x)}{\Phi\left(-\dfrac{x}{\sqrt{n}}\right)} \to 1, \qquad \frac{1-F_n(x)}{1-\Phi\left(\dfrac{x}{\sqrt{n}}\right)} \to 1,$$

for $0 \leqq x \leqq \sqrt{(m/2-1)n\log n}$, *and necessary in order that* (1.10) *hold for* $0 \leqq x \leqq \sqrt{(m+1)n\log n}$.

The methods developed in the theory of large deviations turn out to be useful also in proving global limit theorems, [7], [11]–[14] (the latter may be, by the way, regarded as special forms of theorems on large deviations).

Theorem 5. *Let* $c_3 < \infty$ *and let there exist a subscript* n_0 *such that* $F_{n_0}(x)$ *has an absolutely continuous component. Then*

$$\int_{-\infty}^{\infty}\left|p_n(x) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right||x|^3\,dx = \frac{|\alpha_3|}{6\sqrt{2\pi n}}\int_{-\infty}^{\infty} x^4|x^2-3|e^{-x^2/2}\,dx + o\left(\frac{1}{\sqrt{n}}\right),$$

where $p_n(x) = \sqrt{n}F_n'(x\sqrt{n})$ *and* $\alpha_3 = \mathbf{E}\xi_1^3$.

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2. Proof of Theorem 1

There is no loss of generality in assuming that $y > (nc_mK_m)^{1/m}$. Set

$$F^{(y)}(x) = \begin{cases} F(x), & x \leqq y,\\ F(y), & x > y.\end{cases}$$

Let $F_n^{(y)}(x)$ be the n-fold convolution of $F^{(y)}(x)$,

$$F_{nh}^{(y)}(x) = \int_{-\infty}^{x} e^{hu}\,dF_n^{(y)}(u), \qquad F_h^{(y)}(x) = F_{1h}^{(y)}(x).$$

Evidently,

$$F_n(x)-F_n^{(y)}(x) \leqq n(1-F(y)) \tag{2.1}$$

and

$$F_n^{(y)}(\infty)-F_n^{(y)}(x) = \int_x^\infty e^{-hu}\,dF_{nh}(u) = R^n(y, h)\int_x^\infty e^{-hu}\,d\bar{F}_{nh}^{(y)}(u), \tag{2.2}$$

where

$$R(y, h) = \int_{-\infty}^{y} e^{hu}\,dF(u), \qquad F_{nh}^{(y)}(u) = \frac{\bar{F}_{nh}^{(y)}(u)}{R^n(y, h)}.$$

Set

$$R_1(h) = \int_{-\infty}^{1/h} e^{hu}\,dF(u), \qquad R_2(y, h) = \int_{1/h}^{y} e^{hu}\,dF(u).$$

Evidently,

$$R_1(h) = \int_{-\infty}^{1/h} dF(u) - h\int_{1/h}^{\infty} u\,dF(u) + \frac{h^2}{2}\int_{-\infty}^{1/h} e^{h\theta(u)}u^2\,dF(u), \qquad 0 \leqq \frac{\theta(u)}{u} \leqq 1,$$

$$1-\int_{-\infty}^{1/h} dF(u) < h^2\int_{-\infty}^{\infty} u^2\,dF(u) = h^2, \qquad \int_{1/h}^{\infty} u\,dF(u) < h.$$

Hence,

$$|R_1(h)-1| < 2h^2. \tag{2.3}$$

Further,

$$R_2(y, h) = [1-F(u)]e^{hu}\Big|_{y}^{1/h} + h\int_{1/h}^{y} [1-F(u)]e^{hu}\,du. \tag{2.4}$$

Clearly,

$$\int_{1/h} [1-F(u)]e^{hu}\,du < c_m\int_{1/h}^{y} \frac{e^{hu}}{u^m}\,du = c_m h^{m-1}\int_1^{yh} \frac{e^u}{u^m}\,du,$$

$$\int_1 \frac{e^u}{u^m}\,du = \frac{e^u}{u^m}\Big|_1^v + m\int_1^v \frac{e^u}{u^{m+1}}\,du < \frac{e^v}{v^m} + mv\max\left[\frac{e^v}{v^{m+1}},\ e\right]$$

$$= \frac{e^v}{v^m}\left[1+m\max\left[1,\ \frac{v^{m+1}}{e^{v-1}}\right]\right] < \left[1+\frac{m(m+1)^{m+1}}{e^m}\right]\frac{e^v}{v^m}.$$

Thus

$$h\int_{1/h}^{y} [1-F(u)]e^{hu}\,du < \left[1+\frac{m(m+1)^{m+1}}{e^m}\right]c_m\frac{e^{hy}}{y^m}. \tag{2.5}$$

Clearly,

$$1-F\left(\frac{1}{h}\right) < c_m h^m < c_m\left(\frac{m}{ey}\right)^m e^{hy}. \tag{2.6}$$

It follows from (2.4)–(2.6) that

$$R_2(y, h) < \left[1+\frac{(m+1)^{m+2}}{e^m}\right]\frac{e^{hy}c_m}{y^m}. \tag{2.7}$$

Let $h_m(y)$ be the solution of the equation

$$nK_m c_m e^{hy} = y^m, \tag{2.8}$$

with $K_m = 1+(m+1)^{m+2}/e^m$. Clearly,

$$h_m(y) = \frac{m \log y - \log (nK_m c_m)}{y}. \tag{2.9}$$

If $h_m(y) \geqq y^{-1}$, then on setting $h = h_m(y)$ in (2.2) and using the estimates (2.1), (2.3) and (2.7), we deduce the assertion of the theorem. If $h_m(y) < y^{-1}$, one must consider $F^{(1/h_m(y))}(u)$ instead of $F^{(y)}(u)$.

3. Proof of Theorem 2

We shall use the notations introduced in the proof of Theorem 1 without specific mention and we shall confine ourselves to the case $x > 0$, since the proof is completely analogous for the case $x < 0$.

Without loss of generality, we may assume that $x > \sqrt{n}$.

Consider first the case $\sqrt{n} > c_3 N_3 \exp\{e^3/3\}$, where $N_3 = 6^3 K_3$. Clearly,

$$F_n^{(y)}(x) = \Phi_h(x) + \Psi_h^{(y)}(x), \tag{3.1}$$

where

$$\Phi_h(x) = \begin{cases} \int_{-\infty}^{x} e^{hu}\, dF(u), & x \leqq \frac{1}{h}, \\ \int_{-\infty}^{1/h} e^{hu}\, dF(u), & x > \frac{1}{h}, \end{cases} \qquad \Psi_h^{(y)}(x) = \begin{cases} 0, & x \leqq \frac{1}{h}, \\ \int_{1/h}^{x} e^{hu}\, dF^{(y)}(u), & x > \frac{1}{h}. \end{cases}$$

Let $\Phi_{nh}(x)$ be the n-fold convolution of $\Phi_h(x)$. From (3.1), it follows that

$$F_{nh}^{(y)}(x) - \Phi_{nh}(x) < nR^{n-1}(y, h) R_2(y, h). \tag{3.2}$$

We observe at once that $F_{nh}^{(y)}(x) - \Phi_{nh}(x)$ is monotone increasing with increasing x. Further,

$$\int_{x}^{\infty} e^{-hu}\, d\Phi_{nh}(u) = R_1^n(h) e^{-hx} \int_{0}^{\infty} e^{-h\sigma(h)\sqrt{n}u}\, d\bar{\Phi}_{nh}(u), \tag{3.3}$$

where

$$\sigma^2(h) = \frac{\int_{-\infty}^{1/h} x^2 e^{hx}\, dF(x)}{R_1(h)} - \left[\frac{\int_{-\infty}^{1/h} x e^{hx}\, dF(x)}{R_1(h)}\right]^2,$$

$$\bar{\Phi}_{nh}(u) = \frac{\Phi_{nh}(x + u\sigma(h)\sqrt{n})}{R_1^n(h)}.$$

Set

$$\bar{R}_1(h) = \int_{-\infty}^{1/h} x e^{hx}\, dF(x), \qquad \bar{\bar{R}}_1(h) = \int_{-\infty}^{1/h} x^2 e^{hx}\, dF(x), \qquad m(h) = \frac{\bar{R}_1(h)}{R_1(h)}.$$

Clearly,

$$\bar{R}_1(h) = -\int_{1/h}^{\infty} u\, dF(u) + h\int_{-\infty}^{1/h} u^2\, dF(u) + \frac{h^2}{2}\int_{-\infty}^{1/h} u^3 e^{h\theta(u)}\, dF(u), \quad 0 \leqq \frac{\theta(u)}{u} \leqq 1, \tag{3.4}$$

$$1 - \int_{-\infty}^{1/h} u^2\, dF(u) < c_3 h, \qquad \int_{1/h}^{\infty} u\, dF(u) < c_3 h^2. \tag{3.5}$$

From (3.4) and (3.5), it follows that $|\bar{R}_1(h)-h| < 4c_3h^2$. Hence, employing (2.3), we conclude that

$$|\bar{R}_1(h)-hR_1(h)| < 4c_3h^2+2h^3. \tag{3.6}$$

Clearly, $m(h)-h = [\bar{R}_1(h)-hR_1(h)]/R_1(h)$. From (2.3) and (3.6) it follows that

$$|m(h)-h| < (8c_3+2)h^2, \qquad h \leqq \tfrac{1}{2}. \tag{3.7}$$

Clearly,

$$\sigma^2(h)-1 = \frac{\bar{\bar{R}}_1(h)-R_1(h)}{R_1(h)} - \frac{\bar{R}_1^2(h)}{R_1^2(h)}. \tag{3.8}$$

From

$$\bar{R}_1(h) = -\int_{1/h}^{\infty} u\,dF(u)+h\int_{-\infty}^{1/h} u^2 e^{h\theta(u)}\,dF(u),$$

$$\bar{\bar{R}}_1(h) = \int_{-\infty}^{1/h} u^2\,dF(u)+h\int_{-\infty}^{1/h} u^3 e^{h\vartheta(u)}\,dF(u), \qquad 0 \leqq \frac{\theta(u)}{u} \leqq 1,$$

we conclude that $|\bar{R}_1(h)| < eh$ and $|\bar{\bar{R}}_1(h)-1| < (e+1)c_3h$. The last inequality and (2.3) imply that $|\bar{\bar{R}}_1(h)-R_1(h)| < (e+1)c_3h+2h^2$. Now by (3.8) and (2.3), we have

$$|\sigma^2(h)-1| < 2[(e+1)c_3+e^2+1]h \tag{3.9}$$

for $h < \frac{1}{2}$.

Let A be the set of values of the function $m(h)$, $h > 0$. Consider the equation $u = m(h)$, with $u \in A$. Owing to (3.7),

$$h < 2m(h) \text{ for } h < 1/2a, \tag{3.10}$$

where $a = 8c_3+2$. Therefore, for any $u < 1/4a$, $u \in A$, the equation $u = m(h)$ has a solution $h(u)$ such that

$$|h(u)-u| < 4au^2. \tag{3.11}$$

Consider now the values $x < n/4a$, $x/n \in A$, for which

$$h_3\left(\frac{x}{6}\right) > \frac{2x}{n}. \tag{3.12}$$

For such x, clearly, $h_3(x/6) > h(x/n)$. It is not hard to see that

$$\left|R_1(h)-1-\frac{h^2}{2}\right| < 3c_3h^3. \tag{3.13}$$

By (3.10),

$$h\left(\frac{x}{n}\right) < 2\frac{x}{n}. \tag{3.14}$$

Therefore,

$$\left|h^2\left(\frac{x}{n}\right) - \frac{x^2}{n^2}\right| < 12a\frac{x^3}{n^3}. \tag{3.15}$$

Thus,

$$\left|R_1\left(h\left(\frac{x}{n}\right)\right)-1-\frac{x^2}{2n^2}\right| < (24c_3+6a)\frac{x^3}{n^3} < 9a\frac{x^3}{n^3}. \tag{3.16}$$

Hence, using the estimate $nR_2(y, h_3(y)) < 1$, we obtain

$$(3.17)\qquad R^n\left(\frac{x}{6}, h\left(\frac{x}{n}\right)\right) < \left[R_1\left(h\left(\frac{x}{n}\right)\right) + R_2\left(\frac{x}{6}, h_3\left(\frac{x}{6}\right)\right)\right]^n < \exp\left\{1 + \frac{x^2}{2n} + 9a\frac{x^3}{n^2}\right\}.$$

By virtue of (3.7) and (3.11),

$$(3.18)\qquad R_2\left(\frac{x}{6}, h\left(\frac{x}{n}\right)\right) < \frac{N_3 c_3}{x^3}\exp\left\{\frac{x^2}{6n} + \tfrac{4}{3}a\frac{x^3}{n^2}\right\}.$$

Because of the monotonicity of $F_{nh}^{(y)}(u) - \Phi_{nh}(u)$, (3.2), (3.17) and (3.18) imply

$$(3.19)\qquad \int_x e^{-h(x/n)u}\, d[F_{nh(x/n)}^{(x/6)}(u) - \Phi_{nh(x/n)}(u)] < \frac{N_3 c_3 n}{x^3}\exp\left\{1 - \frac{x^2}{3n} + 10a\frac{x^3}{n^2}\right\}.$$

Employing Esseen's improvement of Lyapunov's theorem, we have

$$(3.20)\qquad \left|\int_0^\infty e^{-h\sigma(h)\sqrt{nu}}\, d\bar{\Phi}_{nh}(u) - \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-h\sigma(h)\sqrt{nu} - \frac{1}{2}u^2}\, du\right| < \frac{Cc_3(h)}{\sqrt{n}\sigma^{3/2}(h)}$$

for $h = h(x/n)$, where C is an absolute constant and

$$c_3(h) = \int_{-\infty}^{1/h} |u|^3 e^{hu}\, dF(u)/R_1(h).$$

Clearly,

$$(3.21)\qquad c_3(h) < ec_3, \qquad h < \frac{1}{6c_3}.$$

By (3.9) and (3.14),

$$(3.22)\qquad \sigma^2\left(h\left(\frac{x}{n}\right)\right) > \frac{1}{2}$$

for $x < n/12a$. Further, on account of (3.9), (3.11), and (3.14),

$$(3.23)\qquad \left|h\sigma(h) - \frac{x}{n}\right| < \left|h - \frac{x}{n}\right| + h|\sigma(h) - 1| < \{4a + 8[(e+1)c_3 + e^2 + 1]\}\frac{x^2}{n^2} < 16a\frac{x^2}{n^2}$$

for $h = h(x/n)$.

Therefore,

$$(3.24)\qquad \left|\frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left\{-h\sigma(h)\sqrt{n}u - \frac{u^2}{2}\right\} du - \exp\left\{\frac{x^2}{2n}\right\}\left(1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right)\right| \leqq \sqrt{n}\left|h\sigma(h) - \frac{x}{n}\right|\left|\sup_{v\geqq 0}\frac{d}{dv}\left[\exp\left\{\frac{v^2}{2}\right\}(1 - \Phi(v))\right]\right| < 16\frac{x^2 a}{n^{3/2}}, \qquad h = h\left(\frac{x}{n}\right).$$

We have here made use of the estimate

$$\left| \frac{d}{dv} \left[\exp\left\{\frac{v^2}{2}\right\} (1-\Phi(v)) \right] \right| < 1/\sqrt{2\pi} < 1,$$

which can be deduced by straightforward differentiation.

Now, by (2.3),

$$\left| \sum_{k=2}^{\infty} \frac{1}{k} [1-R_1(h)]^k \right| < \frac{2h^4}{1-2h^2}, \qquad 2h^2 < 1. \tag{3.25}$$

From (3.13) and (3.25), we conclude

$$\left| \log R_1(h) - \frac{h^2}{2} \right| < 5c_3 h^3, \qquad h < \tfrac{1}{2}. \tag{3.26}$$

In view of (3.10),

$$\left| \frac{h^2}{2} - \frac{hx}{n} + \frac{x^2}{2n^2} \right| < 8a^2 \frac{x^4}{n^4} < 2a \frac{x^3}{n^3}, \qquad h = h\left(\frac{x}{n}\right). \tag{3.27}$$

From (3.26), (3.27), and (3.14), it follows that

$$\left| n \log R_1(h) - hx + \frac{x^2}{2n} \right| < 7a \frac{x^3}{n^2}, \qquad h = h\left(\frac{x}{n}\right). \tag{3.28}$$

By (3.28)

$$hx - n \log R_1(h) > \frac{x^2}{4n}, \qquad h = h\left(\frac{x}{n}\right), \tag{3.29}$$

for $x < n/28a$. From (3.20)–(3.22), (3.24), (3.28) and (3.29), we conclude that

$$\begin{aligned} &\left| R_1^n(h) e^{-hx} \int_0^{\infty} e^{-h\sigma(h)\sqrt{nu}} d\Phi_{nh}(u) - \left(1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right) \right| \\ &< e^{-x^2/4n} \left(\frac{3Cec_3}{\sqrt{n}} + 16a \frac{x^2}{n^{3/2}} \right) + (e^{7a(x^3/n^2)} - 1)\left(1 - \Phi\left(\frac{x}{\sqrt{}}\right)\right), \quad h = h\left(\frac{x}{n}\right), \end{aligned} \tag{3.30}$$

for $x < n/28a$. If we set $x = u\sqrt{n}$, we can rewrite (3.12) in the form $u^2 < 9 \log u + 3 \log (\sqrt{n}/N_3 c_3)$. Therefore, inequality (3.12) holds under any circumstances for $x \sqrt{3n \log (\sqrt{n}/N_3 c_3)}$. Since $\sqrt{n}/N_3 c_3 > e$, $c_3 \geqq 1$ and $c_3 > a/10$, the inequality

$$x < \frac{n}{N_3 c_3} < \frac{10n}{N_3 a} \tag{3.31}$$

holds for such values of x.

Further, $\sqrt{3 \log (\sqrt{n}/N_3 c_3)} < \sqrt[3]{3\sqrt{n}/N_3 c_3} < \sqrt[3]{\sqrt{n}/7a}$ and therefore

$$7a \frac{x^3}{n^2} < 1 \tag{3.32}$$

for $x < \sqrt{3n \log (\sqrt{n}/N_3 c_3)}$.

Using the estimates (2.1), (3.31) and (3.32) and the inequality $x^\alpha e^{-x} < e^{-\alpha}\alpha^\alpha$, we conclude from (2.2), (3.3), (3.12) and (3.30) that

$$\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| < \frac{L_1 c_3 n}{x^3}, \qquad \frac{x}{n} \in A, \tag{3.33}$$

for $x < \sqrt{3n \log (\sqrt{n}/N_3 c_3)}$, where L_1 is an absolute constant.

Consider now the values $x < \sqrt{3n \log(\sqrt{n}/N_3 c_3)}$ for which $x/n \notin A$. For any function $f(h)$, set $\Delta f(h) = f(h+) - f(h-)$. The functions $R_1(h)$ and $\bar{R}_1(h)$ are continuous on the right with $\Delta R_1(h) = -e\Delta F(1/h)$ and $\Delta \bar{R}_1(h) = -(e/h)\Delta F(1/h)$.

It is not hard to see that

$$\Delta m(h) = \frac{R_1(h-)\Delta \bar{R}_1(h) - \bar{R}_1(h)\Delta R_1(h)}{R_1(h)R_1(h-)} = e\Delta F\left(\frac{1}{h}\right)\frac{\bar{R}_1(h) - R_1(h-)/h}{R_1(h)R_1(h-)}.$$

The inequality $|\bar{R}_1(h) - h| < 4c_3 h^2$ implies that $0 < \bar{R}_1(h) < 1/2c_3 < \frac{1}{2}$ for $h < 1/4c_3$. On the other hand, on account of (2.3), we have $R_1(h) < \frac{1}{2}$ for $h < \frac{1}{2}$. Therefore for $h < 1/4c_3$,

$$-\frac{4e}{n}\Delta F\left(\frac{1}{h}\right) < \Delta m(h) \leqq 0. \tag{3.34}$$

Suppose $h_0 < 1/2a$ is such that $m(h_0) < x/n < m(h_0-)$. By (3.10), $h_0 < 2m(h_0) < 2x/n$. Hence, taking (3.34) into consideration, we obtain

$$|\Delta m(h_0)| < 16ec_3 \frac{x^2}{n^2}. \tag{3.35}$$

Now choose h_1 so that $x/n < m(h_1) < m(h_0-) + \Delta m(h_0)/2$. Set $x_0 = nm(h_0)$ and $x_1 = nm(h_1)$. Let $\alpha > 0$ be such that $F_n(x) = \alpha F_n(x_0) + (1-\alpha)F_n(x_1)$. Clearly,

$$\begin{aligned} &\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right)\right| \\ &< \alpha\left| F_n(x_0) - \Phi\left(\frac{x_0}{\sqrt{n}}\right)\right| + (1-\alpha)\left| F_n(x_1) - \Phi\left(\frac{x_1}{\sqrt{n}}\right)\right| \\ &+ \alpha\left|\Phi\left(\frac{x}{\sqrt{n}}\right) - \Phi\left(\frac{x_0}{\sqrt{n}}\right)\right| + (1-\alpha)\left|\Phi\left(\frac{x}{\sqrt{n}}\right) - \Phi\left(\frac{x_1}{\sqrt{n}}\right)\right|. \end{aligned} \tag{3.36}$$

By (3.35) and (3.36),

$$\begin{aligned} \left|\Phi\left(\frac{x}{\sqrt{n}}\right) - \Phi\left(\frac{x_0}{\sqrt{n}}\right)\right| &< 16ec_3 \frac{x^2}{n^{3/2}} \exp\left\{-\frac{\left(x - 16ec_3 \frac{x^2}{n}\right)^2}{2n}\right\} \\ &< 16ec_3 \frac{x^2}{n^{3/2}} e^{-x^2/3n}. \end{aligned} \tag{3.37}$$

Similarly,

$$\left|\Phi\left(\frac{x}{\sqrt{n}}\right) - \Phi\left(\frac{x_1}{\sqrt{n}}\right)\right| < 24ec_3\, e^{-x^2/2n}. \tag{3.38}$$

From (3.33), it follows that

$$\left| F_n(x_0) - \Phi\left(\frac{x_0}{\sqrt{n}}\right)\right| < \frac{8L_1 c_3 n}{x^3}, \tag{3.39}$$

since for $x < \sqrt{3n \log(\sqrt{n}/N_3 c_3)}$ by (3.35) $x_0 > x/2$, and

$$\left| F_n(x_1) - \Phi\left(\frac{x_1}{\sqrt{n}}\right)\right| < \frac{L_1 c_3 n}{x^3}. \tag{3.40}$$

Using the estimates (3.36) – (3.40), we conclude that

$$\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| < \frac{L_2 c_3 n}{x^3}, \tag{3.41}$$

for all $x < \sqrt{3n \log(\sqrt{n}/N_3 c_3)}$, where L_2 is an absolute constant.

We shall now find a lower bound for $u > 1$ for which

$$1 - \Phi(u) \leqq \frac{M}{\sqrt{n} u^3}, \tag{3.42}$$

where M is a constant.

The inequality (3.42) holds in every case for u satisfying the inequality $e^{-u^2/2} \leqq M/\sqrt{n}u^2$. Hence, $u^2(1 - 4 \log u/u^2) \geqq \log(n/M^2)$, and since $4 \log u/u^2 < \frac{1}{3}$ for u $> e^{3/2}$, we have

$$u \geqq \sqrt{\tfrac{3}{2} \log \frac{n}{M^2}} \tag{3.43}$$

for $n > M^2 \exp\{2e^3/3\}$.

Thus, (3.42) holds at least for u satisfying (3.43).

Letting $M = N_3 c_3$ and using (3.41), we find that

$$|1 - F_n(x)| < 13(L_2 + N_3) c_3 n / x^3$$

for

$$\sqrt{3n \log(\sqrt{n}/M)} < x \leqq 4\sqrt{n \log(\sqrt{n}/M)}$$

and therefore,

$$\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| < \frac{(13L_2 + 14N_3) c_3 n}{x^3}. \tag{3.44}$$

We now treat the values of $x > 4\sqrt{n \log(\sqrt{n}/N_3 c_3)}$ and we make use of Theorem 1 after having set $y = x/2$, $m = 3$. As a preliminary, we estimate $1/x\sqrt{n} \log(x^3/nc_3 K_3)$. To this end, let $x = u\sqrt{n}$. Then the expression being estimated assumes the form $[3 \log u + \log(\sqrt{n}/c_3 K_3)]/u$. For $x > 4\sqrt{n \log(\sqrt{n}/N_3 c_3)}$, we have $u > 4\sqrt{\log(\sqrt{n}/N_3 c_3)}$. It is not hard to see that $3 \log u/u < 1.1$ for $u > 4$. Therefore,

$$\frac{1}{x} \sqrt{n} \log \frac{x^3}{K_3 c_3 n} < 1.1 + \tfrac{1}{4} \sqrt{\log \frac{\sqrt{n}}{K_3 c_3}} + \frac{\log 6}{4}. \tag{3.45}$$

Hence by Theorem 1 and some simple computations, we find that $1 - F_n(x) < L_3 c_3 n/x^3$ for $x > 4\sqrt{n \log(\sqrt{n}/N_3 c_3)}$, where L_3 is an absolute constant. Consequently,

$$\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| < \frac{(L_3 + N_3) c_3 n}{x^3}. \tag{3.46}$$

Consider now the case $\sqrt{n}/N_3 c_3 \leqq L_0 = \exp\{e^3/3\}$. If $x^3 \leqq c_3 n N_3$, then evidently

$$\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right) \right| \leqq 1 \leqq \frac{c_3 N_3 n}{x^3}. \tag{3.47}$$

But if $x^3 > c_3 n N_3$, then

$$0 < \sqrt{n}\,\frac{3\log x - \log K_3 c_3 n}{x} < \frac{3\sqrt{n}\log \frac{x}{\sqrt{n}}}{x} + \log\frac{\sqrt{n}}{N_3 c_3} + 3\log 6$$
$$< 3 + 3\log 6 + \log L_0.$$

Letting $y = x$ in Theorem 1, we find

$$1 - F_n(x) < \frac{L_4 c_3 n}{x^3}, \qquad x^3 > c_3 n N_3, \tag{3.48}$$

where L_4 is an absolute constant.

Clearly,

$$1 - \Phi\left(\frac{x}{\sqrt{n}}\right) < \frac{4n^{3/2}}{x^3} < \frac{4N_3 c_3 n L_0}{x^3}. \tag{3.49}$$

From (3.39) and (3.49), it follows that

$$\left| F_n(x) - \Phi\left(\frac{x}{\sqrt{n}}\right)\right| < \frac{L_5 c_3 n}{x^3}, \qquad x > \sqrt{n},$$

where L_5 is an absolute constant.

The proof of Theorem 2 is complete.

4. Proof of Theorems 3 and 4

Without loss of generality, we may let $B(g) = 1$ in (1.5). We first prove the sufficiency of the condition of Theorem 3. As in the proof of Theorem 2, we confine ourselves to values of $x > \sqrt{n}$.

Let $x_1(h)$ be a solution of the equation

$$g'(x) = \frac{h(1+\alpha)}{2}, \qquad h > 0. \tag{4.1}$$

For sufficiently small h, such a solution exists and by the monotonicity of $g'(x)$ is unique. On account of (1.5),

$$g(x) < g(1)x^{\alpha}. \tag{4.2}$$

Therefore $g'(x) < \alpha g(1)x^{\alpha-1}$, and hence

$$x_1(h) < \left[\frac{1+\alpha}{2\alpha}\,\frac{h}{g(1)}\right]^{1/(\alpha-1)}. \tag{4.3}$$

Let

$$R_1(h) = \int_{-\infty}^{x_1(h)} e^{hx}\,dF(x), \qquad R_2(y, h) = \int_{x_1(h)}^{y} e^{hx}\,dF(x).$$

We retain the notations of Sections 2 and 3 for the quantities defined in terms of $R_1(h)$ and $R_2(y, h)$.

Clearly,

$$R_2(y, h) = [1 - F(x)]e^{hx}\Big|_{y}^{x_1(h)} + h\int_{x_1(h)}^{y} e^{hx}[1 - F(x)]\,dx. \tag{4.4}$$

For $y > x_1(h) > 1/h$,

$$\int_{x_1(h)}^{y} e^{hx}[1-F(x)]\,dx < \frac{c(g)}{h}\int_{x_1(h)h}^{yh} e^{x-g(x/h)}\,dx, \tag{4.5}$$

where $c(g) = \int_0^\infty e^{g(x)}\,dF(x)$. Introduce the function $v = x-g(x/h)$. Clearly,

$$dv = \left[1-\frac{1}{h}g'\left(\frac{x}{h}\right)\right]dx.$$

For $x \geqq x_1(h)h$, $dx \leqq 2(1-\alpha)^{-1}dv$ and therefore

$$\int_{x_1(h)h}^{yh} e^{x-g(x/h)}\,dx < \frac{2}{1-\alpha}e^{yh-g(y)}. \tag{4.6}$$

Since $hx-g(x)$ is monotone increasing for $x \geqq x_1(h)$, we have

$$[1-F(x_1(h))]e^{hx_1(h)} < c(g)e^{yh-g(y)}.$$

Now taking (4.5) and (4.6) into consideration, we conclude from (4.4) that

$$R_2(y, h) < \frac{3-\alpha}{1-\alpha}c(g)e^{yh-g(y)}. \tag{4.7}$$

Consider now $R_1(h)$. Obviously,

$$R_1(h) = \int_{-\infty}^{1/h} e^{hx}\,dF(x)+\int_{1/h}^{x_1(h)} e^{hx}\,dF(x). \tag{4.8}$$

Let us estimate

$$\int_{1/h}^{x_1(h)} x^k e^{hx}\,dF(x),$$

on the assumption that $x_1(h) > 1/h$. First of all, it is not hard to see that

$$\int_{1/h}^{x_1(h)} x^k e^{hx}\,dF(x) < c(g)\left[\frac{e^{1-g(1/h)}}{h^k}+\int_{1/h}^{x_1(h)} x^{k-1}[hx+k]e^{hx-g(x)}\,dx\right]. \tag{4.9}$$

The function $hx-g(x)$ assumes a maximum value at one of the endpoints of the interval $[1/h, x_1(h)]$. By (1.5),

$$hx_1(h) = \frac{2x_1(h)}{1+\alpha}g'(x_1(h)) < \frac{2\alpha}{1+\alpha}g(x_1(h)).$$

Therefore,

$$g(x_1(h))-hx_1(h) > \frac{1-\alpha}{1+\alpha}g(x_1(h)) > \frac{1-\alpha}{1+\alpha}g\left(\frac{1}{h}\right). \tag{4.10}$$

Further, by (1.6) and (4.3),

$$x_1^\beta(h)\exp\left\{-g\left(\frac{1}{h}\right)\right\} = o(h^m) \tag{4.11}$$

for any β and $m > 0$. From (4.9)–(4.11), it follows that

$$\int_{1/h}^{x_1(h)} x^k e^{hx}\,dF(x) = o(h^m). \tag{4.12}$$

Expanding e^{hx} in the first integral of (4.8) and using the estimate (4.12), we find that, for any $m > 0$,

$$R_1(h) = 1+\sum_{k=2}^{m}\alpha_k\frac{h^k}{k!}+O(h^{m+1}), \tag{4.13}$$

$$\bar{R}_1(h) = \sum_{k=2}^{m}\alpha_k\frac{h^{k-1}}{(k-1)!}+O(h^m), \tag{4.14}$$

where $\alpha_k = \mathbf{E}\xi_1^k$. From (4.13) and (4.14), it follows that

$$m(h) = \frac{\bar{R}_1(h)}{R_1(h)} = \sum_{k=2}^{m}\gamma_k\frac{h^{k-1}}{(k-1)!}+O(h^m), \tag{4.15}$$

where γ_k is the k-th cumulant of the random variable ξ_1. Analogous reasoning shows that

$$\sigma(h) = 1+O(h). \tag{4.16}$$

In Section 3, it was proved that the equation $u = m(h)$ has

$$h(u) = u+O(u^2) \tag{4.17}$$

as a solution for sufficiently small $u \in A$. If $h(u)$ is expressed in the form

$$h(u) = u+\sum_{k=2}^{m}\lambda_k u^k+\varphi(u),$$

where the λ_k are the coefficients of the series which result when the series (4.15) for $m(h)$ is inverted, and if this expression is substituted in the equation $u = m(h)$ having been first represented as

$$u = \sum_{k=2}^{m+1}\gamma_k\frac{h^{k-1}}{(k-1)!}+O(h^{m+1}),$$

then the estimate $\varphi(u) = O(h^{m+1})$ is obtained instantly. Therefore in view of (4.17), $\varphi(u) = O(u^{m+1})$. Thus,

$$h(u) = u+\sum_{h=2}^{m}\lambda_k u^k+O(u^{m+1}). \tag{4.18}$$

Using (1.5), we can easily show that

$$\beta^\alpha g(x) < g(\beta x) \tag{4.19}$$

for any $0 < \beta < 1$. Therefore,

$$\frac{x}{n} \leqq \frac{g(x)}{x} < 2^{(1+\alpha)/2}\frac{g(2^{-(1+\alpha)/2\alpha}x)}{x} \tag{4.20}$$

for $x \leqq \Lambda(n)$.

Consider now the values of $x \leqq \Lambda(n)$ for which $x/n \in A$. By (4.2),

$$\Lambda(n) < [g(1)n]^{1/(2-\alpha)}. \tag{4.21}$$

Therefore, for sufficiently large n, $h(x/n)$ exists and, by (4.17),

$$h\left(\frac{x}{n}\right) = \frac{x}{n}+O\left(\frac{x^2}{n^2}\right). \tag{4.22}$$

Let $h_g(y)$ be the solution of the equation

$$n \exp \{yh - g(y)\} = 1. \tag{4.23}$$

Obviously, $h_g(y) = (g(y) - \log n)/y$. By condition (1.6),

$$h_g(y) > \frac{g(y)}{y}\left(1 - \frac{2}{\rho(\sqrt{n})}\right). \tag{4.24}$$

From (4.20), (4.22), and (4.24), it follows that $h(x/n) < h_g(2^{-(1+\alpha)/2\alpha}x)$ for sufficiently large n. Hence, by (4.13) and (4.23),

$$R\left(y, h\left(\frac{x}{n}\right)\right) = O\left(\exp\left\{\frac{1}{2^{1-\varepsilon}}\frac{x^2}{n}\right\}\right), \tag{4.25}$$

for $y = \max\,[2^{-(1+\alpha)/2\alpha}x, x_1(h)]$ and any $\varepsilon > 0$. Letting $y = \max\,[2^{-(1+\alpha)/2\alpha}x, x_1(h)]$ and $h = h(x/n)$ in (4.7), we have

$$R_2\left(y, h\left(\frac{x}{n}\right)\right) = O\left(\exp\left\{\frac{x^2}{2^{(1+\alpha)/2\alpha - \varepsilon_n}} - g\left(\frac{x}{2^{(1+\alpha)/2\alpha}}\right)\right\}\right) \tag{4.26}$$

for any $\varepsilon > 0$.

From (3.2), using (4.25) and (4.26), we deduce the estimate

$$\begin{gathered}\int_x^\infty e^{-hu}\, d[F_{nh}^{(y)}(u) - \Phi_{nh}(u)] = O\left(n \exp\left\{-g\left(\frac{x}{2^{(1+\alpha)/2\alpha}}\right)\right\}\right),\\ h = h\left(\frac{x}{n}\right), \quad y = \max\,[2^{-(1+\alpha)/2\alpha}x, x_1(h)], \quad \Phi_{nh}(u) = \int_{-\infty}^u e^{hv}\, dF^{(x_1(h))}(v).\end{gathered} \tag{4.27}$$

Taking (4.18) and (4.21) into consideration, we can easily show that

$$xh\left(\frac{x}{n}\right) = \frac{x^2}{n} + \frac{x^3}{n^2}\sum_{k=2}^{[1/(1-\alpha)]} \lambda_k \left(\frac{x}{n}\right)^{k-2} + o(1) \tag{4.28}$$

for $x \leqq \Lambda(n)$.

We now let $h = h(x/n)$ in (3.3) and we use Cramér's reasoning (see [1]) taking (4.28) into account. As a result, we obtain

$$\int_x^\infty e^{-hu}\, d\Phi_{nh}(u) = \left[1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right] \exp\left\{\frac{x^3}{n^2}\lambda^{[\alpha/(1-\alpha)]}\left(\frac{x}{n}\right)\right\}(1 + o(1)), \tag{4.29}$$

where

$$\lambda^{[\alpha/(1-\alpha)]}(u) = \sum_{k=2}^{[1/(1-\alpha)]} \lambda_k u^{k-2}.$$

From (4.20) and (4.21), it follows that

$$n \exp\left\{-g\left(\frac{x}{2^{(1+\alpha)/2\alpha}}\right)\right\} = o\left(\left(1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right) \exp\left\{\frac{x^3}{n^2}\lambda^{[\alpha/(1-\alpha)]}\left(\frac{x}{n}\right)\right\}\right) \tag{4.30}$$

for $x \leqq \Lambda(n)$.

With the help of the estimates (4.26), (4.27), and (4.30), we find from (2.2) and (4.29) that

$$1-F_n^{(y)}(x) = \left[1-\Phi\left(\frac{x}{\sqrt{n}}\right)\right] \exp\left\{\frac{x^3}{n^2}\lambda^{[\alpha/(1-\alpha)]}\left(\frac{x}{n}\right)\right\}(1+o(1)), \tag{4.31}$$

$$y = \frac{x}{3^{(1+\alpha)/2\alpha}},$$

for $x \leqq \Lambda(n)$, $x/n \in A$.

By virtue of (4.30) and the inequality $1-F(y) < c(g)\exp\{-g(y)\}$, (4.31) and (2.1) imply (1.8) for $x/n \in A$.

Passage to values of $x/n \notin A$ is effected similarly to what was done in the proof of Theorem 2.

Let us now prove the necessity of the condition of Theorem 3. Suppose that (1.8) holds for $x \leqq 2\Lambda(n)$.

Clearly,

$$1-F_n(x) > (1-F(x))(1-F_{n-1}(0)).$$

Therefore, for sufficiently large n,

$$1-\Phi\left(\frac{2^{(3+\alpha)/4}\Lambda(n)}{\sqrt{n}}\right) > \tfrac{1}{4}(1-F(2\Lambda(n))). \tag{4.32}$$

Let $g_1(x) = -\log(1-F(x))$. From (4.32) and (4.19), it follows that

$$g_1(2\Lambda(n)) > \frac{2^{(1+\alpha)/2}\Lambda^2(n)}{n} = 2^{(1+\alpha)/2}g(\Lambda(n)) > 2^{(1-\alpha)/2}g(2\Lambda(n)) \tag{4.33}$$

for sufficiently large n.

Consider the function $y(x)$ determined by the equation $y^2 = xg(y)$. It is easy to see that

$$\frac{dy}{dx} = \frac{g(y)}{2y-xg'(y)} < \frac{g(y)}{(2-\alpha)y}. \tag{4.34}$$

Setting $x = n$ and using the estimate (4.2), we find that

$$\Lambda(n+1)-\Lambda(n) < \frac{g(\Lambda(n))}{(2-\alpha)\Lambda(n)} < \frac{g(1)}{\Lambda^{1-\alpha}(n)}.$$

Hence

$$\lim_{n\to\infty}\frac{\Lambda(n+1)}{\Lambda(n)} = 1. \tag{4.35}$$

Because of (4.19),

$$\frac{g(2\Lambda(n+1))}{g(2\Lambda(n))} < \left[\frac{\Lambda(n+1)}{\Lambda(n)}\right]^{\alpha}. \tag{4.36}$$

From (4.33), (4.35), and (4.36), it follows that

$$g_1(2\Lambda(n)) > 2^{(1-\alpha)/3}g(2\Lambda(n+1))$$

for sufficiently large n. Therefore,

$$g_1(x) > 2^{(1-\alpha)/3}g(x) \tag{4.37}$$

for sufficiently large x.

From (4.37), it follows that $\int_0^\infty e^{g(x)}dF(x) < \infty$. Similarly, it can be shown that $\int_{-\infty}^0 e^{g(|x|)}dF(x) < \infty$. Thus, $\mathbf{E}\exp g(|\xi_1|) < \infty$, q.e.d.

As to Theorem 4, its proof is entirely analogous to that of Theorem 3. In the proof of the sufficiency part of Theorem 4, $R_1(h)$ and $R_2(h)$ are defined just as in Sections 2 and 3. It is not hard to see that (1.10) holds for

$$\sqrt{n} \leqq x \leqq \sqrt{(m/2-1)n \log n}$$

if, for these values of x,

$$h\left(\frac{x}{n}\right) < h_m\left(\frac{x}{3}\right) \tag{4.38}$$

and

$$\frac{x^2}{n} < m \log x - \log n, \tag{4.39}$$

beginning with a certain n. Relation (4.39) clearly holds if

$$u^2 < \left(\frac{m}{2}-1\right) \log n + m \log u, \tag{4.40}$$

where $u = x/\sqrt{n}$. If $u \geqq 1$, then (4.40) is satisfied for $u < \sqrt{(m/2-1) \log n}$. Thus, the condition $\sqrt{n} \leqq x \leqq \sqrt{(m/2-1)n \log n}$ implies (4.39). On the other hand, (4.38) holds if

$$\frac{x^2}{n} < \frac{3m}{2} \log x - \frac{3 \log n}{2}.$$

Therefore (4.38) follows from (4.39).

Assume now that

$$1-F_n(x) = [1-\Phi(x)](1+o(1))$$

for $|x| \leqq \sqrt{(m+1)n \log n}$. Employing similar reasoning to that used to obtain (4.33), one can show that

$$(m+1) \log n < 2g_1(\sqrt{(m+1)n \log n})$$

for sufficiently large n.

Hence, $g_1(x) > (m+\frac{1}{2}) \log x$ for sufficiently large x. This inequality clearly implies that $\int_0^\infty x^m dF(x) < \infty$. In a similar fashion, it can also be proved that $\int_{-\infty}^0 |x|^m dF(x) < \infty$.

5. Proof of Theorem 5

Suppose for simplicity that $n_0 = 1$. Then $F(x)$ can be represented as $F(x) = aF_1(x)+(1-a)F_2(x)$, $0 < a \leqq 1$, where $F_1(x)$ is absolutely continuous and $F_1'(x) < L < \infty$. Let (the symbol $*$ stands for convolution)

$$\tilde{F}_n(x) = F_n(x)-(1-a)^n F_2^{*n}(x)-na(1-a)^{n-1}F_1(x) * F_2^{*(n-1)}(x),$$

$$f(t) = \int e^{itx} dF(x), \qquad g_n(t) = \int e^{itx} dF_n(x).$$

Clearly,

$$g_n(t) = f^n(t)-(1-a)^n f_2^n(t)-na(1-a)^{n-1} f_1(t) f_2^{n-1}(t)$$

where $f_j(t) = \int e^{itx} dF_j(x)$, $j = 1, 2$. Hence, (5.1) $$|g_n(t)| < \frac{n^2a^2}{2}(a|f_1(t)|+(1-a)|f_2(t)|)^{n-2}|f_1(t)|^2.$$

Further,

(5.2) $$|g_n(t)-f^n(t)| < (n+1)(1-a)^{n-1}.$$

It is known that

$$\left|f^n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\left(1+\frac{\alpha_3}{\sqrt{n}}(it)^3\right)\right| < \frac{\delta(n)}{T_{3n}}|t|^3e^{-t^2/4}$$

for $|t| \leqq T_{3n} = \sqrt{n}/24c_3$ (see, for example, [10], § 41), where $\delta(n)$ depends only on n and $\lim_{n\to\infty}\delta(n) = 0$.

Therefore, for $|t| \leqq T_{3n}$,

(5.3) $$\left|g_n\left(\frac{t}{\sqrt{n}}\right)-e^{-t^2/2}\left(1+\frac{\alpha_3}{6\sqrt{n}}(it)^3\right)\right| < \frac{\delta(n)}{T_{3n}}|t|^3e^{-t^2/4}+(n+1)(1-a)^{n-1}.$$

With the help of (5.1) and (5.3), it is not hard to show that

(5.4) $$\tilde{p}_n(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\left(1+\frac{\alpha_3}{6\sqrt{n}}(x^3-3x)\right)+R_n(x),$$

where $\tilde{p}_n(x) = \sqrt{n}\tilde{F}_n'(x\sqrt{n})$ and $\sup_x R_n(x) = o(1/\sqrt{n})$ (cf. [10], § 47).

Hereafter, we shall use the notations of Sections 2 and 3. Let

$$\tilde{\Phi}_{nh}(u) = \Phi_{nh}(u)-(1-a)^nF_{2h}^{*n}(u)-na(1-a)^{n-1}F_{2h}^{*n-1}(u)*F_{1h}(u),$$

where

$$F_{ih}(u) = \begin{cases}\int_{-\infty}^{u}e^{hy}dF_i(y), & u \leqq \frac{1}{h},\\ \int_{-\infty}^{1/h}e^{hy}dF_i(y), & u > \frac{1}{h},\end{cases} \qquad i = 1, 2.$$

Let $x/n \in A$. If $h = h(x/n)$ ($h(u)$ is a solution of the equation $u = m(h)$), then, as easily seen,

(5.5) $$\tilde{\Phi}_{nh}'(x) = R_1^n(h)\tilde{\tilde{\Phi}}_{nh}'(0),$$

where $\tilde{\tilde{\Phi}}_{nh}(u) = \tilde{\Phi}_{nh}(u+nm(h))/R_1^n(h)$. Set

$$f_h(t) = \frac{1}{R_1(h)}\int_{-\infty}^{1/h}e^{itx}d\Phi_h(x).$$

Choose a $B > 0$ so that $F_1(B)-F_1(-B) > 0$. It is not hard to show that

(5.6) $$1-|f_h(t)|^2 > \frac{2e^{-2Bh}}{R_1^2(h)}\int_{-B}^{B}\int_{-B}^{B}\sin^2\frac{t(u-v)}{2}F_1'(u)F_1'(v)\,du\,dv, \qquad \frac{1}{B} \geqq h \geqq 0.$$

By (5.6), there exists, for any positive ε and η, $0 < \rho(\varepsilon, \eta) < 1$, such that for $|t| > \varepsilon$,

(5.7) $$|f_h(t)| < \rho(\varepsilon, \eta)$$

uniformly with respect to $0 < h < \eta$.

Now

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|f_{1h}(t)|^2\,dt = \frac{1}{R_1^2(h)}\int_{-\infty}^{1/h} e^{2hx}F_1'^2(x)\,dx < \frac{e^2L}{R_1^2(h)}, \tag{5.8}$$

where

$$f_{1h}(t) = \frac{1}{R_1(h)}\int e^{itx}\,dF_{1h}(x).$$

With the help of (5.1), (5.3), (5.7), and (5.8) and using reasoning standard in the proof of local limit theorems, we can easily show that

$$\tilde{\bar{\Phi}}'_{nh}(\sigma(h)\sqrt{n}u) = \frac{1}{\sigma(h)\sqrt{2\pi n}}\,e^{-u^2/2}+O\left(\frac{1}{n}\right) \tag{5.9}$$

uniformly with respect to h in any finite interval.

By (3.9) and (3.14), $\sigma(h) = 1+O(x/n)$, $h = h(x/n)$. Therefore, (5.5) and (5.9), by virtue of (3.28) and (3.32), imply that

$$e^{-hx}\tilde{\Phi}'_{nh}(x) = \frac{1}{\sqrt{2\pi n}}\,e^{-x^2/2n}+O\left(\frac{x^2}{n^{3/2}}\,e^{-x^2/2n}\right), \tag{5.10}$$

for $x < \Delta_n\sqrt{n}$, $x/n \in A$, and $h = h(x/n)$, where $\Delta_n = \sqrt{3\log(\sqrt{n}/N_3c_3)}$.

Consider now values of x for which $x/n \notin A$. In Section 3, it was shown that in this case there exists an $x_0 > x$ such that

$$x_0-x < 16ec_3\,\frac{x^2}{n}. \tag{5.11}$$

Evidently,

$$\tilde{\Phi}'_{nh}(x) = R_1^n(h)\tilde{\bar{\Phi}}'_{nh}(x-x_0), \qquad h = h(x_0/n). \tag{5.12}$$

But by (3.32) and (5.11),

$$\exp\left\{-\frac{(x-x_0)^2}{2n}\right\} = 1+O\left(\frac{x^4}{n^3}\right),$$

$$\exp\{-hx\} = \exp\left\{-\frac{x^2}{n}\right\}\left[1+O\left(\frac{x^4}{n^3}\right)\right], \qquad h = h\left(\frac{x_0}{n}\right),$$

for $x < \Delta_n\sqrt{n}$.

Therefore,

$$e^{-hx}\tilde{\Phi}'_{nh}(x) = \frac{1}{\sqrt{2\pi n}}\,e^{-x^2/2n}+O\left(\frac{x^2}{n^{3/2}}\,e^{-x^2/2n}\right), \quad h = h\left(\frac{x_0}{n}\right), \quad x < \Delta_n\sqrt{n}. \tag{5.13}$$

Evidently,

$$\frac{\partial}{\partial u}F_{nh}^{(y)}(u) = \frac{\partial}{\partial u}\sum_{k=0}^{n} C_n^k\,\Phi_{kh} * \Psi_{(n-k)h}^{(y)}(u), \tag{5.14}$$

where $\Psi_{kh}^{(y)}$ is the k-fold convolution of $\Psi_h^{(y)}$. By (5.9),

$$n\frac{\partial}{\partial u}\tilde{\Phi}_{(n-1)h} * \Psi_h^{(y)}(u) = O(\sqrt{n}R^n(y,h)R_2(y,h)). \tag{5.15}$$

Now

$$\frac{\partial}{\partial u}\sum_{k=1}^{n-2} C_n^k \tilde{\Phi}_{kh} * \Psi^{(y)}_{(n-k)h}(u) = O(n^2 R^n(y,h) R_2^2(y,h)). \tag{5.16}$$

Set $h_n(x) = h(x/n)$ if $x/n \in A$ and $h_n(x) = h(x_0/n)$ if $x/n \notin A$ (x_0 satisfies (5.11)).

From (5.15) and (5.16), it follows by (3.17), (3.18), and (3.32) that

$$e^{-hx}\frac{\partial}{\partial x}\sum_{k=1}^{n-1} C_n^k \tilde{\Phi}_{kh} * \Psi^{(y)}_{(n-k)h}(x) = O\left(\frac{\sqrt{n}}{x^3}e^{-x^2/3n} + \frac{n^2}{x^6}e^{-x^2/6n}\right) \tag{5.17}$$

for $x < \Delta_n\sqrt{n}$, $y = x/6$ and $h = h_n(x)$. For $u < n/h$, it is clear that $\Psi^{(y)}_{nh}(u) = 0$. By (3.14),

$$\frac{n}{h(x/n)} > \frac{n^2}{2x} > \Delta_n\sqrt{n} \tag{5.18}$$

for $x < \Delta_n\sqrt{n}$ and $n > 5$. Finally,

$$\begin{aligned} &e^{-hu}\frac{\partial}{\partial u}\sum_{k=1}^{n} C_n^k[\Phi_{kh} - \tilde{\Phi}_{kh}] * \Psi^{(y)}_{(n-k)h}(u) \\ &\quad = e^{-hu}\frac{\partial}{\partial u}\left[\sum_{k=1}^{n} C_n^k(1-a)^k F_{2h}^{*k} * \Psi^{(y)}_{(n-k)h}(u)\right. \\ &\qquad \left. + a\sum_{k=1}^{n} C_n^k k(1-a)^{k-1} F_{1h} * F_{2h}^{*(k-1)} * \Psi^{(y)}_{(n-k)h}(u)\right] \\ &\quad < \frac{\partial}{\partial u}\left[(1-a)F_2(u) + \int_{1/h}^{u} dF(v)\right]^{*n} + naLe^{1-hu}\left[(1-a)F_{2h}\left(\frac{1}{h}\right) + R_2(y,h)\right]^n. \end{aligned} \tag{5.19}$$

It is not hard to see that, for $h = h_n(x)$ and $y = x/6$,

$$n[(1-a)F_{2h}(1/h) + R_2(y,h)]^n = o((1-a+\varepsilon)^n), \tag{5.20}$$

where ε is an arbitrarily small positive quantity.

Let

$$G(u,h) = (1-a)F_2(u) + \int_{1/h}^{u} dF(v).$$

Clearly,

$$\int_0^{\Delta_n\sqrt{n}} x^2 \frac{\partial}{\partial x} G^{*n}(x, h_n(x))\,dx < \Delta_n^3 n^{3/2}[2-a-F(m_n)]^n, \tag{5.21}$$

where $m_n = \min_{0 \leq x < \Delta_n\sqrt{n}} 1/h_n(x)$. Set

$$Q^{(y)}(u) = \begin{cases} F(u) - F(y), & u \geqq y, \\ 0, & u < y, \end{cases} \qquad Q_k^{(y)}(u) = Q^{(y)*k}(u).$$

Obviously,

$$\frac{d}{du}F_n(u) - \frac{\partial}{\partial u}F_n^{(y)}(u) < \sum_{k=0}^{n-1} C_n^k \frac{\partial}{\partial u} F_k * Q^{(y)}_{n-k}(u). \tag{5.22}$$

Set

$$p_{1n}(x) = \frac{\partial}{\partial x}F_{n-1} * Q^{(x/6)}(x), \qquad p_{2n}(x) = \frac{\partial}{\partial x}\sum_{k=1}^{n-2} C_n^k \tilde{F}_k * Q^{(x/6)}_{n-k}(x).$$

It is not hard to see that

$$\int_B^\infty x^3 p_{1n}(x)\,dx \leqq \int_{B/6}^\infty dF(u)\int_0^{6u} x^3\,dF_{n-1}(x-u) < 216\int_{B/6}^\infty u^3\,dF(u).$$

Therefore, for any $B > 0$, we have

$$\int_B^\infty x^3 p_{1n}(x\sqrt{n})\,dx = o\left(\frac{1}{n^2}\right). \tag{5.23}$$

Now, $p_{2n}(x) < Ln^2(1-F(x/6))^2/2$. Therefore,

$$\int_B^\infty x^3 p_{2n}(x\sqrt{n})\,dx = O\left(\frac{1}{nB^2}\right). \tag{5.24}$$

It is not hard to show that

$$\begin{aligned}&\int_{B\sqrt{n}}^{\Delta_n\sqrt{n}} x^3 \frac{\partial}{\partial x}\sum_{k=1}^{n-2} C_n^k(F_k-\tilde{F}_k) * Q_{n-k}^{(x/6)}(x)\,dx\\ &\quad< \int_{B\sqrt{n}}^{\Delta_n\sqrt{n}} x^3 \frac{\partial}{\partial x}\sum_{k=1}^{n-2} C_n^k(F_k-\tilde{F}_k) * Q_{n-k}^{(B\sqrt{n})}(x)\,dx = o((1-a+\varepsilon)^n)\end{aligned} \tag{5.25}$$

for any $\varepsilon > 0$.

Clearly, $Q_n^{(y)}(x) = 0$ for $x < ny$. Hence, for $n > 6$,

$$\frac{\partial}{\partial x} Q_n^{(x/6)}(x) = 0. \tag{5.26}$$

Clearly,

$$\begin{aligned}&\left|\int_0^\infty \left| p_n(x) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right| x^3\,dx - \frac{|\alpha_3|}{6\sqrt{2\pi n}}\int_0^\infty x^4|x^2-3|e^{-x^2/2}\,dx\right|\\ &\quad< \int_0^B x^3\left|\tilde{p}_n(x) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2} - \frac{\alpha_3}{6\sqrt{2\pi n}}(x^3-3x)e^{-x^2/2}\right| dx\\ &\quad+ \int_0^B |p_n(x)-\tilde{p}_n(x)|x^3\,dx + \int_B^{\Delta_n}\left|\sqrt{n}\frac{\partial}{\partial x}F_n^{(y)}(x\sqrt{n}) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right| x^3\,dx\\ &\quad+ \frac{|\alpha_3|}{6\sqrt{2\pi n}}\int_B^\infty x^4(x^2-3)e^{-x^2/2}\,dx + \int_B^{\Delta_n}\left(p_n(x) - \sqrt{n}\frac{\partial}{\partial x}F_n^{(y)}(x\sqrt{n})\right)x^3\,dx\\ &\quad+ \int_{\Delta_n}^\infty p_n(x)x^3\,dx + \frac{1}{\sqrt{2\pi}}\int_{\Delta_n}^\infty x^3 e^{-x^2/2}\,dx, \qquad y = \frac{x}{6}.\end{aligned} \tag{5.27}$$

Since $\partial F_n^{(y)}(u)/\partial u = e^{-hu}\partial F_{nh}^{(y)}(u)/\partial u$, (5.10), (5.13), (5.14), (5.17), (5.18)–(5.21) imply that

$$\begin{aligned}&\int_B^{\Delta_n}\left|\sqrt{n}\frac{\partial}{\partial x}F_n^{(y)}(x\sqrt{n}) - \frac{1}{\sqrt{2\pi}}e^{-x^2/2}\right| x^3\,dx\\ &\quad= O\left(\frac{1}{\sqrt{n}}\int_B^\infty x^2 e^{-x^2/6}\,dx\right) + o((1-a+\varepsilon)^n).\end{aligned} \tag{5.28}$$

Then by (5.22)–(5.26),

$$(5.29)\quad \int_B^{\Delta_n} \left(p_n(x)-\sqrt{n}\,\frac{\partial}{\partial x}\,F_n^{(y)}(x\sqrt{n})\right) x^3\,dx = O\left(\frac{1}{\sqrt{nB^2}}\right)+o\left(\frac{1}{\sqrt{n}}\right)+o((1-a+\varepsilon)^n).$$

Letting $y = x/2$ in Theorem 1 and using the estimate (3.45), we find that

$$(5.30)\qquad 1-F_n(x) = O(n(1-F(x/2))+O(n^{5/2}/x^6)$$

for $x > 4\sqrt{n\log(\sqrt{n}/N_3c_3)}$. Let Δ_n' denote $4\sqrt{\log(\sqrt{n}/N_3c_3)}$. It easily follows from (5.30) that

$$(5.31)\qquad \int_{\Delta'_n}^{\infty} x^3\,dF_n(x\sqrt{n}) = o\left(\frac{1}{\sqrt{n}}\right).$$

Clearly, $e^{-\Delta_n{}^2/2} = O(n^{-3/4})$. Hence,

$$(5.32)\qquad \int_{\Delta_n}^{\infty} x^3 e^{-x^2/2}\,dx = o\left(\frac{1}{\sqrt{n}}\right)$$

and

$$(5.33)\qquad 1-\Phi(\Delta_n) = o\left(\frac{1}{\Delta_n^3\sqrt{n}}\right).$$

Using the estimates (2.1), (3.19), (3.30), and (5.33), we can easily show that $1-F_n(\Delta_n\sqrt{n}) = o(1/\Delta_n^3\sqrt{n})$ and therefore,

$$(5.34)\qquad \int_{\Delta_n}^{\Delta'_n} x^3\,dF_n(x\sqrt{n}) = o\left(\frac{1}{\sqrt{n}}\right).$$

From (5.27)–(5.29), (5.4), (5.31), (5.32) and (5.34) on setting $B = \sqrt[5]{\sqrt{n}/R_n}$, we deduce that

$$\int_0^{\infty} \left| p_n(x)-\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\right| x^3\,dx = \frac{|\alpha_3|}{6\sqrt{2\pi n}}\int_0^{\infty} x^4|x^2-3|e^{-x^2/2}\,dx+o\left(\frac{1}{\sqrt{n}}\right).$$

Similarly,

$$\int_{-\infty}^{0} \left| p_n(x)-\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\right| |x|^3\,dx = \frac{|\alpha_3|}{6\sqrt{2\pi n}}\int_0^{\infty} x^4|x^2-3|e^{-x^2/2}\,dx+o\left(\frac{1}{\sqrt{n}}\right).$$

The theorem is proved.

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