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LARGE DEVIATIONS OF SUMS OF INDEPENDENT RANDOM VARIABLES

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This paper deals with numerous variants of bounds for probabilities of large deviations of sums of independent random variables in terms of ordinary and generalized moments of individual summands. A great deal of attention is devoted to the study of the precision of these bounds. In this connection comparisons are made with precise asymptotic results. At the end of the paper various applications of the bounds for probabilities of large deviations to the strong law of large numbers, the central limit theorem and to certain other problems are discussed.

0. Introduction. Let $X_1, X_2, \cdots, X_n$ be independent random variables and put $F_i(x) = P(X_i < x)$. Let us also put

$$\sigma_i^2 = \text{Var}\, X_i, \qquad B_n^2 = \Sigma_1^n \sigma_i^2,$$

$$A_t^+ = \Sigma_1^n \textstyle\int_{u>0} u^t\, dF_i(u), \qquad A_t = \Sigma_1^n E|X_i|^t.$$

If the X_i are identically distributed, then their common distribution function will be denoted by $F(x)$ and their variance by σ^2. We shall denote the distribution function of the standard Gaussian law by $\Phi(x)$.

Let $S_n = \Sigma_i^n X_i$. It is customary to consider as large deviations the values $S_n \geqslant x > 0$ (or $S_n \leqslant -x$) for which the probability $P(S_n \geqslant x)$ (or $P(S_n \leqslant -x)$) is small. It is also possible to examine large deviations of the type $|S_n| \geqslant x$.

In the theory of large deviations there are two possible approaches: (1) the study of the asymptotic behavior of $P(S_n \geqslant x)$ as $n \to \infty$ and $x \to \infty$; and (2) the derivation of bounds (particularly upper bounds) for $P(S_n \geqslant x)$. In this article we shall concentrate mainly on the second approach. At the same time, we shall also consider the asymptotic forms of many of the bounds discussed, partly in order to demonstrate their precision. A very large part of the asymptotic theory of large deviations is obtained under the assumption that the summands X_i are identically distributed.

We shall now try to give the briefest possible description of the inherent regularities. First of all, the asymptotic behavior of $P(S_n \geqslant x)$ depends, on the one hand, on the speed with which $x \to \infty$ as $n \to \infty$, and on the other hand, on the speed with which $1 - F(x)$ decreases as $x \to \infty$.

If $EX_1 = 0$ and $\sigma^2 < \infty$ (for the sake of simplicity we shall assume that $\sigma^2 = 1$), then there exists a monotonically increasing function $\phi(n, F)$ of n (the distribution

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function F assumes here the role of a parameter) such that for $0 < x \leqslant \phi(n, F)$ and $n \to \infty$,

$$(0.1) \qquad P(S_n \geqslant x) = \left[1 - \Phi\left(\frac{x}{n^{\frac{1}{2}}}\right)\right] \exp\left\{\frac{x^3}{n^2}\lambda^{[s]}\left(\frac{x}{n}\right)\right\}(1 + o(1)).$$

Here $\lambda^{[s]}(u)$ is a partial sum of the so-called Cramér series containing the first s terms (for the definition of the Cramér series see, for example, [43], pages 270–271), where s depends on the rate at which $1 - F(x)$ decreases.

If Cramér's condition is fulfilled, i.e., if there exists an h_0 such that

$$(0.2) \qquad Ee^{hX_1} < \infty,$$

for $|h| < h_0$, then $s = \infty$ and $\phi(n, F) = \alpha(n)n^{\frac{1}{2}}$, where $\alpha(n) \to 0$ arbitrarily slowly as $n \to \infty$ (see [10] and [38]).

If

$$Ee^{g(|X_1|)} < \infty,$$

where $g(x)$ is a continuous function with a monotonically decreasing continuous derivative satisfying the conditions

$$0 < g'(x) < \frac{\alpha g(x)}{x}, \quad \alpha < 1, \quad x > B(g)$$

and

$$g(x) > \rho(x)\log x,$$

where $\rho(x)$ tends to ∞ arbitrarily slowly when $x \to \infty$, then $s = [\alpha/(1 - \alpha)]$ (we shall assume by definition that $\lambda^{[0]}(u) = 0$), and $\phi(n, F)$ is the solution of the equation $x^2 = ng(x)$ (see [22], [23], [26], [30], [32], [39], and [40]). Finally, if there exists a t exceeding 2 such that

$$E|X_1|^t < \infty,$$

then $s = 0$ and $\phi(n, F) = ((t/2 - 1)n \log n)^{\frac{1}{2}}$ (see [30]).

If $x > \phi(n, F)$, then, in order to obtain asymptotic expressions for $P(S_n \geqslant x)$, it is necessary for the distribution function $F(x)$ to behave in a sufficiently regular way as $x \to \infty$. (An exception is the case where condition (0.2) is fulfilled and $x \sim cn$ (see [10], [38], and [41])). For example, if

$$1 - F(x) = \frac{l(x)}{x^t}(1 + o(1)), \qquad t > 2,$$

as $x \to \infty$, where $l(x)$ is a slowly varying function, then

$$(0.3) \qquad P(S_n \geqslant x) = n(1 - F(x))(1 + o(1))$$

for $x \geqslant b\phi(n, F)$, where b is any number greater than $2^{\frac{1}{2}}$ (see [27]).

If

$$1 - F(x) = e^{\chi(x)}(1 + o(1)),$$

where $\chi(x)$ changes regularly (we are not defining exactly in what sense) and

$$\chi(x)/\log x \to -\infty, \qquad \chi(x)/x \to 0$$

when $x \to \infty$, then the representation (0.3) is valid for $x \geqslant \phi_1(n, F)$, where in general

$$\limsup_{n\to\infty} \frac{\phi_1(n, F)}{\phi(n, F)} = \infty;$$

(see [26] and [29]). For example, if

$$\chi(x) = -x^{\alpha}(0 < \alpha < 1),$$

then

$$\phi_1(x, F) = \rho(n) n^{1/2(1-\alpha)},$$

where $\rho(n)$ tends arbitrarily slowly to ∞ (see [26]). For

$$\phi(n, F) < x < \phi_1(n, F)$$

there exists a special representation which is a combination of (0.1) and (0.3) (for a more detailed discussion of this see [26] and [32]). The results of the article [32] are also discussed below in Section 2.

As to the upper bounds for $P(S_n \geqslant x)$, they do not require any limitations on the distribution of the summands X_i, except perhaps for the existence of moments of one kind or another. The bounds for $P(S_n \geqslant x)$ arise from the famous Chebyshev inequality

$$P(S_n \geqslant x) \leqslant B_n^2/x^2. \tag{0.4}$$

A detailed listing of inequalities of the Chebyshev type may be found, for example in Savage's survey [58].

In the present article the point of departure is D. X. Fuk's and S. V. Nagaev's work [17], the results of which are given in Section 1.

The inequalities that are discussed in Sections 2 through 4 may at first glance seem complex and not easily applicable. This is to some extent negated by the rather numerous applications of these inequalities that already exist. Some of these are discussed in Section 6. We note that the bounds for $P(S_n > x)$ which will be discussed in the present article allow generalization to $\max_{k<n} S_k$ (see [7], [16], and [33]) and to martingales (see [16]). The extension of these bounds to Markov chains appears to be of interest.

To return now to the system of notation, everywhere below $x > 0$ is an arbitrary number, $Y = \{y_1, \cdots, y_n\}$ is any set of n positive numbers, and $y \geqslant \max\{y_1, \cdots, y_n\}$. We shall designate by $A(t; \cdot, \cdot)$, $B^2(\cdot, \cdot)$, and $\mu(\cdot, \cdot)$ sums of the corresponding absolute moments of order t truncated at the levels indicated in the parentheses, second moments, and mathematical expectations respectively. The symbol Y signifies that the moments being summed are truncated correspondingly on the levels $y_1, y_2, \cdots, y_n$. For example,

$$A(t;\, -Y, 0) = \Sigma_1^n \int_{-y_i < u < 0} |u|^t \, dF_i(u),$$

$$B^2(-Y, Y) = \Sigma_1^n \int_{|u| < y_i} u^2 \, dF_i(u),$$

$$\mu(-\infty, Y) = \Sigma_1^n \int_{u < y_i} u \, dF_i(u).$$

The symbols $c, c_1, c_2, \cdots$ designate constants—not necessarily the same at each occurrence. The symbol $c(\delta)$ denotes a constant which depends only on δ. If a constant is absolute, it is explicitly specified as such.

1. Bounds in terms of exponential moments. In [17] three types of bounds are distinguished depending on the order of the moments involved. These types are described by Theorems 1.1 through 1.3, formulated below.

THEOREM 1.1. *Suppose that* $0 < t \leq 1$. *Then we have the inequality,*

$$P(S_n \geq x) \leq \Sigma_1^n P(X_i > y_i) + P_1, \tag{1.1}$$

where

$$P_1 = \exp\left\{ \frac{x}{y} - \frac{x}{y} \log\left(\frac{xy^{t-1}}{A(t;\, 0, Y)} + 1 \right) \right\}.$$

If

$$xy^{t-1} > A(t;\, 0, Y),$$

then

$$P(S_n \geq x) \leq \Sigma_1^n P(X_i > y_i) + P_2, \tag{1.2}$$

where

$$P_2 = \exp\left\{ \frac{x}{y} - \frac{A(t;\, 0, Y)}{y^t} - \frac{x}{y} \log\left(\frac{xy^{t-1}}{A(t;\, 0, Y)} \right) \right\}.$$

Obviously $P_2 \leq P_1$.

THEOREM 1.2. *For* $1 \leq t \leq 2$ *we have the inequality*

$$P(S_n \geq x) \leq \Sigma_1^n P(X_i > y_i) + P_3, \tag{1.3}$$

where

$$P_3 = \exp\left\{ \frac{x}{y} - \left(\frac{x - \mu(-Y, Y)}{y} + \frac{A(t;\, -Y, Y)}{y^t} \right) \cdot \log\left(\frac{xy^{t-1}}{A(t;\, -Y, Y)} + 1 \right) \right\}.$$

Let us turn now to the case $t \geq 2$. We use the notations

$$P_4 = \exp\left\{ \beta\frac{x}{y} - \left(\left(1 - \frac{\alpha}{2}\right)\frac{x}{y} - \frac{\mu(-\infty, Y)}{y} \right) \cdot \log\left(\frac{\beta xy^{t-1}}{A(t;\, 0, Y)} + 1 \right) \right\},$$

$$P_5 = \exp\left\{ \left(\beta - \frac{t\alpha}{2} \right)\frac{x}{y} - \left(\beta\frac{x}{y} - \frac{\mu(-\infty, Y)}{y} \right) \cdot \log\left(\frac{\beta xy^{t-1}}{A(t;\, 0, Y)} + 1 \right) \right\},$$

$$P_6 = \exp\left\{ -\frac{\alpha x(\alpha x/2 - \mu(-\infty, Y))}{e^t B^2(-\infty, Y)} \right\}.$$

THEOREM 1.3. *Suppose $t \geq 2$, $0 < \alpha < 1$, and $\beta = 1 - \alpha$. If*

$$\max\left[t, \log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right] \geq \frac{\alpha x y}{e^t B^2(-\infty, Y)}, \tag{1.4}$$

then

$$P(S_n \geq x) \leq \Sigma_1^n P(X_i > y_i) + P_6. \tag{1.5}$$

If

$$\max\left[t, \log\left(\frac{\beta x y^{t-1}}{A(t; 0, Y)} + 1\right)\right] < \frac{\alpha x y}{e^t B^2(-\infty, Y)}, \tag{1.6}$$

then

$$P(S_n \geq x) < \Sigma_1^n P(X_i > y_i) + P_4. \tag{1.7}$$

If, instead of (1.6), *at least one of the conditions $\beta \geq t\alpha/2$ and $\beta x \geq \mu(-\infty, Y)$ is fulfilled, then*

$$P(S_n \geq x) < \Sigma_1^n P(X_i > y_i) + P_5. \tag{1.7a}$$

PROOF OF THEOREM 1.3. Define

$$\tilde{X}_i = X_i \quad \text{if} \quad X_i \leq y_i,$$
$$= 0 \quad \text{if} \quad X_i > y_i,$$

and

$$\tilde{S}_n = \Sigma_1^n \tilde{X}_i.$$

It is not difficult to see that

$$P(S_n \geq x) \leq P(\tilde{S}_n \geq x) + \Sigma_1^n P(X_i > y_i). \tag{1.8}$$

On the other hand, for all positive h,

$$P(\tilde{S}_n \geq x) \leq e^{-hx} E e^{h\tilde{S}_n} = e^{-hx} \Pi_1^n E e^{h\tilde{X}_i}. \tag{1.9}$$

We now require the following

LEMMA 1.4. *Let X be a random variable satisfying*

$$P(X > b) = 0,\ b > 0,\ \mu = EX,\ \beta = EX^2,\ \alpha_t = \textstyle\int_{u \geq 0} u^t \, dF(u),$$

where $F(u) = P(X < u)$. Then for positive h,

$$\log E e^{hX} \leq h\mu + e^t \beta h^2/2, \tag{1.10}$$

if $h \leq t/b$, and

$$\log E e^{hX} \leq h\mu + e^t \beta h^2/2 + \alpha_t (e^{hb} - 1 - hb)/b^t, \tag{1.11}$$

if $h > t/b$.

PROOF. First of all,

$$E e^{hX} = 1 + h\mu + \textstyle\int_{u \leq b} (e^{hu} - 1 - hu) \, dF(u). \tag{1.12}$$

Suppose that $h < t/b$. Then

(1.13) $\int_{u<b}(e^{hu} - 1 - hu)\,dF(u) < \int_{u<t/h}(e^{hu} - 1 - hu)\,dF(u) < e^t\beta h^2/2.$

From (1.12), (1.13) and the inequality

(1.14) $$\log z < z - 1$$

the bound (1.10) follows.

Let us assume now that $h > t/b$. Then

(1.15) $$\int_{u<b}(e^{hu} - 1 - hu)\,dF(u) = \int_{u<t/h} + \int_{u>t/h}.$$

The function $(e^{hu} - 1 - hu)/u^t$ increases for $u > t/h$, and therefore

(1.16) $$\int_{u>t/h} = \int_{u>t/h}\frac{e^{hu} - 1 - hu}{u^t}u^t\,dF(u) < \alpha_t(e^{hb} - 1 - hb)/b^t.$$

From (1.12) and (1.15), with the aid of (1.14) and (1.16) and the second of the inequalities (1.13), we obtain the bound (1.11).

Let us continue the proof of Theorem 1.3. Applying Lemma 1.4 to each of the factors $Ee^{h\bar{X}_i}$ in the right-hand side of (1.9) we obtain

(1.17) $$P(\tilde{S}_n > x) < \exp\left\{h(\mu(-\infty, Y) - x) + \frac{1}{2}e^tB^2(-\infty, Y)h^2\right\}$$

if $0 < h < t/y$ and

(1.18) $$P(\tilde{S}_n > x) < \exp\left\{h(\mu(-\infty, Y) - x) + \frac{1}{2}e^tB^2(-\infty, Y)h^2 + \frac{e^{hy} - 1 - hy}{y^t}A(t; 0, Y)\right\}$$

if $h > t/y$.

Define

$$f(h) = \frac{1}{2}e^tB^2(-\infty, Y)h^2 - hx,$$

$$f_1(h) = \frac{1}{2}e^tB^2(-\infty, Y)h^2 - \alpha hx, \qquad 0 < \alpha < 1,$$

$$f_2(h) = \frac{e^{hy} - 1 - hy}{y^t}A(t; 0, Y) - \beta hx, \qquad \beta = 1 - \alpha.$$

Further define

$$h_1 = \alpha x/e^tB^2(-\infty, Y), \qquad h_2 = \frac{1}{y}\log(\beta xy^{t-1}/A(t; 0, Y) + 1).$$

We note that h_1 and h_2 satisfy equations $f_1'(h) = 0$ and $f_2'(h) = 0$, respectively. By virtue of the convexity of the functions $f_1(h)$ and $f_2(h)$ this implies that h_1 and h_2 are their points of minimality.

Let us first examine the case $h_1 \leqslant t/y$. Assuming in (1.17) that $h = h_1$ we conclude that

$$P(\tilde{S}_n \geqslant x) \leqslant P_6. \tag{1.19}$$

Now let $h_2 \geqslant h_1 > t/y$. We assume $h = h_1$ on the right-hand side of (1.18). Then

$$P(\tilde{S}_n \geqslant x) \leqslant \exp\{h_1 \mu(-\infty, Y) + f_1(h_1) + f_2(h_1)\}.$$

Since $f_2(0) = 0, f_2(h_2) < 0$. Consequently also $f_2(h_1) < 0$. This means that (1.19) is fulfilled in this case. Thus for $h_1 \leqslant \max[t/y, h_2]$ the bound (1.19) is valid.

If we observe that the condition $h_1 < \max[t/y, h_2]$ is equivalent to (1.4) and take into account the inequality (1.8); we see that the condition (1.4) leads to the bound (1.5).

Let $h_1 > \max[t/y, h_2]$. Then, either $h_1 > h_2 > t/y$, or $h_2 \leqslant t/y < h_1$. In the former case

$$\begin{aligned} f_1(h_2) + f_2(h_2) + h_2 \mu(-\infty, Y) &< h_2(e^t B^2(-\infty, Y)h_1/2 - x) \\ &\quad + (e^{h_2 y} - 1)A(t; 0, Y)/y^t + h_2 \mu(-\infty, Y) \\ &= \beta x/y - h_2((1 - \alpha/2)x - \mu(-\infty, Y)) \\ &= \beta x/y - \alpha x h_2/2 - (\beta x - \mu(-\infty, Y))h_2 \\ &< (\beta - t\alpha/2)x/y - (\beta x - \mu(-\infty, Y))h_2. \end{aligned}$$

Now taking $h = h_2$ on the right-hand side of (1.18), we obtain

$$P(\tilde{S}_n \geqslant x) < P_4 < P_5.$$

It now remains to examine the condition $h_2 \leqslant t/y < h_1$. In this case

$$f(h_2) < h_2(e^t B^2(-\infty, Y)h_1/2 - x) = (\alpha/2 - 1)x h_2.$$

Consequently

$$f(h_2) + h_2 \mu(-\infty, Y) < h_2(\mu(-\infty, Y) - (1 - \alpha/2)x).$$

Taking $h = h_2$ in (1.17), we conclude that

$$P(\tilde{S}_n \geqslant x) < \exp\{h_2(\mu(-\infty, Y) - (1 - \alpha/2)x)\} < P_4.$$

If $\beta x \geqslant \mu(-\infty, Y)$, then

$$\begin{aligned} f(t/y) + \mu(-\infty, Y)t/y &< (e^t B^2(-\infty, Y)h_1/2 - x)t/y + \mu(-\infty, Y)t/y \\ &= ((\alpha/2 - 1)x + \mu(-\infty, Y))t/y \\ &= (\mu(-\infty, Y) - \beta x)t/y - \alpha x t/2y \\ &\leqslant (\mu(-\infty, Y) - \beta x)h_2 - \alpha x t/2y. \end{aligned}$$

Now taking $h = t/y$ in (1.17), we obtain

$$P(\tilde{S}_n \geqslant x) < P_5. \tag{1.20}$$

However, if $\beta x < \mu(-\infty, Y)$ but, on the other hand, $\beta > t\alpha/2$, then $P_5 > 1$ and the bound (1.20) becomes trivial.

Thus, if the condition $h_1 > \max[t/y, h_2]$ is fulfilled, which is equivalent to (1.6), then

$$P(\tilde{S}_n > x) < P_4.$$

If, in addition, at least one of the inequalities

$$\beta x > \mu(-\infty, Y), \qquad \beta > t\alpha/2,$$

holds, then

$$P(\tilde{S}_n > x) < P_5.$$

In order to complete the proof of (1.7) and (1.7a) it is only necessary to use inequality (1.8).

The proofs of Theorems 1.1 and 1.2 are analogous, though significantly simpler. Therefore, we do not reproduce them here, referring the reader to the work [17] already cited above.

We turn now to several corollaries to Theorems 1.1 through 1.3. From Theorem 1.1 obviously follows:

COROLLARY 1.5. *Suppose* $A_t^+ < \infty, 0 < t \leq 1$. *Then*

$$P(S_n > x) \leq \Sigma_1^n P(X_i > y_i) + \left(eA_t^+/xy^{t-1}\right)^{x/y}. \tag{1.21}$$

Observing that

$$\mu(-Y, Y) \leq A_t^+/\left(\min_{1 \leq i \leq n} y_i\right)^{t-1},$$

we infer the next corollary from (1.3).

COROLLARY 1.6. *If* $A_t^+ < \infty$, $1 \leq t \leq 2$, *and* $EX_i = 0$, $i = 1, \cdots, n$, *then, for* $y^t \geq 4A_t^+$ *and* $x > y$,

$$P(S_n \geq x) \leq \Sigma_1^n P(X_i > y) + \left(e^2A_t^+/xy^{t-1}\right)^{x/2y}. \tag{1.22}$$

We turn now to the corollaries of Theorem 1.3. Setting $\beta = t/(t+2)$ in (1.7a) we obtain

COROLLARY 1.7. *Suppose* $t \geq 2$, $EX_i = 0$, $i = 1, \cdots, n$, $\beta = t/(t+2)$, *and* $\alpha = 1 - \beta$. *Then*

$$P(S_n \geq x) \leq \Sigma_1^n P(X_i > y_i) + \exp\left\{-\alpha^2x^2/2e^tB^2(-\infty, Y)\right\} + \left(A(t; 0, Y)/\beta xy^{t-1}\right)^{\beta x/y}. \tag{1.23}$$

Setting $y_i = \beta x$, $i = 1, \cdots, n$ in (1.23) and bounding $P(X_i > y)$ by Chebyshev's inequality, we obtain

COROLLARY 1.8. *If* $EX_i = 0$ *and* $A_t^+ < \infty$, $t > 2$, *then*

$$P(S_n > x) < c_t^{(1)} A_t^+ x^{-t} + \exp\{-c_t^{(2)} x^2 / B_n^2\}, \tag{1.24}$$

where $c_t^{(1)} = (1 + 2/t)^t$ *and* $c_t^{(2)} = 2(t+2)^{-2} e^{-t}$.

We shall now compare the inequalities given above with the corresponding asymptotic results.

THEOREM 1.9. *Let* $X_1, X_2, \cdots, X_n$ *be identically distributed and suppose that* $1 - F(x) = l(x)x^{-t}(1 + o(1))$ *as* $x \to \infty$, *where* $l(x)$ *is a slowly varying function and* $t > 2$. *If, in addition,* $EX_1 = 0$, $\sigma^2 = 1$, *and* $E|X_1|^{2+\delta} < \infty$ *for some* $\delta > 0$, *then*

$$P(S_n > x) = \left(1 - \Phi\left(x/n^{\frac{1}{2}}\right)\right)(1 + o(1)) + n(1 - F(x))(1 + o(1)) \tag{1.25}$$

for $n \to \infty$ *and* $x > n^{\frac{1}{2}}$.

This result is due to A. V. Nagaev [27]. Asymptotic relations of this type were obtained earlier under more restrictive conditions by Yu. V. Linnik [22] and S. V. Nagaev [28] and [29].

It is not difficult to see that if $n^{\frac{1}{2}} < x < a(n \log n)^{\frac{1}{2}}$, where $a < (t-2)^{\frac{1}{2}}$, then

$$n(1 - F(x)) = o\left(1 - \Phi\left(x/n^{\frac{1}{2}}\right)\right),$$

and if, on the other hand, $x > b(n \log n)^{\frac{1}{2}}$, $b > (t-2)^{\frac{1}{2}}$, then

$$1 - \Phi\left(x/n^{\frac{1}{2}}\right) = o(n(1 - F(x))).$$

Thus,

$$P(S_n > x) = \left(1 - \Phi\left(x/n^{\frac{1}{2}}\right)\right)(1 + o(1)), \tag{1.25a}$$

if $x < a(n \log n)^{\frac{1}{2}}$, and $a < (t-2)^{\frac{1}{2}}$, and

$$P(S_n > x) = n(1 - F(x))(1 + o(1)), \tag{1.25b}$$

if $x > b(n \log n)^{\frac{1}{2}}$ and $b > (t-2)^{\frac{1}{2}}$. On the other hand, for $n \to \infty$ and $x \to \infty$,

$$n(1 - F(x))/P(\max_{1 < i < n} X_i > x) \to 1,$$

if only $n(1 - F(x)) \to 0$. Clearly, the latter condition is fulfilled if $x > b(n \log n)^{\frac{1}{2}}$.

This makes possible the following interpretation of relation (1.25b): for sufficiently large values of x the sum S_n exceeds x essentially because one of the summands X_i, $i = 1, \cdots, n$, assumes a value exceeding x. On the other hand, equation (1.25a) shows that for relatively small x the contribution of an individual summand to a value of S_n exceeding x is small in comparison to S_n itself. Thus the nature of large deviations is different for large and for moderate values of x.

The philosophy set forth, useful not only for explaining (1.25a) and (1.25b), served also as the basis for the proof of (1.25) in the previously mentioned paper [27].

We shall now give a direct probabilistic proof of (1.25b) under the assumption that $x \to \infty$ faster than $(n \log n)^{\frac{1}{2}}$. Although this proof is based on the same initial considerations as in [27], we shall rely substantially on the bound (1.23), which appeared after [27]. Let us assume for simplicity that $l(x) \equiv 1$.

The following equation will serve as the point of departure:

$$(1.26)\qquad P(S_n \geq x) = nP(S_n \geq x, X_n > y, \max_{1 \leq k \leq n-1} X_k \leq y) + P(S_n \geq x, \cup_{1 \leq i < k \leq n}\{X_i > y, X_k > y\}) + P(S_n \geq x, \max_{1 \leq k \leq n} X_k \leq y) = nP_{1n} + P_{2n} + P_{3n}.$$

Now let $x = x_n$ and $x_n/(n \log n)^{\frac{1}{2}} \to \infty$. Let $y = y_n \equiv x_n/\alpha_n$, where $\alpha_n = (\log n)^{\frac{1}{2}}$. Then it is easy to see that

$$P_{2n} < \binom{n}{2} P^2(X_1 > y_n).$$

From this follows

$$P_{2n} = O(n^2/y_n^{2t}).$$

Furthermore,

$$n^2/y_n^{2t} = n^2\alpha_n^{2t}/x_n^{2t} < n^{1-t/2}\alpha_n^{2t}(n/x_n^t).$$

Consequently,

$$(1.27)\qquad P_{2n} = o(nP(X_1 > x_n)).$$

Let us now turn to the probability P_{1n}. It is not hard to see that for all x_n' and z_n

$$(1.28)\qquad P_{1n} = \int_{-\infty}^{\infty} P(X_1 \geq \max[y_n, x_n - u])\, dP(S_{n-1} < u, \max_{1 \leq k \leq n-1} X_k \leq y_n) = \int_{-z_n}^{x_n'} + \int_{-\infty}^{-z_n} + \int_{x_n'}^{\infty} = I_1 + I_2 + I_3.$$

We now choose $x_n' \to \infty$ so that

$$x_n/x_n' \to \infty, \qquad x_n'/(n \log n)^{\frac{1}{2}} \to \infty, \qquad y_n/x_n' \to 0.$$

It is clear that

$$(1.29)\qquad P(S_{n-1} \geq x_n', \max_{1 \leq k \leq n-1} X_k \leq y_n) \leq P(S_{n-1}' \geq x_n'),$$

where

$$S_n' = \Sigma_1^n X_i',$$
$$X_i' = X_i, \qquad X_i \leq y_n,$$
$$= 0, \qquad X_i > y_n.$$

We note that

$$\lim_{y \to \infty} \int_{0 \leq u \leq y} u^t\, dF(u)/\log y = t.$$

Therefore, by virtue of (1.23),

$$(1.30)\qquad P(S_{n-1}' \geq x_n') \leq \exp\{-\alpha^2 x_n'^2/2ne^t\} + ((2tn \log y_n)/\beta x_n' y_n^{t-1})^{\beta x_n'/y_n} = \Pi_1 + \Pi_2,$$

where $\beta = t/(t+2)$ and $\alpha = 1 - \beta$. For sufficiently large n,

$$x_n'^t n^{-t/2} \exp\{-\alpha^2 x_n'^2/2e'n\} < \exp\{-\alpha^2 x_n'^2/4e'n\} < n^{-t}.$$

Consequently,

(1.31) $$\Pi_1 = o(P(X_1 > x_n)).$$

We note that $y_n/n^{\frac{1}{2}} \to \infty$. Therefore, for all $\varepsilon, 0 < \varepsilon < t$, and for sufficiently large n,

$$\Pi_2 < \left(ny_n^{\varepsilon - t}\right)^{\beta\alpha_n'} = n^{\beta\alpha_n'} y_n^{(\varepsilon - t)\beta\alpha_n' + t}\alpha_n^t / x_n^t$$
$$< n^{((\varepsilon - t)/2 + 1)\beta\alpha_n' + t/2}\alpha_n^t / x_n^t,$$

where $\alpha_n' = x_n'/y_n$.

If $t - \varepsilon > 2$, then

$$\lim_{n\to\infty} \alpha_n^t n^{((\varepsilon - t)/2 + 1)\beta\alpha_n' + t/2} = 0.$$

Thus

(1.32) $$\Pi_2 = o(P(X_1 > x_n)).$$

From the bounds (1.30), (1.31) and (1.32) follows

(1.33) $$P(S_{n-1}' \geq x_n') = o(P(X_1 > x_n)).$$

Comparing (1.29) and (1.33) we conclude that

(1.34) $$P(S_{n-1} \geq x_n', \max_{1 \leq k \leq n-1} X_k \leq y_n) = o(P(X_1 > x_n)).$$

Since

$$I_3 \leq P(S_{n-1} \geq x_n', \max_{1 \leq k \leq n-1} X_k \leq y_n),$$

it follows by (1.34) that

(1.35) $$I_3 = o(P(X_1 > x_n)).$$

From Chebyshev's inequality we have

(1.36) $$P(S_{n-1} \leq -z_n, \max_{1 \leq k \leq n-1} X_k \leq y_n) \leq P(S_{n-1} \leq -z_n) \leq n/z_n^2.$$

We select z_n in such a way that $z_n/n^{\frac{1}{2}} \to \infty$ and $x_n/z_n \to \infty$. Then, by virtue of

(1.37) $$I_2 = o(P(X_1 > x_n))$$

and

(1.38) $$P(X_1 > x_n + z_n)/P(X_1 > x_n) \to 1.$$

Further,

(1.39) $$P(X_1 > x_n - x_n')/P(X_1 > x_n) \to 1.$$

From (1.38) and (1.39) it follows that

$$\lim_{n\to\infty} \sup_{-z_n \leq u \leq x_n'} |P(X_1 > x_n - u)/P(X_1 > x_n) - 1| = 0.$$

For sufficiently large n

$$P(X_1 \geq \max[y_n, x_n - u]) = P(X_1 \geq x_n - u)$$

for all $u < x_n'$, and, therefore,

(1.40) $I_1 = P(X_1 > x_n)P(-z_n < S_{n-1} < x_n', \max_{1<k<n} X_k < y_n)(1 + o(1))$.

It is not difficult to see that

(1.41) $$P(\max_{1<k<n-1} X_k < y_n) = 1 + O(n/y_n^t) = 1 + o(1).$$

From (1.34), (1.36) and (1.41) follows

(1.42) $$P(-z_n < S_{n-1} < x_n', \max_{1<k<n-1} X_k < y_n) = 1 + o(1).$$

Comparing (1.40) and (1.42) we conclude that

(1.43) $$I_1 = P(X_1 > x_n)(1 + o(1)).$$

Taking into account (1.35), (1.37) and (1.43), we obtain from (1.28)

(1.44) $$P_{1n} = P(X_1 > x_n)(1 + o(1)).$$

Finally,

$$P_{3n} < P(S_n' > x_n) < P(S_{n-1}' > x_n - y_n).$$

From this it follows, by virtue of (1.33), that

(1.45) $$P_{3n} = o(nP(X_1 > x_n)).$$

From (1.26), with the aid of (1.44) and the bounds (1.27) and (1.45), we obtain

(1.46) $$P(S_n > x_n) = nP(X_1 > x_n)(1 + o(1)).$$

Since

$$1 - \Phi(x_n) = o(n/x_n^t),$$

the representation (1.25) for $P(S_n > x_n)$ is valid.

On the other hand, in consequence of (1.35) and (1.44),

$$\begin{aligned} P_{1n} &= I_1 + I_2 + o(P(X_1 > x_n)) \\ &= P(S_n > x_n, X_n > y_n, \max_{1<k<n-1} X_k < y_n, S_{n-1} < x_n')(1 + o(1)). \end{aligned}$$

Using this representation, as well as the identity (1.26) and the bounds (1.27) and (1.45), we obtain

(1.47) $$\begin{aligned} &P(S_n > x_n) \\ &= P\left(S_n > x_n, \cup_{k=1}^n \{X_k > y_n, \max_{j \neq k} X_j < y_n, \Sigma_{j \neq k} X_j < x_n'\}\right)(1 + o(1)) \end{aligned}$$

in place of (1.46). Since $x_n/x_n' \to \infty$, equation (1.47) shows that the level x_n is actually exceeded because of the fact that one of the random variables X_k, $1 < k < n$, assumes a much larger value than the other $X_j, j \neq k$, $1 < j < n$, while the total contribution $X_j, j \neq k$, to the sum S_n is significantly less than that of X_k.

Equation (1.25) explains the presence of the summands $\Sigma_1^n P(X_i > y_i)$ and $\exp\{-\alpha^2 x^2/2e^t B^2(-\infty, Y)\}$ on the right-hand side of inequality (1.23). Moreover, because of (1.25) one may get the impression that the term $(A(t; 0, Y)/\beta x y^{t-1})^{\beta x/y}$

is superfluous. We shall now construct an example which shows that, in general, this is not so.

Let the random variables $X_1, X_2, \cdots, X_n$ be identically distributed and assume only the values $-1, 0, 1$ and suppose $P(X_1 = -1) = P(X_1 = 1) = p$. Let $y_i = y > 1$. Then

$$B_n^2 = 2np, \qquad A(t; 0, Y) = np < npy^t$$

and, consequently,

$$(1.48) \qquad (A(t; 0, Y)/xy^{t-1})^{x/y} < \exp\left\{\frac{x}{y}\log(npy/x)\right\}.$$

Let $x/y = m$ and $np = \lambda$. Obviously, for $x > \lambda$,

$$e^{m\log(\lambda/m)} > e^{-x\log(x/\lambda)}.$$

Consequently, for all α, $0 < \alpha < 1$,

$$(1.49) \quad \exp\{-\alpha^2x^2/2e^tB_n^2\} = \exp\{-\alpha^2x^2/4e^t\lambda\} = o(\exp\{m\log(\lambda/m)\}),$$

if $m \to \infty$, $x/\lambda \to \infty$ and $\lambda < m$. It follows from (1.23), by virtue of (1.48) and (1.49), that, for sufficiently large m and x/λ,

$$(1.50) \qquad P(S_n > x) < 2\exp\{\beta m(\log(\lambda/m) - \log\beta)\}.$$

Let us assume now that $m^2/n \to 0$ and $\lambda/m \to 0$. We shall also take m to be an integer. It is not difficult to see that

$$(1.51) \qquad P(S_n > x) > \binom{n}{m}p^m(1-2p)^{n-m}.$$

Also

$$(1.52) \qquad \begin{aligned} m^2/n \to 0 &\Rightarrow m!n^{-m}\binom{n}{m} \to 1, \\ \lambda m/n \to 0 &\Rightarrow (1-2p)^{-m} \to 1, \\ \lambda^2/n \to 0 &\Rightarrow (1-2p)^n e^{2\lambda} \to 1. \end{aligned}$$

It is clear that, under our assumptions, $\lambda m/n \to 0$ and $\lambda^2/n \to 0$. Therefore, it follows from (1.51) and (1.52) that, for sufficiently large n,

$$(1.53) \qquad P(S_n > x) > \frac{\lambda^m}{2m!}e^{-2\lambda}.$$

Now, applying Stirling's formula, we conclude that, for sufficiently large n and m,

$$(1.54) \qquad P(S_n > x) > \exp\{m/2 + m\log(\lambda/m)\}.$$

Comparing (1.49) and (1.54) we see that

$$\exp\{-\alpha^2x^2/2e^tB_n^2\} = o(P(S_n > x))$$

if $m \to \infty$, $m^2/n \to 0$, $\lambda/m \to 0$ (provided $y > 1$, $\lambda/m \to 0 \Rightarrow x/\lambda \to \infty$).

At the same time the right-hand sides of inequalities (1.50) and (1.54) for β close to 1 differ little from each other (in a definite sense, of course). This means that the bound (1.23) is close to the optimal one.

The fact that, in our example, X_1 assumes only three values, is entirely consistent with the fact that the extremum of a functional, involving independent random variables, is reached for random variables that admit a finite number of values (see, for example, [19]). As to inequalities (1.21) and (1.22), their asymptotic analog is Heyde's result [18], which showed that if $X_1, X_2, \cdots, X_n$ are identically distributed and the distribution of the random variable S_n/b_n converges to the stable law with the exponent α, $0 < \alpha < 2$, $\alpha \neq 1$ (which satisfies some additional restrictions), then

$$P(|S_n| > x_n b_n) = nP(|X_1| > x_n b_n)(1 + o(1))$$

for $n \to \infty$ and $x_n \to \infty$. Here b_n are the normalizing constants.

Let us cite another inequality of the type (1.3).

THEOREM 1.10. *Suppose* $B^2(-\infty, Y) < \infty$. *Then*

$$(1.55)\quad P(S_n > x) < \Sigma_1^n P(X_i > y_i) + \exp\{x/y - (x/y - \mu(-\infty, Y)/y + B^2(-\infty, Y)/y^2) \times \log(xy/B^2(-\infty, Y) + 1)\}.$$

Let us formulate two corollaries to Theorem 1.10.

COROLLARY 1.11. *Suppose* $EX_i = 0$, $i = 1, \cdots, n$. *Then*

$$(1.56)\qquad P(S_n > x) < \Sigma_1^n P(X_i > y) + (B_n^2/xy)^{x/y} e^{x/y}.$$

COROLLARY 1.12. *If* $X_i < L$ *and* $EX_i = 0$, $i = 1, \cdots, n$, *then*

$$P(S_n > x) < \exp\{x/L - (x/L + B_n^2/L^2)\log(xL/B_n^2 + 1)\}.$$

The last inequality was obtained independently by Bennett [3] and Hoeffding [20].

In conclusion we note that, with the aid of the arguments that led us to the asymptotic representation (1.46), we may obtain lower bounds for $P(S_n > x)$ without demanding at the same time that the X_i be identically distributed or that $P(X_i > x)$ behave in a regular fashion. Let us assume for the sake of simplicity that $EX_i = 0$, $i = 1, \cdots, n$ and $B_n^2 < \infty$. It is not difficult to see that

$$(1.57)\qquad P(S_n > x) > \Sigma_{j=1}^n P(S_n > x, X_j > x, \max_{i\neq j} X_i < x) = \Sigma_1^n P_j.$$

Now

$$P_j > \int_{u>-x} P(X_j > \max[x, x - u])\, dP(S_n^j < u, \max_{i\neq j} X_i < x)$$
$$> P(X_j > 2x) P(S_n^j > -x, \max_{i\neq j} X_i < x),$$

where

$$S_n^j = \Sigma_{i\neq j} X_i = S_n - X_j.$$

It is clear that

$$P\left(S_n^j \geq -x, \max_{i\neq j} X_i < x\right) \geq P\left(S_n^j \geq -x\right) - P\left(\max_{i\neq j} X_i > x\right).$$

On the other hand,

$$P\left(S_n^j \geq -x\right) \geq 1 - B_n^2/x^2$$

and

$$P\left(\max_{i\neq j} X_i > x\right) \leq B_n^2/x^2.$$

Therefore,

$$P_j \geq \tfrac{1}{2} P(X_j \geq 2x) \tag{1.58}$$

if $x \geq 2B_n$. From (1.57) and (1.58) it follows for $x \geq 2B_n$ that

$$P(S_n \geq x) \geq \tfrac{1}{2}\Sigma_1^n P(X_j \geq 2x).$$

2. Bounds in terms of generalized moments. Let $g(x)$ be a nondecreasing function such that $g(x) \to \infty$ as $x \to \infty$. Let us assume

$$b_{gi} = \int_{0+}^{\infty} e^{g(u)}\, dF_i(u), \qquad B_{gn} = \Sigma_1^n b_{gi}.$$

If $\lim_{x\to\infty} g(x)/\log x = \infty$ and $b_{gi} < \infty$, then the bound

$$P(X_i \geq x) \leq e^{-g(x)} b_{gi} \tag{2.1}$$

is for large values of x more precise than the bound

$$P(X_i \geq x) \leq x^{-t} \int_0^{\infty} u^t\, dF_i(u).$$

This suggests that in terms of generalized pseudomoments b_{gi} it is possible to obtain more precise bounds for $P(S_n \geq x)$ than the bounds in Section 1, at least for large values of x.

We shall study separately the cases $\lim_{x\to\infty} g(x)/x = 0$ and $\liminf_{x\to\infty} g(x)/x > 0$.

Let us turn first to the former case. In order to better imagine what bound to expect here, we shall analyze one asymptotic result from [32]. Let X_i be identically distributed, $EX_1 = 0$, $\sigma^2 = 1$. Let us assume that as $x \to \infty$

$$1 - F(x) = e^{\chi(x)}(1 + o(1)),$$

where $\chi(x)$ is a nonincreasing function which is defined for $x > 0$ and satisfies these conditions:

(i) $\lim_{x\to\infty} x\chi'(x)/\log x = -\infty$.

(ii) There exists an α, $0 < \alpha < 1$, such that $\alpha\chi(x)/x \leq \chi'(x)$.

(iii) $l\chi''(x) \leq -\chi'(x)/x \leq L\chi''(x)$.

(iv) $0 \leq -\chi'''(x) \leq L_1\chi''(x)/x$.

Here l, L and L_1 are positive constants.

Assume that

$$E|X_1|^{N(\alpha)} < \infty,$$

where $N(\alpha) = [(3 - 2\alpha)/(1 - \alpha)]$. Let us introduce the notation

$$K(u) = \Sigma_2^{N(\alpha)} \chi_k u^k,$$

where χ_k are the semiinvariants (cumulants) of the random variable X_1. Let $\lambda_\alpha(z)$ be the segment of the Cramér series consisting of the first $N(\alpha) - 3$ terms.

Let us now examine the equation

$$K'(-\chi'((1 - u)x)) = ux/n \tag{2.2}$$

in the range $0 < u < 1$. Denote by $\beta = \beta(x, n)$ the smallest root of this equation (if, of course, it has at least one solution). Further let $\Lambda(n)$ be the positive root of the equation

$$\chi(x) + x^2/n = 0. \tag{2.3}$$

(The definition of $\Lambda(n)$ is proper since equation (2.3) has exactly one strictly positive root.)

Write

$$P_1(x) = n(1 - \chi''((1 - \beta)x)n)^{-\frac{1}{2}}(1 - F((1 - \beta)x))$$
$$\cdot \exp\{-(\beta x)^2/2n + \lambda_\alpha(\beta x/n)(\beta x)^3/n^2\},$$
$$P_2(x) = \left(1 - \Phi\left(x/n^{\frac{1}{2}}\right)\right)\exp\{\lambda_\alpha(x/n)x^3/n^2\}.$$

THEOREM 2.1. *If* $\lim_{n\to\infty} xn^{-1/(2-\alpha)} = \infty$, *then*

$$P(S_n > x) = P_1(x)(1 + o(1)). \tag{2.4}$$

If $\limsup_{n\to\infty} xn^{-1/(2-\alpha)} < \infty$ *and* $\limsup_{n\to\infty} n\chi''((1 - \beta)x) < 1$, *then*

$$P(S_n > x) = (P_1(x) + P_2(x))(1 + o(1)). \tag{2.5}$$

If $\liminf_{n\to\infty} n\chi''((1 - \beta)x) > 1$ *and* $\Lambda(n) \leq x$ *or* $x \leq \Lambda(n)$, *then*

$$P(S_n > x) = P_2(x)(1 + o(1)). \tag{2.6}$$

The implicit function $\beta = \beta(x, n)$ plays a part, as we see, in the definition of $P_1(x)$. Therefore, one can determine the precise behavior of $P_1(x)$ as $x \to \infty$ only after having studied how $\beta(x, n)$ behaves for large values of x and n. We therefore turn now to the study of $\beta(x, n)$.

First of all, since $\chi(x)$ does not increase, $\chi'(x) \leq 0$ and consequently, by virtue of condition (ii), $\chi(x) \leq 0$. Therefore, condition (ii) may be rewritten in the form

$$\chi'(x)/\chi(x) \leq \alpha/x.$$

Integrating this inequality from 1 to $y > 1$ we get

$$\log(-\chi(y)) - \log(-\chi(1)) \leq \alpha \log y,$$

i.e.,

(2.7) $$-\chi(x) \leq -\chi(1)x^{\alpha}, \qquad x \geq 1.$$

This means that

(2.8) $$-\chi'(x) \leq -\alpha\chi(1)x^{\alpha-1}, \qquad x \geq 1.$$

Further, there exists $u_0 > 0$ such that, for all u, $0 \leq u \leq u_0$

(2.9) $$K'(u) \leq 2u.$$

Let x_0 be the root of the equation $-\chi'(x) = u_0$. If $(1-u)x \geq x_0$, then, in consequence of (2.9),

(2.10) $$K'(-\chi'((1-u)x)) \leq -2\chi'((1-u)x).$$

Let $\beta_1 = \beta_1(x, n)$ be the smallest root of the equation

(2.11) $$2\chi'((1-u)x) + ux/n = 0, \qquad 0 < u < 1,$$

and let $\beta_2 = \beta_2(x, n)$ be the smallest root of the equation

(2.12) $$2\alpha\chi(1)((1-u)x)^{\alpha-1} + ux/n = 0, \qquad 0 < u < 1.$$

If the condition

(2.13) $$(1-\beta_2)x \geq 1$$

is satisfied, then, according to (2.8),

(2.14) $$\beta_2 \geq \beta_1.$$

Let us now rewrite equation (2.12) in the form

(2.15) $$u(1-u)^{1-\alpha} = n\gamma_0 x^{\alpha-2},$$

where $\gamma_0 = -2\alpha\chi(1)$.

The function $u(1-u)^{1-\alpha}$ reaches its maximum at $u = 1/(2-\alpha)$, where the value is

$$\gamma_1 = (1-\alpha)^{1-\alpha}(2-\alpha)^{\alpha-2} \leq 1.$$

Clearly, for $x^{2-\alpha} \geq n\gamma_o/\gamma_1$ equation (2.15) has a solution. It is not difficult to see that

$$\beta_2 \leq 1/(2-\alpha).$$

Suppose

(2.16) $$x \geq \max\left[(2-\alpha)x_0/(1-\alpha), (2-\alpha)/(1-\alpha), (n\gamma_0/\gamma_1)^{1/(2-\alpha)}\right].$$

Then equation (2.15) has a solution, (2.13) is satisfied and, in addition, $(1-\beta_2)x \geq x_0$. By virtue of (2.14) this means that $(1-\beta_1)x \geq x_0$. Making use, now, of (2.10), we obtain

$$K'(-\chi'((1-\beta_1)x)) \leq -2\chi'((1-\beta_1)x) = \beta_1 x/n.$$

It follows from this that equation (2.2) has a solution in the interval $0 < u \leq \beta_1$.

Thus, if x satisfies (2.16), then

$$\beta < \beta_2 < 1/(2-\alpha).$$

It is not difficult to see that $\beta_2(x, n) \to 0$, if $nx^{\alpha-2} \to 0$. Consequently, also

(2.17) $$\beta(x, n) \to 0,$$

if $nx^{\alpha-2} \to 0$. From (2.8) and (2.17) it follows that for $nx^{\alpha-2} \to 0$,

(2.18) $$\chi'((1-\beta)x) \to 0.$$

For sufficiently small u,

$$K'(u) > u/2.$$

Therefore, from (2.2) and (2.18) it follows that, if $nx^{\alpha-2} \to 0$, then

(2.19) $$\beta x/n > -\chi'((1-\beta)x)/2$$

for n exceeding some n_0.

On the other hand, if the inequality (2.19) is satisfied then by virtue of condition (iii),

$$n\chi''((1-\beta)x) < \frac{2\beta(1-\beta)^{-1}}{l}.$$

Thus,

(2.20) $$n\chi''((1-\beta)x) \to 0,$$

if $nx^{\alpha-2} \to 0$.

From (2.2), by virtue of (2.18), it follows that, for $nx^{\alpha-2} \to 0$,

$$\beta x/n \to 0.$$

This means that

(2.21) $$\limsup_{nx^{\alpha-2}\to 0} \exp\{-(\beta x)^2/2n + \lambda_\alpha(\beta x/n)(\beta x)^3/n^2\} < 1.$$

From the representation (2.4) and the relations (2.20) and (2.21) there follows the existence of constants c and c_1 such that

(2.22) $$P(S_n > x) < cnP(X_1 > x/2)$$

if $x > c_1 n^{1/(2-\alpha)}$.

Let us turn now to the case $\Lambda(n) < x < c_1 n^{1/(2-\alpha)}$. It is evident that

$$x < c_1 n^{1/(2-\alpha)} \Rightarrow \lim_{n\to\infty} x/n = 0.$$

This means that in the range $x < c_1 n^{1/(2-\alpha)}$,

(2.23) $$\exp\{-(\beta x)^2/2n + \lambda_\alpha(\beta x/n)(\beta x)^3/n^2\} < c\exp\{-(\beta x)^2/4n\},$$

$$P_2(x) < ce^{-x^2/4n}.$$

Two cases are possible: $0 < \beta < \frac{1}{2}$ and $\frac{1}{2} < \beta < 1$. Let us bound $n\exp\{-(\beta x)^2/4n\}$ for $\frac{1}{2} < \beta < 1$. To this end let us examine the behavior of $\Lambda(n)$ as $n \to \infty$. In consequence of condition (ii),

(2.24) $$\Lambda(n)/n > -\chi'(\Lambda(n))/\alpha.$$

If $\Lambda(n) > 1$, then, because of (2.3) and (2.7),

$$\Lambda^{2-\alpha}(n) \leqslant -n\chi(1).$$

Thus

$$\Lambda(n) \leqslant \max\left[1, (-n\chi(1))^{1/(2-\alpha)}\right],$$

and, consequently,

(2.25) $$\lim_{n\to\infty} \Lambda(n)/n = 0.$$

The function $-\chi'(x)$, as we have already seen, is nonnegative. By virtue of condition (iii), $\chi''(x) \geqslant 0$. This means that the function $-\chi'(x)$ does not increase. Further, $\chi'(x)$ cannot be identically 0 because this would contradict condition (i).

Therefore, it follows from (2.24) and (2.25) that

$$\lim_{n\to\infty} \Lambda(n) = \infty.$$

Again using condition (i), we obtain

$$-\chi'(\Lambda(n))\Lambda(n) = \rho(n)\log \Lambda(n),$$

where $\rho(n) \to \infty$ as $n \to \infty$. From the last relation and from (2.24) it follows that

(2.26) $$\frac{\Lambda^2(n)}{n} \geqslant \rho(n)\log \Lambda(n)/\alpha.$$

From this it follows, in particular, that

$$\Lambda(n)n^{-\frac{1}{2}} \to \infty$$

as $n \to \infty$.

Assume $\Lambda(n) = \lambda(n)n^{\frac{1}{2}}$. Then (2.26) assumes the form

$$\alpha\lambda^2(n) \geqslant \frac{\rho(n)}{2}\log n + \rho(n)\log \lambda(n)$$

since $\lambda(n) \to \infty$; it then follows from the last inequality that

$$\liminf_{n\to\infty} \lambda^2(n)/\rho(n)\log n \geqslant \tfrac{1}{2}\alpha.$$

Hence for all positive ε,

$$\lim_{n\to\infty} ne^{-\varepsilon\Lambda^2(n)/n} = 0.$$

From this it follows that for $\beta \geqslant \frac{1}{2}$ and $x > \Lambda(n)$,

(2.27) $$ne^{-(\beta x)^2/4n} \leqslant ce^{-x^2/20n}.$$

On the other hand, if $\beta < \frac{1}{2}$, then

(2.28) $$P(X_1 \geqslant (1-\beta)x) \leqslant P(X_1 > x/2).$$

From the representation (2.5) and the bounds (2.23), (2.27) and (2.28) we conclude that if $\Lambda(n) \leqslant x < c_1 n^{1/(2-\alpha)}$ and $n\chi''((1-\beta)x) < \lambda < 1$, then

(2.29) $$P(S_n \geqslant x) \leqslant c(1-\lambda)^{-\frac{1}{2}}\left(e^{-x^2/20n} + nP(X_1 > x/2)\right).$$

Finally, by virtue of (2.6) and (2.23), if

$$x < c_1 n^{-1/(2-\alpha)}, \qquad n\chi''((1-\beta)x) > 1 \qquad \text{and} \quad x > \Lambda(n),$$

then

$$P(S_n > x) < ce^{-x^2/4n}. \tag{2.30}$$

This bound remains valid also for $x \leqslant \Lambda(n)$.

From (2.22), (2.29) and (2.30) it follows that

$$P(S_n > x) \leqslant c(1-\lambda)^{-\frac{1}{2}}\left(e^{-x^2/20n} + nP(X_1 > x/2)\right) \tag{2.31}$$

except, perhaps, for those values of $x > \Lambda(n)$ for which the equation (2.2) does not have a solution or $\lambda \leqslant n\lambda''((1-\beta)x) \leqslant 1$.

In A. V. Nagaev's work [26] it is proved in the special case $\chi(x) = -x^\alpha$ that for any $x > 0$ one of the representations of type (2.4), (2.5) or (2.6) is valid. This enables one to obtain the bound

$$P(S_n > x) \leqslant c\left(e^{-x^2/20n} + nP(X_1 > x/2)\right), \tag{2.32}$$

which is satisfied for all positive x. The bound (2.32) is already close to the bound

$$P(S_n > x) \leqslant c\left(1 - \Phi\left(x/n^{\frac{1}{2}}\right) + nP(X_1 > x)\right),$$

which follows from (1.25).

Our considerations make very plausible an inequality of the type

$$P(S_n > x) \leqslant \exp\left\{-\alpha_1 x^2/B_n^2\right\} + \Sigma_1^n P(X_i > y) + R, \tag{2.33}$$

where R is a correction term which by analogy with (1.23) has the form

$$R = c\left(B_{gn}\exp\{-\alpha_2 g(y)\}\right)^{\beta x/y}.$$

Here c, α_1, α_2, β are constants depending only on the function g. (The summands X_i are not assumed to be identically distributed.)

Regrettably, such a bound is in general not correct even in the case of a regularly varying function g (satisfying, for example, conditions (i)–(iv) mentioned at the beginning of the section). In order to convince ourselves of this, let us examine the following example.

EXAMPLE 2.2. Let $g(u) = u^\alpha$, $0 < \alpha < 1$, and suppose $X_j, j = 1, \cdots n$, are identically distributed, where X_1 can assume only the values $-z, 0, z > 0$ with corresponding probabilities p, $1-2p, p$. Let us hold fixed the values c, β, α_1, α_2 and the ratio $\gamma = x/y$. We shall demonstrate now that it is possible to choose n, z, p, y so that inequality (2.33) is violated.

Let us introduce the additional notation $\lambda = np$. It is not difficult to see that in our case $B_{gn} = \lambda e^{z^\alpha}$ and $B_n^2 = 2\lambda z^2$. Let us now relate z and y through $z^\alpha = \alpha_2 y^\alpha/2$. In addition, let us assume that $\lambda y = 1$. Then

$$\left(B_{gn}\exp\{-\alpha_2 g(y)\}\right)^{\beta x/y} = y^{-\beta\gamma}\exp\{-\alpha_2\beta\gamma y^\alpha/2\}, \tag{2.34}$$

$$\exp\left\{-\alpha_1 x^2/B_n^2\right\} = \exp\left\{-2^{2/\alpha-1}\alpha_1\alpha_2^{-2/\alpha}\gamma^2 y\right\}.$$

It is easy to verify that

(2.35) $$P(S_n \geq x) \geq \binom{n}{m} p^m (1-2p)^{n-m},$$

where $m = [x/z] + 1$. Since m is fixed,

(2.36) $$P(S_n \geq x) \geq \tfrac{1}{2}(m!) y^m$$

for sufficiently large n and y. Comparing (2.34) and (2.36), we see that for large n and y,

$$P(S_n \geq x) > \exp\{-\alpha_1 x^2/B_n^2\} + c\big(B_{gn} \exp\{-\alpha_2 g(y)\}\big)^{\beta x/y}.$$

It is clear from the example we have examined that the bound for $P(S_n \geq x)$, in the case where $g(u)$ is not a logarithmic function, must be more complex than (2.33). We shall now formulate one result in this direction that was recently obtained in [35].

Let us first introduce the necessary definitions and notations. Let $G(\delta)$, $\delta \geq 2$, be the class of functions $g(u)$ over $(0, \infty)$ with continuous nonincreasing derivatives satisfying the conditions:

(a) $g'(u) \to \infty$ as $u \to \infty$;

(b) $g'(u) > \delta/u$.

Put $S(u) = e^{-g(u)} g'(u) u^2$.

Let γ, $\{\gamma_i\}_1^3$, β be positive constants such that $\sum_1^3 \gamma_i = 1$, $\beta = 1 - \gamma_1/2 - \gamma_2/\delta$, $\gamma < 1$, and let a be the solution of the equation $(u+1)/u = e^{u-1}$. It is not hard to see that the function $S(u)$ is strictly decreasing and therefore has an inverse. Indeed,

$$\log(u^2 e^{-g(u)}) = 2 \log u - g(u).$$

By virtue of condition (b),

$$\frac{d}{du}(2 \log u - g(u)) = 2/u - g'(u) \leq 0.$$

Thus the function $u^2 e^{-g(u)}$ is nonincreasing. Consequently the function $S(u)$ is also nonincreasing.

Let us now show that the function $S(u)$ cannot have intervals of constancy. Indeed, suppose $S(u)$ is constant over $[a, b]$, where $b > a$. Then $g'(u)$ is constant and $2/u - g'(u)$ is identically 0 for $u \in [a, b]$, i.e., $2/u$ is constant over $[a, b]$, which is impossible.

We denote by $S^{-1}(u)$ the inverse function to $S(u)$.

THEOREM 2.3. *If $g \in G(\delta)$, then for all positive x,*

(2.37)
$$\begin{aligned} P(S_n \geq x) \leq{} & \exp\{\gamma_3/\gamma - \gamma_1 \beta a^2 x^2/2(a+1)B_n^2\} \\ & + \exp\{\gamma_3/\gamma - \beta a x/S^{-1}(\gamma_2 a x/e^a B_{gn})\} \\ & + (\gamma/\gamma_3) B_{gn}^{\beta/\gamma} \exp\{\gamma_3/\gamma - g(\gamma x)\beta/\gamma\} \\ & + \textstyle\sum_1^n (1 - F_i(\gamma x)). \end{aligned}$$

An analogous but less precise bound was obtained earlier in [56].

Let us cite now an example that illustrates the precision of the bound (2.37). An analogous example was constructed in [35].

EXAMPLE 2.4. Let $X_1, X_2, \cdots X_n$ be defined in the same way as in Example 2.2. As before, $\lambda = np$, but $m = x/z$. Let us select α, $0 < \alpha < 1$, $\delta > 2$, and suppose that $z > (\delta/\alpha)^{1/\alpha}$.

Assume now that

$$g(u) = \delta \log u + (1 - \log(\delta/\alpha))\delta/\alpha, \qquad 0 < u < (\delta/\alpha)^{1/\alpha},$$
$$= u^\alpha, \qquad (\delta/\alpha)^{1/\alpha} \leqslant u \leqslant z,$$
$$= z^\alpha + \alpha z^{\alpha-1}(u - z), \qquad z < u \leqslant x.$$

Thus $g(u)$ is defined for $0 < u \leqslant x$ and, as is easy to verify, it satisfies condition (b) in this interval. Obviously it is possible to extend the definition of $g(u)$ for $u > x$ so that conditions (a) and (b) are satisfied.

Let us fix constants γ_i, $i = 1, 2, 3$ on the right-hand side of inequality (2.37). In addition to this, let $\gamma = 1$ and $x = e^{z^\alpha}$, and for the sake of simplicity let m be an integer. Suppose now that $n \to \infty$, $m \to \infty$, $m^2/n \to 0$ and $\lambda = bm$, where b is a constant which will be defined later.

Making use of the inequality (2.35) and Stirling's formula, we conclude that, for n sufficiently large,

$$P(S_n \geqslant x) \geqslant \exp\{m(\log b - 2b + 1)\}/2(2\pi m)^{\frac{1}{2}}. \tag{2.38}$$

It is not difficult to see that

$$B_{gn} \exp\{-g(x)\} = \lambda \exp\{g(z) - g(x)\} = bm \exp\{\alpha z^\alpha - \alpha z^{\alpha-1}e^{z^\alpha}\}.$$

Since $m = z^{-1}e^{z^\alpha}$,

$$\left(B_{gn} \exp\{-g(x)\}\right)^\beta = o(P(S_n \geqslant x)) \tag{2.39}$$

for all positive β. Further,

$$x^2/B_n^2 = m^2/2\lambda = m/2b. \tag{2.40}$$

Let us now subject b to the condition

$$\gamma_1 \beta a^2/4(a + 1)b = 2b - \log b - 1 + \varepsilon,$$

where ε is an arbitrarily small positive number. Then by virute of (2.38) and (2.40),

$$\exp\{-\gamma_1 \beta a^2 x^2/2(a + 1)B_n^2\} = o(P(S_n \geqslant x)). \tag{2.41}$$

The bounds (2.39) and (2.41) show that, under our assumptions,

$$R \equiv \exp\{\gamma_3/\gamma - \beta a x/S^{-1}(\gamma_2 a x/e^a B_{gn})\}$$

becomes the biggest summand in the right side of (2.37). In our case,

$$S(u) = \alpha u^2 z^{\alpha-1} \exp\{-z^\alpha - \alpha z^{\alpha-1}(u - z)\}, \qquad z \leqslant u \leqslant x.$$

Therefore, for any positive ε there exists a $u(\varepsilon)$ such that

$$S(u) < \exp\{-(1 - \varepsilon)(z^\alpha + \alpha z^{\alpha-1}(u - z))\},$$

if $u(\varepsilon) \leqslant z \leqslant u \leqslant x$. Consequently, if

$$\exp\{-(1-\varepsilon)(z^{\alpha}+\alpha z^{\alpha-1}(x-z))\} \leqslant v \leqslant S(z), \qquad u(\varepsilon) \leqslant z,$$

then

$$S^{-1}(v) \leqslant z^{1-\alpha}\alpha^{-1}(1-\varepsilon)^{-1}\log(1/v) + (\alpha-1)z/\alpha. \tag{2.42}$$

Now $B_{gn} = \lambda e^{z^{\alpha}}$, and hence $x/B_{gn} = ze^{-z^{\alpha}}/b$. Therefore, for sufficiently large z,

$$\exp\{-(1-\varepsilon)(z^{\alpha}+\alpha z^{\alpha-1}(x-z))\} \leqslant \gamma_2 ax/e^aB_{gn} \leqslant S(z) = \alpha z^{\alpha+1}e^{-z^{\alpha}}.$$

Making use now of (2.42), we conclude that

$$S^{-1}(\gamma_2 ax/e^aB_{gn}) \leqslant (1+\varepsilon)z,$$

for all positive ε if z is sufficiently large. This shows that

$$R \leqslant \exp\{\gamma_3/\gamma - \beta am/(1+\varepsilon)\} \tag{2.43}$$

for large z.

Comparing the bounds (2.38) and (2.43), we see that they differ essentially only in the constant factor in the exponent. Let us now turn to the case

$$\liminf_{x\to\infty} g(x)/x > 0.$$

Suppose that the function $g(\cdot)$ has a positive nondecreasing derivative. Then, in particular, it follows that $g(\cdot)$ is convex. In addition, there exists a finite or an infinite limit $\lim_{x\to\infty} g(x)/x$.

THEOREM 2.5. *For any* $x > 0$,

$$P(S_n \geqslant x) \leqslant \Pi_1^n(b_j(x/n) + b_{gj})\exp\{-ng(x/n)\}, \tag{2.44}$$

where

$$b_j(x) = e^{g(0)}\int_{u<0} e^{g'(x)u}\, dF_j(u).$$

This bound was recently obtained by S. K. Sakoyan and the author.

PROOF. Obviously,

$$P(S_n \geqslant x) \leqslant e^{-hx}\prod_1^n Ee^{hX_j}. \tag{2.45}$$

Further,

$$\begin{aligned} Ee^{hX_j} &= \int_{-\infty}^{0-} e^{hu}\, dF_j(u) + \int_{0-}^{\infty} e^{hu-g(u)}e^{g(u)}\, dF_j(u) \\ &\leqslant \int_{-\infty}^{0-} e^{hu}\, dF_j(u) + b_{gj}\sup_{u\geqslant 0} e^{hu-g(u)}. \end{aligned} \tag{2.46}$$

Let us introduce $f(u) = hu - g(u)$. Obviously $f'(u) = h - g'(u)$. If $h = g'(x/n)$, then $f'(x/n) = 0$, i.e.,

$$\sup_{u\geqslant 0} e^{hu-g(u)} = e^{hx/n-g(x/n)}. \tag{2.47}$$

By virtue of the convexity of $g(\cdot)$

$$g'(x/n)x/n > g(x/n) - g(0).$$

Consequently, as $h = g'(x/n)$,

$$e^{hx/n-g(x/n)} \geqslant e^{-g(0)}.$$

This shows that

(2.48) $\int_{-\infty}^{0-} e^{hu}\, dF_j(u)$

$$< \exp\{g(0) + hx/n - g(x/n)\}\int_{-\infty}^{0-} e^{hu}\, dF_j(u), \qquad h = g'(x/n).$$

From (2.46), by virtue of (2.47) and (2.48), it follows that

$$Ee^{hX_j} < (b_j(x/n) + b_{gj})\exp\{hx/n - g(x/n)\}, \qquad h = g'(x/n).$$

Taking $h = g'(x/n)$ in (2.45) and making use of the bound just obtained, we arrive at the result of the theorem.

The bound (2.44) is better that the trivial bound $P(S_n > x) < 1$ only, generally speaking, for sufficiently large x. Indeed, let $X_j, j = 1, \cdots, n$, be identically distributed with $EX_1 = 0$ and $\sigma^2 = 1$. Clearly, $b_{gi} > e^{g(0)}$. Therefore, in our case

$$\Pi_1^n(b_j(x/n) + b_{gj}) > e^{n(g(0)+\varepsilon)}, \qquad \varepsilon > 0.$$

This means that the right side of (2.44) may tend to 0 as $n \to \infty$ only under the condition $\liminf_{n\to\infty} x/n > 0$. At the same time, according to Chebyshev's inequality,

$$\lim_{n\to\infty} P(S_n > x) = 0$$

if $x/n^{\frac{1}{2}} \to \infty$.

The parameter n which enters on the right side of (2.44) does not represent the essential characteristics of the distribution of the sum S_n, if only because the latter does not change with the addition of summands which equal 0 with probability 1. Therefore, the bound (2.44) is more precise when the X_j are identically distributed. The only case where the right side of (2.44) does not depend on n is $g(u) = Tu$.

At the same time inequality (2.44) has a very clear probabilistic interpretation. Let n be fixed while $x \to \infty$. Then

$$\Pi_1^n(b_j(x/n) + b_{gj}) \to \Pi_1^n b_{gj}.$$

Therefore, for all positive ε,

(2.49) $$P(S_n > x) < (1 + \varepsilon)\Pi_1^n b_{gj} e^{-ng(x/n)},$$

if x is sufficiently large. On the other hand,

(2.50) $$P(S_n > x) > \Pi_1^n P(X_j > x/n),$$

because

$$\cap_1^n\{X_j > x/n\} \subseteq \{S_n > x\}.$$

Further, for all j, $1 < j < n$,

(2.51) $$b_{gj} e^{-g(x/n)} > P(X_j > x/n).$$

If

$$P(X_j > y) \sim b_{gj} e^{-g(y)}$$

as $y \to \infty$ (we are not stating more precisely in what sense equivalence is meant), then by (2.49) and (2.50),

$$P(S_n > x) \sim \Pi_1^n P(X_j > x/n). \tag{2.52}$$

This shows that for large values of x all summands X_j contribute approximately equally to the value of the sum $S_n > x$.

On the other hand, if one assumes from the beginning that the equivalence relation (2.52) is valid, then, on account of (2.51), the inequality (2.44) becomes completely transparent.

Our discussion demonstrates that the bound (2.44) is not as weak as it may seem at first glance. This is also confirmed by exact asymptotic results. We cite one such result obtained by A. V. Nagaev in [25]. Let $h(u)$ be a strictly increasing function, $h(0) = 0$, and let $m(u)$ be an inverse function to $h(u)$. Suppose that the functions $h'(u)$, $m'(u)$, $u^2h'(u)$, $u^2m'(u)$ are convex. Let

$$q(x) = \left(\frac{h'(x)}{2\pi}\right)^{\frac{1}{2}} \exp\{-\textstyle\int_0^x h(u)\,du\}.$$

We shall say that the absolutely continuous distribution function $F(x)$ lies in $Q[h(x)]$ if $p(x) = F'(x)$ is square integrable and

$$p(x) = q(x)(1 + o(1))$$

for $x \to \infty$. Let $f(s) = \int_{-\infty}^{\infty} e^{su}\,dF(u)$, where $F(x) \in Q[h(x)]$. Then $f(s)$ is defined for $0 < s < s_0$, where $s_0 = \lim_{u\to\infty} h(u)$. Let us consider the function $M(s) = f'(s)/f(s)$. Since $M'(s) > 0$ for $0 < s < s_0$, the function $M(s)$ has an inverse which we designate as $H(u)$. It is not difficult to see that $H(u)$ is defined for u in $[0, \infty)$, where $H(\infty) = s_0$.

THEOREM 2.6. *Let X_j, $j = 1, \cdots, n$ be identically distributed with $EX_1 = 0$ and $\sigma^2 < \infty$, and suppose*

$$F(u) \equiv P(X_1 < u) \in Q[h(u)].$$

Then

$$P(S_n > xn) = \frac{1}{H(x)}\left(\frac{H'(x)}{2\pi n}\right)^{\frac{1}{2}} \exp\{-n\textstyle\int_0^x H(u)\,du\}(1 + o(1))$$

as $x \to \infty$.

The role of the function $g(x)$ is here taken by $\int_0^x H(u)\,du$. It is interesting to compare the bound (2.44) with Bernstein's inequality ([4], page 162). However fast $g(u)$ increases as $u \to \infty$, the most that Bernstein's inequality can yield is the bound

$$P(S_n > x) < \exp\{-x^2/2B_n^2\}. \tag{2.53}$$

It is clear that if $g(x)/x^2 \to \infty$ when $x \to \infty$, then the bound (2.44) for large x is more precise than (2.53).

Suppose $\lim_{x\to\infty} g(x)/x = \infty$, $b_{gj} < \infty$, $EX_j = 0$, and $\int_{-\infty}^0 u^2\, dF_j(u) < \infty$, $j = 1, \cdots, n$. Put

$$g_j(T) = \int_0^\infty e^{Tu} u^2\, dF_j(u) + \int_{-\infty}^0 u^2\, dF_j(u).$$

It is evident that $g_j(T) < \infty$ for all j, $1 \leq j \leq n$, and positive T. It is not hard to see that if $0 \leq h \leq T$, then

$$Ee^{hX_j} \leq e^{g_j(T)h^2/2}, \qquad j = 1, \cdots, n.$$

Let

$$G(T) = \Sigma_1^n g_j(T).$$

Applying V. V. Petrov's [42] generalization of Bernstein's inequality (see also [43], page (70), we conclude, for all positive T, that

$$P(S_n \geq x) \leq \exp\{-x^2/2G(T)\}, \qquad 0 \leq x \leq TG(T), \tag{2.54}$$

and

$$P(S_n \geq x) \leq \exp\{-Tx/2\}, \qquad x \geq TG(T). \tag{2.55}$$

Thus the bound (2.54) is best for relatively small x, beyond which the bound (2.55) begins to be operative. Whatever T may be, for sufficiently large x the bound (2.44) is more precise than (2.55). We note that in the case $g(u) = Tu$ the bound (2.44) transforms itself into an exponential of the type (2.55).

3. Bounds in terms of moment products without repetitions. The common element in the inequalities discussed above is that averaged characteristics appear on the right hand sides. Therefore, the precision of the bounds depends to a considerable extent on the individual properties of the distributions of the summands.

Let us investigate the inequality (1.56) from this point of view. Assuming $n = 3$, $y = x/2$, we get the bound

$$P(S_3 \geq x) \leq \Sigma_1^3 P(X_i \geq x/2) + 4e^2\left(\Sigma_1^3 \sigma_i^2/x^2\right)^2.$$

Obviously,

$$\left\{\Sigma_1^3 X_i \geq x\right\} \subseteq \cup_1^3\{X_i \geq x/2\} \cup \left(\left\{\Sigma_1^3 X_i \geq x\right\} \cap \cap_1^3\{X_i < x/2\}\right).$$

On the other hand,

$$\left\{\Sigma_1^3 X_i \geq x\right\} \cap \cap_1^3\{X_i < x/2\} \subseteq \cup_j \cap_{i\neq j}\{X_i \geq x/4\}.$$

Consequently,

$$\begin{aligned} P(S_3 \geq x) &\leq \Sigma_1^3 P(X_i \geq x/2) + \Sigma_{i<k} P(X_i \geq x/4) P(X_k \geq x/4) \\ &\leq \Sigma_1^3 P(X_i \geq x/2) + 256\Sigma_{i<k}\sigma_i^2\sigma_k^2/x^4. \end{aligned}$$

The summand $256\, \sigma_i^2\sigma_k^2/x^4$ obviously arises from the case in which the level x is exceeded because of the large values taken by the random variables X_i and X_k. Let us now write $4e^2(\Sigma_1^3\sigma_i^2/x^2)^2$ in the form

$$4e^2\left(\Sigma_1^3\sigma_i^4/x^4 + 2\Sigma_{i<k}\sigma_i^2\sigma_k^2/x^4\right).$$

The summands σ_i^4/x^4 do not have such an intuitive probabilistic interpretation as $\sigma_i^2\sigma_k^2/x^4$, and, therefore, they give the impression of being superfluous. On the other hand, if one of the dispersions σ_i^2 is large in comparison with the others, then $\Sigma_1^3\sigma_i^4/x^4$ may greatly exceed $2\Sigma_{i<k}\sigma_i^2\sigma_k^2/x^4$.

Thus the precision of the bound (1.56) is at its best when the summands are identically distributed and it may decrease considerably if the distributions F_i differ greatly. Our reasoning makes the bound

$$P(S_n > x) < \Sigma_1^n P(X_i > y_i) + c\Sigma_{i<k}\sigma_i^2\sigma_k^2/x^4,$$

very plausible, c being an absolute constant. A bound of this type was obtained in [33].

In order to formulate this bound we require some additonal notation. For a random finite set of numbers $\{u_i\}_{i\in I}$ we take

$$\Sigma_p\{u_k\}_I = \Sigma u_{k_1}u_{k_2}\cdots u_{k_p},$$

where the summation extends over all $k_1, k_2, \cdots, k_p$ that belong to I and satisfy $k_1 < k_2 < \cdots < k_p$. Let

$$D_p = \Sigma_p\{\sigma_k^2\}_{J_n}, \qquad J_n = \{1, 2, \cdots, n\},$$

and

$$D_p(Y) = \Sigma_p\left\{\textstyle\int_{|u|<y_k} u^2\, dF_k(u)\right\}_{J_n}.$$

THEOREM 3.1. *Suppose* $p \geq 1$, $n \geq 1$, *and* $y_i < x/\Delta_p$, $i = 1, \cdots, n$. *Then*

$$P(|S_n| \geq x) \leq \Sigma_1^n P(|X_i| > y_i) + \left(\Gamma_p D_p + \Delta_p^{2p} D_p(Y)\right)/x^{2p}, \tag{3.1}$$

where

$$\Delta_p = \frac{\alpha_p + a_p}{\alpha_p}(2p-1)^{\frac{1}{2}}, \qquad \Gamma_p = p!/(1-\alpha_p),$$

and α_p *is an arbitrary number from the interval* (0, 1). *For* a_p *we may choose* $(p!)^{2p+2}$.

We note that the right-hand side of inequality (3.1) tends to $P(|X_1| > y_1)$ as $\sigma_j \to 0, j = 2, \cdots, n$. This attests once again to the fact that the bound (3.1) may turn out to be considerably more precise than (1.56), There is an analogous extension of the inequality (1.23) (see [34]).

Let us put

$$\alpha_k^{(t)} = \textstyle\int_0^\infty u^t\, dF_k(u), \qquad D_p^{(t)} = \Sigma_p\{\alpha_k^{(t)}\}_{J_n}.$$

THEOREM 3.2. *Suppose* $X_k \leq b$, $EX_k = 0$, $k = 1, \cdots, n$, p *is any positive integer and* γ *is any number between* 0 *and* 1. *Then for* $\gamma a/2 \geq (2p-1)b > 0$ *and*

$t > 0$,

(3.2) $P(S_n \geqslant a) \leqslant \max\left[\exp\{(\gamma-1)\gamma a^2/e^t\beta_p D_p^{1/p}\}, \left((D_p^{(t+1)})^{1/p}/\gamma ab^t\right)^{(1-\gamma)a/b}\right.$

where $\beta_p = 2(2p-1)$.

Suppose

$$D_p^{(t)}(Y) = \Sigma_p\{\alpha_k^{(t)}(y_k)\}_{J_n},$$

where

$$\alpha_k^{(t)}(y_k) = \int_{0<u<y_k} u^t\, dF_k(u).$$

COROLLARY 3.3. *Suppose* X_k, $k = 1, \cdots, n$, *are symmetrically distributed, p is any positive integer,* $0 < \gamma < 1$, *and* $t > 0$. *Then for* $\gamma x/2 \geqslant (2p-1)y$, $y \geqslant \max_{1\leqslant k\leqslant n} y_k$,

(3.3) $$P(S_n \geqslant x) \leqslant \Sigma_1^n P(X_k > y_k) + P_7,$$

where P_7 *is equal to the value of the right side of* (3.2) *in which* D_p *and* $D_p^{(t+1)}$ *are replaced by* $D_p(Y)$ *and* $D_p^{(t+1)}(Y)$ *respectively.*

Let us put

$$\bar{D}_p^{(t)}(Y) = \Sigma_p\{\bar{\alpha}_k^{(t)}(y_k)\}_{J_n},$$

where

$$\bar{\alpha}_k^{(t)}(y_k) = \int_{\mu_k<u<y_k} (u-\mu_k)^t\, dF_k(u),$$

$$\mu_k = \int_{u<y_k} u\, dF_k(u),$$

$$\bar{D}_p(Y) = \Sigma_p\left\{\int_{u<y_k}(u-\mu_k)^2\, dF_k(u)\right\}_{J_n}.$$

COROLLARY 3.4. *If* $\gamma(x-\mu(-\infty, Y))/2 \geqslant (2p-1)(y-\mu^*)$, *where p is any positive number,* $0 < \gamma < 1$, *and* $\mu^* = \min_{1\leqslant k\leqslant n}\mu_k$, *then, for any positive* y_k *satisfying the condition* $\max_{1\leqslant k\leqslant n} y_k \leqslant y$,

$$P(S_n \geqslant x) \leqslant \Sigma_1^n P(X_k > y_k) + P_8,$$

where P_8 *is equal to the value of the right side of* (3.2) *in which* D_p *and* $D_p^{(t+1)}$ *are replaced by* $\bar{D}_p(Y)$ *and* $\bar{D}_p^{(t+1)}(Y)$ *respectively for* $a = x - \mu(-\infty, Y)$ *and* $b = y - \mu^*$.

The reason for the appearance of $D_p^{(t)}$ instead of $A(t; 0, Y)$ in the right side of inequality (3.2) is the same as that for the presence of D_p and $D_p(Y)$ in inequality (3.1). It remains for us to comment on the substitution of $B^2(-\infty, Y)$ for $D_p^{1/p}$.

At first glance it may even seem that the inequality (3.2) contradicts the central limit theorem since, according to the latter,

(3.4) $$P(S_n > x) > \frac{1}{10}e^{-x^2/B^2}, \qquad B^2 = B_n^2,$$

for x that is not too large in comparison to B.

But $D_p^{1/p}$ may be significantly smaller than B^2 only in the case where one of the dispersions, say σ_1^2, is comparable in size with B^2. Indeed, let the dispersions $\sigma_i^2, i = 1, \cdots, n$ be ordered by size: $\sigma_1^2 \geqslant \sigma_2^2 \geqslant \cdots \geqslant \sigma_n^2$. Assume that $\bar{\sigma}^2 = \Sigma_1^{p-1}\sigma_i^2$, $\alpha = D_p/B^{2p}$, $\beta = \bar{\sigma}^2/B^2$. We shall see the connection between α and β.

Let us introduce the notation

$$B^2(i_1, i_2, \cdots, i_s) = \Sigma_{j \in J_n; j \neq i_1, \cdots, i_s}\sigma_j^2,$$

$$D_p(i_1, i_2, \cdots, i_s) = \Sigma_p\{\sigma_j^2\}_{j \in J_n; j \neq j_1, \cdots, j_s}.$$

It is not difficult to see that for all $s \leqslant p - 1$

$$B^2(i_1, i_2, \cdots, i_s) \geqslant (1 - \beta)B^2 \geqslant (1 - \beta)B^2(i_1, i_2, \cdots, i_{s-1}), \tag{3.5}$$

if the i_j are pairwise distinct. Further,

$$D_p = \frac{1}{p}\Sigma_1^n\sigma_i^2 D_{p-1}(i) = \alpha B^{2p} = \alpha\Sigma_1^n\sigma_i^2 B^{2(p-1)}. \tag{3.6}$$

From (3.5) and (3.6) there follows the existence of an i_1 such that

$$D_{p-1}(i_1) \leqslant \alpha p B^{2(p-1)} \leqslant \alpha p(1 - \beta)^{1-p}B^{2(p-1)}(i_1).$$

Repeating the reasoning, we conclude that there exists an i_2 for which

$$D_{p-2}(i_1, i_2) \leqslant \alpha p(p - 1)(1 - \beta)^{3-2p}B^{2(p-2)}(i_1, i_2)$$

and so on. Finally, we come to a set $i_1, i_2, \cdots, i_{p-1}$ such that

$$D_1(i_1, i_2, \cdots, i_{p-1}) \equiv B^2(i_1, i_2, \cdots, i_{p-1})$$
$$\leqslant \alpha p!\,(1 - \beta)^{p(1-p)/2}B^2(i_1, i_2, \cdots, i_{p-1}).$$

This means that

$$\alpha p!\,(1 - \beta)^{p(1-p)/2} \geqslant 1,$$

and thus

$$\beta \geqslant 1 - (\alpha p!)^{2/p(p-1)}.$$

We see that for $\alpha \to 0$, $\bar{\sigma}^2/B^2 \to 1$. Incidentally, the Ljapunov ratio is $L \geqslant \sigma_1^3/B^3$ and, therefore, cannot tend to 0 as $\alpha \to 0$. The ratio $B^2/D_p^{1/p}$ may be considered as a measure of the variance of the distribution of the summands.

Let us now return to inequality (3.4). Assume that the random variables X_i, $i = 1, 2, \cdots$, are bounded and identically distributed and that $EX_1 = 0$. Let us consider a sequence $\{X_n^o\}_1^\infty$ of random variables which does not depend on the sequence $\{X_i\}_1^\infty$.

We require that $EX_n^o = 0$ and

$$\lim_{n\to\infty} P(|X_n^o| > \varepsilon B_n) = 0$$

for all $\varepsilon > 0$, where

$$B_n^2 = (n - 1)\text{Var } X_1 + \text{Var } X_n^o,$$

and

$$\lim_{n\to\infty} \operatorname{Var} X_n^o / B_n^2 = 1.$$

In the given case X_n^o plays the role of X_n for each n and therefore, according to the general system of notation introduced in Section 0, $S_n = X_n^o + \Sigma_1^{n-1} X_i$. It is not hard to see that the sequence S_n/B_n converges to 0 in probability. Consequently, inequality (3.4) cannot be satisfied for values of x comparable to B_n when n is sufficiently large. At the same time, one of the dispersions in the case examined, namely $\operatorname{Var} X_n^o$, is comparable in magnitude to B_n^2.

4. **Generalizations to the multivariate case.** Let $X_1, X_2, \cdots, X_n$ be independent random variables with values in a separable Banach space $\mathfrak{X}$. Let us put $A_t = \Sigma_1^n E|X_i|^t$ and $B_n^2 = A_2 = \Sigma_1^n E|X_i|^2$. Here and below $|X_i|$ denotes the norm of X_1. Put $S_n = \Sigma_1^n X_i$ as before.

The problem of large deviations for the sum S_n may be presented in the following way. Let G be a region in $\mathfrak{X}$ not containing 0, and put $\rho(G) = \inf\{|x|, x \in G\}$. Let us consider the probability $P\{S_n \in G\}$. It is clear that, if $\rho(G)$ is large, then the probability $P(S_n \in G)$ is small. It is desirable to bound this probability in terms of the numerical characteristics of the individual summands X_i – for example, in terms of moments of the form $E|X_i|^t$.

The simplest case is where $G = \{x : |x| \geq r\}$, that is, G is the complement of a sphere in $\mathfrak{X}$. We will examine this case below. It is evident that

$$P(S_n \geq r) \leq E|S_n|^2/r^2. \tag{4.1}$$

If $\mathfrak{X}$ is a Hilbert space and $EX_i = 0$ for $1 \leq i \leq n$, then $E|S_n|^2 = B_n^2$. This allows us to rewrite inequality (4.1) in the form

$$P(|S_n| \geq r) \leq \frac{B_n^2}{r^2}. \tag{4.2}$$

This inequality coincides in form with inequality (0.4).

There arises, naturally, the problem of generalizing the inequalities discussed in Sections 1, 2 and 3. In the first place, the problem of generalizing the classical Bernstein inequality to random variables with values in a Banach space was raised by Yu. V. Prohorov. Initially it was solved for the finite dimensional space R^k. In 1968, Yu. V. Prohorov [50] obtained the following bound.

THEOREM 4.1. *Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random vectors with values in R^k; suppose $EX_i = 0$ and $|X_1| < L$. Then for $n \geq k$*

$$P(|S_n| \geq r) \leq c \exp\{-r^2/8e^2L^2n\}, \tag{4.3}$$

where

$$c = 1 + \left(e^{5/12}/\pi 2^{\frac{1}{2}}\right)\sigma^2/L^2, \qquad \sigma^2 = E|X_1|^2.$$

This bound was refined by A. V. Prohorov (see [44], [45] and [46]).

The condition $n \geq k$ (conditions of this type appear also in the works of A. V.

Prohorov) does not seem very natural, especially if one compares the bound (4.3) with the inequality (4.2) which, generally speaking, is less precise, but, on the other hand, does not depend on dimensionality. And indeed, as it soon became apparent, the bound $n > k$ in (4.3) is connected, not with the essence of the matter, but with the method of proof.

In 1970, V. V. Yurinskii [61] proposed a clever method which permits the reduction of multivariate bounds to univariate ones. This method freed Yurinskii of limitations connected with dimensionality and he immediately obtained the Bernstein inequality for the Hilbert space case. For identically distributed random variables X_i, Yurinskii's result takes the following form.

THEOREM 4.2. *Suppose $EX_1 = 0$, $|X_1| < L$. Then*

$$P\left(|S_n| > rn^{\frac{1}{2}}\right) \leq 2\exp\left\{-\frac{r^2}{2\sigma^2}\left(1-\frac{rL'}{2\sigma^2 n^{\frac{1}{2}}}\right)\right\}, \tag{4.4}$$

if $0 < r < (\sigma^2/L')n^{\frac{1}{2}}$ and

$$P\left(|S_n| > rn^{\frac{1}{2}}\right) < 2\exp\left\{-rn^{\frac{1}{2}}/4L'\right\},$$

if $r > \sigma^2 n^{\frac{1}{2}}/L'$, where $\sigma^2 = E|X_1|^2$, $L' = \frac{1}{2}L(1 + (1 + 4\sigma^2/L^2)^{\frac{1}{2}})$.

Yurinskii's method made it possible also to derive an inequality having the form (1.24) in R^k (see [12]).

THEOREM 4.3. (S. S. Ebralidze, 1971). *Let X_i take values in R^k and suppose $EX_i = 0$, $i = 1, \cdots, n$. Then*

$$P(|S_n| > r) < 4\exp\{-K_1 r^2/B_n^2\} + K_2 A_3/r^3, \tag{4.5}$$

where K_1 and K_2 are absolute constants. (One may take, for example, $K_1 = \frac{1}{24}$ and $K_2 = 30000$.)

Since the right side of the bound (4.5) does not depend on the dimensionality, the latter is valid also for separable Hilbert spaces.

In 1974 Yurinskii [62] also succeeded in obtaining an exponential bound in the case of random variables in a separable Banach space.

THEOREM 4.4. *Let X_i, $i = 1, \cdots, n$ satisfy the condition*

$$E|X_i|^m < \frac{m!}{2}E|X_i|^2 L^{m-2}, \qquad m > 2.$$

Then

$$P(|S_n| > r) < \exp\{-(r^2/8B_n^2 - r\beta/4B_n^2)/(1 + a/2)\}, \tag{4.6}$$

where $a = rL/B_n^2$, $\beta = E|S_n|$.

The right side of inequality (4.6) depends heavily on the ratio β/B_n. Suppose that $EX_i = 0$, $i = 1, \cdots, n$ and $\mathfrak{X}$ is of type 2. That is, suppose that there exists an A such that for all n and all $X_1, X_2, \cdots, X_n$,

$$E|S_n|^2 \leq AB_n^2. \tag{4.7}$$

Then the inequality (4.6) may be rewritten in the form

(4.8) $$P(|S_n| > r) < \exp\left\{-\left(r^2/8B_n^2 - A^{\frac{1}{2}}r/4B_n\right)/(1 + a/2)\right\}.$$

Condition (4.7) plays an important role in research devoted to the law of large numbers and the central limit theorem in Banach spaces (see, for example, [15] and [21]).

Recently, F. I. Pinelis, combining methods of [17] and [62], extended inequality (3.2) to separable Banach spaces.

THEOREM 4.5. *Suppose* $|X_j| < L, j = 1, \cdots, n, p$ *is a natural number*, $0 < \gamma < 1$, *and* $\gamma(r - E|S_n|)/2 > (2p - 1)2L$. *Then, for all positive* t,

$$P(|S_n| > r) < \max\Big[\left(D_p^{(t+1)}/L^{p(t+1)}\right)^{\beta},$$
$$\exp\left\{(\gamma - 1)(t^{-1} + e^t)^{-1}(r - E|S_n|)L/2\left(D_p^{(2)}\right)^{1/p}\right\}\Big]$$

where $\beta = (1 - \gamma)(r - E|S_n|)/pL$ *and* $D_p^{(t)} = \Sigma_p\{E|X_k|^t\}_{J_n}$.

For the definition of the sum without repetitions of Σ_p see Section 3.

5. Various applications. Let $X_1, X_2, \cdots, X_n, \cdots$ be an infinite sequence of independent random variables. We shall say that the strong law of large numbers is satisfied if

$$P\left[\lim_{n\to\infty}\left(\frac{S_n}{n} - \operatorname{med}\left(\frac{S_n}{n}\right)\right) = 0\right] = 1.$$

To study the applicability of the strong law of large numbers, one can, without loss of generality, consider random variables X_n which are symmetrically distributed (see, for example, [48], Section 1).

Let

$$I_r = \{n : 2^r + 1 < n < 2^{r+1}\} \quad \text{and} \quad \chi_r = \frac{1}{2^r}\Sigma_{n\in I_r}X_n.$$

Yu. V. Prohorov [47] proved that the strong law of large numbers is satisfied if and only if, for all positive ε,

(5.1) $$\Sigma_0^\infty P(\chi_r > \varepsilon) < \infty.$$

Thus, the problem of finding necessary and sufficient conditions for the strong law of large numbers is reduced to the obtaining of upper and lower bounds for probabilities of large deviations of the sums $\Sigma_{n\in I_r}X_n$.

Using bounds for large deviations discussed in Sections 1, 2 and 3, we can formulate without undue difficulty variants of sufficient conditions for the strong law of large numbers. Let us mention several results obtained according to this prescription. We first introduce the following notation, in which the summation always extends over the k in I_r:

$$K(t, \delta_r, r) = 2^{-rt}\Sigma\int_{u<2^r\delta_r} u^t \, dF_k(u),$$

$$K_{t,r} = 2^{-rt}\Sigma E|X_k|^t,$$

$$H(\delta_r, r) = 2^{-2r}\Sigma\int_{|u|<2^r\delta_r} u^2 \, dF_k(u),$$

$$H_r = 2^{-2r}\Sigma\sigma_k^2.$$

By inequalities (1.5) and (1.7a), if $\beta = t/(t+2)$, then

$$(5.2) \quad P(\chi_r > \varepsilon) < \Sigma P(X_k > 2^r\delta_r) + \left(\varepsilon_1\delta_r^{t-1}/K(t, \delta_r, r) + 1\right)^{-\varepsilon_1/\delta_r} + \exp\{-\varepsilon_2/H(\delta_r, r)\},$$

where $\varepsilon_1 = \varepsilon t/(t+2)$, $\varepsilon_2 = 2\varepsilon^2/e^t(t+2)^2$, and $t > 2$. It is not hard to see that if

$$\Sigma_1^\infty\left(\varepsilon_1\delta_r^{t-1}/K(t, \delta_r, r) + 1\right)^{-\varepsilon_1/\delta_r}$$

and

$$\Sigma_1^\infty \exp\{-\varepsilon_2/H(\delta_r, r)\}$$

converge for all $\varepsilon > 0$, then the series formed from them by the replacement of ε_1 and ε_2 by ε also converge for all positive ε, and vice versa. For this reason we obtain from condition (3.1) and inequality (5.2):

THEOREM 5.1. *Let there exist a sequence of positive numbers* $\{\delta_r\}$ *such that*

$$(5.3) \qquad \Sigma_1^\infty\Sigma_{k\in I_r} P(X_k > 2^r\delta_r) < \infty,$$

$$(5.4) \qquad \Sigma_1^\infty\left(\varepsilon\delta_r^{t-1}/K(t, \delta_r, r) + 1\right)^{-\varepsilon/\delta_r} < \infty, \qquad t > 2,$$

and

$$(5.5) \qquad \Sigma_1^\infty \exp\{-\varepsilon/H(\delta_r, r)\} < \infty$$

for all positive ε. *Then the strong law of large numbers holds for the sequence* $\{X_k\}$.

COROLLARY 5.2. *Suppose* $E|X_k|^t < \infty$ *for all* k *and for* $t > 2$. *Then the combined conditions* (5.3),

$$(5.6) \qquad \Sigma_1^\infty\left(\varepsilon\delta_r^{t-1}/K_{t,r} + 1\right)^{-\varepsilon/\delta_r} < \infty,$$

and

$$(5.7) \qquad \Sigma_1^\infty e^{-\varepsilon/H_r} < \infty, \qquad \text{all} \quad \varepsilon > 0,$$

are sufficient for the strong law of large numbers.

Taking $\delta_r = \varepsilon/\beta$ in (5.3) and (5.6) we get:

COROLLARY 5.3. *Suppose* $t > 2$ *and* $\beta > 1$. *Then the combination of conditions* (5.7),

$$(5.8) \qquad \Sigma_1^\infty P(X_k > k\varepsilon) < \infty, \qquad \text{all} \quad \varepsilon > 0,$$

and

$$\Sigma_1^\infty (K_{t,r})^\beta < \infty \tag{5.9}$$

is sufficient for the strong law of large numbers.

If $t = 2$, then $K_{t,r} = H_r$, and, therefore, condition (5.9) implies condition (5.7) so that we obtain:

COROLLARY 5.4. *If condition* (5.8) *is satisfied and if there exists a* $\beta \geqslant 1$ *such that*

$$\Sigma_1^\infty H_r^\beta < \infty, \tag{5.10}$$

then the strong law of large numbers holds.

Taking $\beta = 1$ in (5.9) and making use of Chebyshev's inequality to bound $P(X_k > k\varepsilon)$, we get:

COROLLARY 5.5. *Conditions* (5.7) *and*

$$\Sigma_1^\infty E|X_k|^t / k^t < \infty, \qquad t > 2, \tag{5.11}$$

are sufficient for the strong law of large numbers.

It is not hard to see that the combination of conditions (5.7) and (5.11) is less restrictive than the condition

$$\Sigma_1^\infty E|X_k|^t / k^{t/2+1} < \infty, \tag{5.12}$$

introduced by Brunk [8] (see also [47]). Indeed,

$$\Sigma_1^\infty H_r^{t/2} < \infty \Rightarrow \Sigma_1^\infty \exp\{-\varepsilon/H_r\} < \infty$$

for all positive ε. On the other hand,

$$H_r^{t/2} = \left(2^{-2r}\Sigma_{k\in I_r} EX_k^2\right)^{t/2} \leqslant 2^{-r(t/2+1)}\Sigma_{k\in I_r} E|X_k|^t.$$

Thus, from condition (5.12) we obtain condition (5.7). We note that in conditions (5.9) and (5.10) $K_{t,r}$ and H_r may be replaced by $K(t, \delta, r)$ and $H(\delta, r)$ respectively, where δ is arbitrarily small but fixed. Theorem 5.1 and and its corollary are taken from [17].

Let us turn now to Theorem 3.2 for the purpose of obtaining sufficient conditions for the strong law of large numbers. We are obliged to introduce some additional notation. Let

$$D_p^{(t)}(r, \delta) = \Sigma_p\{\beta_n^{(t)}(\delta)\}_{I_r},$$

where

$$\beta_n^{(t)}(\delta) = \int_{|u| < n\delta} |u|^t \, dF_n(u)$$

(see the definition of Σ_p in Section 3 above). Let us put

$$D_p(r, \delta) = D_p^{(2)}(r, \delta).$$

Making use of inequality (3.3) to bound $P(\chi_r \geqslant \varepsilon)$, we get the following combination of conditions which is sufficient for the strong law of large numbers.

THEOREM 5.6. *If condition* (5.8) *is satisfied and there exist, in addition,* $\delta > 0$, $t > 2$ *and an integer* $p > 1$ *such that*

(5.13) $$\Sigma_1^\infty \exp\{-\varepsilon 2^{2r}/D_p^{1/p}(r,\delta)\} < \infty,$$

for all $\varepsilon > 0$ *and*

(5.14) $$\Sigma_1^\infty D_p^{(t)}(r,\delta)/2^{trp} < \infty,$$

then the strong law of large numbers holds.

If $t = 2$, then condition (5.13) follows from condition (5.14), and we obtain the following result.

COROLLARY 5.7. *If there exist* $\delta > 0$, $t > 2$ *and an integer* $p > 1$ *such that*

(5.15) $$\Sigma_1^\infty D_p(r,\delta)/2^{2pr} < \infty,$$

then the strong law of large numbers holds.

Condition (5.15) is somewhat weaker than Egorov's condition

$$\Sigma_{n=1}^\infty n^{-2p}\Sigma\sigma_{j_1}^2\sigma_{j_2}^2\cdots\sigma_{j_p}^2 < \infty,$$

(see [13]) where the internal summation extends over all $j_1, j_2, \cdots, j_p$ satisfying $0 < j_1 < j_2 < \cdots < j_p \leq n$.

Let us now return to the problem of necessary and sufficient conditions for the strong law of large numbers. Put $X_n^\varepsilon = X_n V_n^\varepsilon$, where V_n^ε is the indicator of event

$$|X_n| \leq n\varepsilon,$$

and let $\chi_r^\varepsilon = 2^{-r}\Sigma_{n\in I_r}X_n^\varepsilon$. It is easy to see that

(5.16) $$|P(\chi_r > \varepsilon) - P(\chi_r^\varepsilon > \varepsilon)| \leq \Sigma_{n\in I_r}P(|X_n| > n\varepsilon).$$

If the series $\Sigma_1^\infty P(X_n > n\varepsilon)$ converges, then

$$\Sigma_1^\infty P(\chi_r^\varepsilon > \varepsilon) < \infty \Rightarrow \Sigma_1^\infty P(\chi_r > \varepsilon) < \infty.$$

Because of (5.1), this means that the combination of conditions (5.8) and

(5.17) $$\Sigma_1^\infty P(\chi_r^\varepsilon > \varepsilon) < \infty$$

for all $\varepsilon > 0$ is sufficient for the strong law of large numbers.

On the other hand, condition (5.8) is necessary for the strong law of large numbers. Turning again to (5.16), we see that condition (5.17) also is necessary for the strong law of large numbers. Thus, the combination of conditions (5.8) and (5.17) is necessary and sufficient for the strong law of large numbers.

Since the random variables X_n^ε are bounded, exponential bounds for $P(\chi_r > \varepsilon)$ are possible. If we have inequalities of the form

$$cf_r(F_{n_{r-1}+1}^\varepsilon, \ldots, F_{n_r}^\varepsilon) \leq P(\chi_r^\varepsilon > \varepsilon) \leq f_r(F_{n_{r-1}+1}^\varepsilon, \ldots, F_{n_r}^\varepsilon),$$

where F_j^ε is a function of the distribution of the random variable X_j^ε and c is an absolute constant, it is clear that the combination of conditions (5.8) and

(5.18) $$\Sigma_1^\infty f_r(F_{n_{r-1}+1}^\varepsilon, \ldots, F_{n_r}^\varepsilon) < \infty$$

is necessary and sufficient for the strong law of large numbers.

Of course, a condition of the type (5.18) is significant only in the case when the evaluation of

$$f_r(F^{\varepsilon}_{n_{r-1}+1}, \ldots, F^{\varepsilon}_{n_r})$$

is less complex than direct evaluation of $P(\chi_r > \varepsilon)$. Let us now formulate the result which we were able to obtain by this method in [31]. Let

$$f_n(h, \varepsilon) = \int_{|u|<n\varepsilon} e^{hu}\, dF_n(u),$$

$$\psi_r(h, \varepsilon) = \Sigma_{n \in I_r} \frac{d}{dh} f_n(h, \varepsilon) / f_n(h, \varepsilon).$$

Define $h_r(\varepsilon)$ as the solution to the equation $\psi_r(h, \varepsilon) = \varepsilon n_r$ if $\sup_h \psi_r(h, \varepsilon) > \varepsilon n_r$; otherwise put $h_r(\varepsilon) = \infty$.

THEOREM 5.8. *The strong law of large numbers holds if and only if*
(i) $\Sigma_1^\infty P(X_n > n\varepsilon) < \infty$,
(ii) $\Sigma_1^\infty e^{-\varepsilon h_r(\varepsilon) n_r} < \infty$,
for all positive ε.

Condition (ii) seems not very effective at first glance. Nevertheless, a criterion for the strong law of large numbers is quite easily obtained from it as a corollary, namely,

$$\Sigma_1^\infty \exp\{-\varepsilon/H_r\} < \infty, \qquad \text{all} \quad \varepsilon > 0$$

for random variables X_n satisfying the additional condition

$$X_n = 0(n/\log\log n)$$

(see [31]). This criterion was at first introduced by Prohorov [49]. The application of Theorem 5.8 is considerably more advantageous than a direct proof.

We also note that, with the aid of Theorem 5.8, it is not difficult to get sufficient conditions of the type that figure in Theorem 5.6 for the strong law of large numbers (see [36]).

Bounds for probabilities of large deviations also play a major role in the proof of the law of the iterated logarithm. Making use of bounds of the type (1.55) for $P(\max_{1<k<n} S_n > x)$, A. I. Sahanenko [55] obtained sufficient conditions for the law of the iterated logarithm which in form are close to Teicher's conditions (see [60]), though weaker. The conditions of Sahanenko also contain as a special case the conditions of Egorov [14].

Let us now touch upon applications to nonuniform bounds on the rate of convergence to the normal law. Assume first for simplicity that the X_i are identically distributed and that $EX_1 = 0$, $\sigma^2 = 1$, and $\beta_3 \equiv E|X_1|^3 < \infty$. Also assume for the moment that $F(x)$ satisfies an additional condition

$$1 - F(x) = \frac{l(x)}{x^t}(1 + o(1)),$$

as $x \to \infty$, where $l(x)$ is a slowly varying function, and $t > 3$. If $x > b(n \log n)^{\frac{1}{2}}$, where $b > (t-2)^{\frac{1}{2}}$, then for sufficiently large n,

$$n(1-F(x)) > 2\left(1-\Phi\left(x/n^{\frac{1}{2}}\right)\right).$$

Turning now to (1.25), we see that for $x > b(n \log n)^{\frac{1}{2}}$,

$$\Phi\left(x/n^{\frac{1}{2}}\right) - P(S_n < x) \geqslant \frac{n}{4}(1-F(x)),$$

if n is sufficiently large.

Thus, we do not lose much in the way of precision if for $x > b(n \log n)^{\frac{1}{2}}$ we make use of the trivial bound

$$(5.19) \qquad \Delta_n(x) \equiv \left|\Phi\left(x/n^{\frac{1}{2}}\right) - P(S_n < x)\right| \leqslant 1 - \Phi\left(x/n^{\frac{1}{2}}\right) + P(S_n \geqslant x).$$

If $F(x)$ varies irregularly as $x \to \infty$, our reasoning loses its precision. Nevertheless, as we shall presently see, it also makes sense in this case to start from inequality (5.19) (for sufficiently large x, of course). The inequality (5.19) reduces the problem of estimating the difference $\Phi(x/n^{\frac{1}{2}}) - P(S_n < x)$ to that of estimating the difference between the probabilities of large deviations for S_n and for a normally distributed random variable.

Taking $t = 3$ in inequality (1.24), we get the bound

$$(5.20) \qquad P(S_n \geqslant x) \leqslant c_1 n\beta_3/x^3 + \exp\left\{-c_2x^2/n\right\},$$

where $c_1 = (1 + 2/3)^3$ and $c_2 = 2/25e^3$.

Let us now find the lower limit of those x for which

$$(5.21) \qquad c_1 n\beta_3/x^3 > \exp\left\{-c_2x^2/n\right\}.$$

This inequality may be put in the form

$$y^2\left(c_2 - \frac{3}{y^2}\log y\right) > \log n^{\frac{1}{2}}/\beta_3 c_1,$$

where $y = x/n^{\frac{1}{2}}$. Since $\log y < y$,

$$c_2 - \frac{3}{y^2}\log y > c_2 - 3/y > c_2/2,$$

for $y > 6/c_2$. Consequently, the inequality (5.21) is satisfied at least for

$$x \geqslant \phi(n) \equiv \max\left[6n^{\frac{1}{2}}/c_2, \left(\frac{2n}{c_2}\log\left(n^{\frac{1}{2}}/c_1\beta_3\right)\right)^{\frac{1}{2}}\right].$$

In consequence of (5.20) this means that

$$(5.22) \qquad P(S_n \geqslant x) \leqslant 2c_1\beta_3 n/x^3,$$

for $x \geqslant \phi(n)$.

Since for $x \geqslant n^{\frac{1}{2}}$

$$1 - \Phi\left(x/n^{\frac{1}{2}}\right) \leqslant e^{-c_2x^2/n},$$

by virtue of (5.21)

$$1 - \Phi(x/n^{\frac{1}{2}}) < c_1 n\beta_3/x^3, \tag{5.23}$$

only if $x \geqslant \phi(n)$. From (5.19), (5.22) and (5.23) it follows that for $x \geqslant \phi(n)$,

$$\Delta_n(x) \leqslant 3c_1 n\beta_3/x^3. \tag{5.24}$$

On the other hand, the method of conjugate distributions leads to the bound

$$\Delta_n(x) \leqslant \frac{\gamma_1 \beta_3}{n^{\frac{1}{2}}} e^{-\gamma_2 x^2/n} + n(1 - F(\gamma_3 x)), \tag{5.25}$$

which is valid for $\beta_3 \leqslant \gamma_4 n^{\frac{1}{2}}$ and $0 \leqslant x \leqslant \psi(n)$, where $\psi(n) = n^{\frac{1}{2}}(n^{\frac{1}{2}}/\beta_3)^{\frac{1}{3}}$, and $\gamma_1, \cdots, \gamma_4$ are absolute constants.

If $\beta_3/n^{\frac{1}{2}}$ is sufficiently small, then $\psi(n) > \phi(n)$. In consequence of (5.24) and (5.25),

$$\Delta_n(x) \leqslant \frac{\gamma_1 \beta_3}{n^{\frac{1}{2}}} e^{-\gamma_2 x^2/n} + \beta_3(3c_1 + \gamma_3^{-3})n/x^3.$$

Therefore,

$$|\Phi(x) - P(S_n/n^{\frac{1}{2}} < x)| < c\beta_3/n^{\frac{1}{2}}(1 + x^3),$$

where c is an absolute constant. An analogous result is of course also valid for $-x$. The bound

$$|\Phi(u) - P(S_n/n^{\frac{1}{2}} < u)| < c\beta_3/n^{\frac{1}{2}}(1 + |u|^3) \tag{5.26}$$

was initially obtained in [30] by using precisely this approach.

From the asymptotic representation (1.25) it is not hard to see the impossibility of replacing the $|u|^3$ in the right-hand side of (5.26) by a higher power of $|u|$.

Utilizing the method of [30], Bikjalis [5] extended the bound (5.26) to random variables that are not identically distributed.

THEOREM 5.9. (Bikjalis, 1966). *Suppose that* $EX_k = 0$, $E|X_k| = 0$, $E|X_k|^3 < \infty$ *for all* k. *Then there exists an absolute constant* c *such that*

$$|\Phi(u) - P(S_n/B_n < u)| \leqslant \frac{cL_n}{1 + |u|^3}, \tag{5.27}$$

where L_n *is the Ljapunov ratio* $\Sigma_1^n E|X_k|^3/B_n^3$.

Somewhat later L. V. Osipov [37], used a completely different approach to generalize the bound (5.26) in another direction.

THEOREM 5.10. (Osipov, 1967). *Let* X_i *be identically distributed and suppose* $EX_1 = 0$, $\sigma^2 = 1$, $E|X_1|^r < \infty$ *for some* $r \geqslant 3$. *Then*

$$|\Phi(u) - P(S_n/n^{\frac{1}{2}} < u)| < \frac{c(r)}{1 + |u|^r}\left(\beta_3/n^{\frac{1}{2}} + E|X_1|^r/n^{(r-2)/2}\right). \tag{5.28}$$

Finally, S. K. Sakoyan [56], using bounds of the type (3.37), recently bounded the difference $\Phi(u) - P(S_n/B_n < u)$ in terms of generalized moments.

Let $\bar{G}(\delta)$, $\delta > 2$, be the class of functions on $(0, \infty)$ with continuous nonincreasing derivative satisfying the conditions

(1) $\lim_{u\to\infty} g'(u) = \infty$;

(2) $\delta/u \leq g'(u) \leq \delta e^{g(u)-1}/u^{\delta+1}$.

Obviously $\bar{G}(\delta) \subset G(\delta)$ (for the definition of $G(\delta)$ see Section 3). Let us put

$$L_{gn} = B_{gn}/B_n^{\delta}.$$

THEOREM 5.11. (Sakoyan, 1974). *If* $EX_k = 0$ *for all* k, $g \in \bar{G}(\delta)$, $\delta > 3$, *and* $L_{gn} \leq 1$, *then for* $u \geq 0$

$$|\Phi(u) - P(S_n/B_n < u)| < c(\delta)(L_n + L_{gn})e^{-g(u)}, \tag{5.29}$$

The bound (5.29) simultaneously generalizes (5.27) and (5.28).

Inequalities for probabilities of large deviations may be used to bound the mean deviation of $P(S_n/B_n < u)$ from $\Phi(u)$—that is, to bound

$$\int_{-\infty}^{\infty} g(|u|)|P(S_n/B_n < u) - \Phi(u)|\, du,$$

where $g(u)$ is a nonnegative function. Let us mention several results obtained through this approach.

THEOREM 5.12. (S. V. Nagaev, 1965). *Let the* X_k *be identically distributed with* $EX_1 = 0$, $\sigma^2 = 1$, $\beta_3 < \infty$ *and suppose there exists an* n_0 *such that the distribution of* S_{n_0} *has an absolutely continuous component. Then*

$$\int_{-\infty}^{\infty}|p_n(u) - 2\pi^{-\frac{1}{2}}e^{-u^2/2}|\,|u|^3\, du$$

$$= \frac{|EX_1^3|}{6(2\pi n)^{\frac{1}{2}}}\int_{-\infty}^{\infty}u^4|u^2 - 3|e^{-u^2/2}\, du + o\left(1/n^{\frac{1}{2}}\right),$$

where $p_n(u)$ *is the density of the distribution of* $S_n/n^{\frac{1}{2}}$.

THEOREM 5.13. (Bikjalis, Yasyunis, 1967). *Let the* X_k *be identically distributed with* $EX_1 = 0$, $\sigma^2 = 1$, $\beta_3 < \infty$. *Then*

$$\int_{-\infty}^{\infty}u^2\left|P\left(S_n < un^{\frac{1}{2}}\right) - \Phi(u) - \frac{EX_1^3}{6(2\pi n)^{\frac{1}{2}}}(1 - u^2)e^{-u^2/2}\right| du = o\left(1/n^{\frac{1}{2}}\right),$$

if X_1 *has a nonlattice distribution.*

THEOREM 5.14. (Sakoyan, 1975). *If* $EX_k = 0$, *for all* k, $g \in \bar{G}(\delta)$, $\delta > 3$, *and* $L_{gn} < 10$, *then*

$$\int_0^{\infty}e^{g(u)}g'(u)|P(S_n < uB_n) - \Phi(u)|\, du \leq c(\delta)(L_n + L_{gn}).$$

We now point out a curious application of inequality (1.55). This concerns an upper bound of $E|S_n|^t$ for $t \geq 2$. Let us assume that $EX_k = 0, k = 1, \cdots, n$. Since for this condition $\mu(-\infty, Y) \leq 0$ and $\mu(-Y, \infty) \geq 0$, the following bound can be deduced from inequality (1.55):

$$P(S_n \geq x) \leq \Sigma_1^n P(|X_k| \geq y) + 2\exp\left\{\frac{x}{y} - \frac{x}{y}\log(xy/B_n^2 + 1)\right\}.$$

Let $y = x/c$, $c > t/2$. Multiplying both parts of the above inequality by tx^{t-1} and integrating with respect to x from 0 to ∞, we get

$$E|S_n|^t \leq c^t A_t + 2te^c \int_0^\infty x^{t-1}(1 + x^2/B_n^2 c)^{-c}\,dx.$$

The integral in the right-hand side of the inequality is equal to

$$2^{-1}c^{t/2}B(t/2, c - t/2)B_n^t,$$

where $B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du$. On the other hand, it is known that for $t \geq 1$

$$E|S_n|^t \geq a_t E(\Sigma_1^n X_k^2)^{t/2},$$

where a_t depends only on t (see [24]).

According to Hölder's inequality,

$$E(\Sigma_1^n X_k^2)^{t/2} \geq B_n^t.$$

Further,

$$(\Sigma_1^n X_k^2)^{t/2} \geq \Sigma_1^n |X_k|^t,$$

from which follows

$$E(\Sigma_1^n X_k^2)^{t/2} \geq A_t.$$

Consequently

$$E|S_n|^t \geq a_t \max[B_n^t, A_t].$$

Thus, the following theorem holds.

THEOREM 5.15. (Nagaev, Pinelis, 1977). *Suppose $t \geq 2$ and $EX_k = 0$, $k = 1, \cdots, n$. Then for all $c > t/2$*

$$E|S_n|^t \leq c^t A_t + tc^{t/2}e^c B(t/2, c - t/2)B_n^t, \tag{5.30}$$

where

$$B(\alpha, \beta) = \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du,$$

and

$$E|S_n|^t \geq a_t \max[B_n^t, A_t],$$

and a_t depends only on t.

Note that the bound of Dharmadhikari and Jogdeo [11] follows from (5.30), except for a constant factor that depends only on t.

Inequality (5.30) may also be derived from the bound for $E|S_n|^m g(S_n)$ obtained by V. V. Sazonov [59] for $m > 2$, $g(x)$ being a nondecreasing, even, nonnegative function that is subject to the condition $x_1/g(x_1) \leq x_2/g(x_2)$, for $0 < x_1 < x_2$.

In addition, other values for the coefficients are obtained according to B_n^t and A_t.

One should mention also the work of Rosén [51], in which it was proved that if, for $k = 1, \cdots, n$ and $m = 1, \cdots, p$,

$$EX_k^{2m} < \lambda_k^{2m}\rho_k,$$

then

$$ES_n^{2p} \leq c(p)\max\left[\left(\Sigma_1^n\lambda_k^2\rho_k\right)^p, \Sigma_1^n\lambda_k^{2p}\rho_k\right].$$

This inequality is evidently a special case of (5.30). In the case of $t = 3$, inequality (5.30) is cited (without proof or explicit expressions for the coefficients) in [54] (page 663) and [1] (page 259, inequality (1.4)). For random variables which are functions on [0, 1], the bound (5.30) is obtained in [52] (see also [53]).

Incidentally, it follows from inequality (5.30) that for $g(u) = t \log u$, the restriction $L_{gn} < 10$ in Theorem 5.14 is superfluous. Indeed, in this case

$$\begin{aligned} \int_0^\infty x^{t-1}|P(S_n < xB_n) - \Phi(x)|\, dx \\ &\leq E|S_n|^t/tB_n^t + \int_0^\infty x^{t-1}(1 - \Phi(x))\, dx \\ &\leq c(t)(L_{gn} + 1). \end{aligned}$$

One of the possible applications of inequality (5.30) is to the study of the convergence of series of the form

$$\Sigma_1^\infty n^t P(S_n > \varepsilon n^s), \qquad s > 0, \quad \varepsilon > 0. \tag{5.31}$$

Let us now consider the following example. Suppose $EX_n = 0$, $n = 1, 2, \cdots$, and $\Sigma_1^\infty E|X_n|^{2t}/n^{t+1} < \infty$. By virtue of (5.30),

$$P(|S_n| > n\varepsilon) \leq E|S_n|^{2t}/\varepsilon^{2t}n^{2t} \leq n^{-2t}c(t)\varepsilon^{-2t}\left(\Sigma_1^n E|X_k|^{2t} + B_n^{2t}\right).$$

On the other hand,

$$B_n^{2t} \leq n^{t-1}\Sigma_1^n E|X_k|^{2t},$$

Consequently,

$$\begin{aligned} \Sigma_1^\infty P(S_n > n\varepsilon)/n &\leq c(t)\varepsilon^{-2t}\Sigma_{n=1}^\infty n^{-t-2}\Sigma_{k=1}^n E|X_k|^{2t} \\ &= c(t)\varepsilon^{-2t}\Sigma_{k=1}^\infty E|X_k|^{2t}\Sigma_{n=k+1}^\infty n^{-t-2} < \infty \end{aligned}$$

for all positive ε, and we have reproved the result of Baum and Katz [2].

Of course, to bound sums having the form (5.31) one can make direct use of the inequalities obtained in Sections 1 and 3. Assume, for example, that the X_k are identically distributed, $EX_1 = 0$, $\sigma^2 < \infty$, $E(X_1^+)^p < \infty$ $(X_1^+ = \max[0, X_1])$.

$\alpha > \frac{1}{2}$, $p\alpha > 1$, and let us bound

$$\Sigma = \Sigma_1^\infty n^{p\alpha-2} P(S_n \geq n^\alpha).$$

First of all, because of (1.56)

$$P(S_n \geq n^\alpha) \leq n(1 - F(n^\alpha/\beta)) + \beta^\beta e^\beta \left(\frac{\sigma^2}{n^{2\alpha-1}}\right)^\beta, \tag{5.32}$$

for all positive β. Let us now write Σ in the form

$$\Sigma = \Sigma_{n^{2\alpha-1} \leq \sigma^2} + \Sigma_{n^{2\alpha-1} > \sigma^2} = \Sigma_1 + \Sigma_2. \tag{5.33}$$

Putting $\beta = \alpha p/(2\alpha - 1)$, we get

$$\Sigma_{n^{2\alpha-1} > \sigma^2} n^{\alpha p - 2} (\sigma^2/n^{2\alpha-1})^\beta < \frac{\pi^2}{6} \sigma^{2(\alpha p - 1)/(2\alpha - 1)}. \tag{5.34}$$

Simple calculations show that

$$n^{\alpha p - 1}(1 - F(n^\alpha/\beta)) < \frac{2^{\alpha p - 1}\beta^p}{\alpha} \int_{(n-1)^\alpha/\beta}^{n^\alpha/\beta} u^{p-1}(1 - F(u))\, du$$

for $n = 2, 3, \cdots$. On the other hand,

$$1 - F(1/\beta) \leq E(X_1^+)^p \beta^p.$$

Therefore

$$\Sigma_1^\infty n^{\alpha p - 1}(1 - F(n^\alpha/\beta)) < (2^{\alpha p - 1}/\alpha p + 1)\beta^p E(X_1^+)^p. \tag{5.35}$$

From (5.32), (5.34) and (5.35) it follows that

$$\Sigma_2 < (2^{\alpha p - 1}/\alpha p + 1)\beta^p E(X_1^+)^p + \frac{\pi^2}{6} \beta^\beta e^\beta \sigma^{2(\alpha p - 1)/(2\alpha - 1)}. \tag{5.36}$$

If $\sigma^2 < 1$, then $\Sigma_1 = 0$. If $\sigma^2 \geq 1$, then

$$\Sigma_1 \leq \Sigma_{1 \leq n^{2\alpha-1} \leq \sigma^2} n^{p\alpha - 2} < \sigma^{2(p\alpha - 1)/(2\alpha - 1)}. \tag{5.37}$$

On the basis of (5.33), (5.36) and (5.37), we conclude that

$$\Sigma_1^\infty n^{p\alpha - 2} P(S_n \geq n^\alpha) < c_1(\alpha, p) E(X_1^+)^p + c_2(\alpha, p)\sigma^{2(p\alpha - 1)/(2\alpha - 1)}, \tag{5.38}$$

where

$$c_1(\alpha, p) = (2^{\alpha p - 1}/\alpha p + 1)\beta^p,$$

$$c_2(\alpha, p) = \frac{\pi^2}{6} e^\beta \beta^\beta + 1, \qquad \beta = \alpha p/(2\alpha - 1).$$

Since inequality (1.56) is valid also for $P(\bar{S}_n \geq n^\alpha)$, where $\bar{S}_n = \max_{1 \leq k \leq n} S_k$ (see, for example, [7]), it is possible to replace S_n by $\bar{S}_n$ on the left-hand side of (5.38). It is indeed such a bound that is obtained in [9] (page 53, inequality (1.9)) (although without explicit expressions for coefficients $c_1(\alpha, p)$ and $c_2(\alpha, p)$). For this however, the authors required lengthier reasoning, since they started by deriving an inequality of the form (1.56).

The examples discussed show that the application of the inequalities of Section 1 allow a simplification and standardization of this subject.

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