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S. V. Nagaev

## LARGE DEVIATIONS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

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# LARGE DEVIATIONS FOR SUMS OF INDEPENDENT RANDOM VARIABLES

S. V. NAGAEV

NOVOSIBIRSK

## 1. INTRODUCTION

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of independent identically distributed random variables with distribution function $F(x)$ and $\mathsf{M}\xi_1 = 0$, $\mathsf{D}\xi_1 = 1$.

Assume that for $x \to \infty$

$$1 - F(x) = e^{\chi(x)}(1 + o(1)),$$

where $\chi(x)$ is a nonincreasing function satisfying the following conditions

(I) $$\lim_{x\to\infty} \chi'(x)\, x/\ln x = -\infty,$$

(II) $$\alpha\, \chi(x)/x \leqq \chi'(x), \quad 0 < \alpha < 1.$$

(III) $$l\, \chi''(x) \leqq -\chi'(x)/x \leqq L\, \chi''(x),$$

(IV) $$0 \leqq -\chi'''(x) < L_1\, \chi''(x)/x,$$

where $l$, $L$, $L_1$ are some positive constants.

Assume that

$$\mathsf{M}|\xi_1|^{N(\alpha)} < \infty,$$

where

$$N(\alpha) = [(3 - 2\alpha)/(1 - \alpha)].$$

Let

$$K(u) = \sum_2^{N(\alpha)} \chi_k u^k,$$

where $\chi_k$ are cumulants of the distribution $F(x)$.

Denote by $\lambda_\alpha(z)$ the part of Cramer's series containing $N(\alpha) - 3$ first terms.

Let $A(n)$ is the solution of the equation

$$\chi(x) + \frac{x^2}{n} = 0\,.$$

THEOREM. *Let*

$$P_1(x) = n(1 - \chi''((1-\beta)\,x)\,n)^{-1/2}\,(1 - F((1-\beta)\,x)) \times$$
$$\times \exp\left\{-\frac{(\beta x)^2}{2n} + \frac{(\beta x)^3}{n^2}\,\lambda_\alpha\left(\frac{\beta x}{n}\right)\right\},$$

*where $\beta$ is the least positive root of the equation*

$$K'(-\chi'((1-\beta)\,x)) = \frac{\beta x}{n}\,, \tag{1.0}$$

$$P_2(x) = \left(1 - \Phi\left(\frac{x}{\sqrt{n}}\right)\right)\exp\left\{\frac{x^3}{n^2}\,\lambda_\alpha\left(\frac{x}{n}\right)\right\}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-u^2/2}\,du$$

*If*

$$\lim_{n\to\infty} xn^{-1/(2-\alpha)} = \infty\,,$$

*then*

$$P(x) \equiv \mathsf{P}\left(\sum_{i=1}^{n} \xi_i > x\right) = P_1(x)\,(1 + o(1))\,. \tag{1.1}$$

*If*

$$\overline{\lim_{n\to\infty}}\, xn^{-1/(2-\alpha)} < \infty \quad \textit{and} \quad \overline{\lim_{n\to\infty}}\, n\,\chi''((1-\beta)\,x) < 1\,,$$

*then*

$$P(x) = (P_1(x) + P_2(x))\,(1 + o(1))\,. \tag{1.2}$$

*If*

$$\underline{\lim_{n\to\infty}}\, n\,\chi''((1-\beta)\,x) \geqq 1 \quad \textit{and} \quad x \geqq A(n)\,,$$

*then*

$$P(x) = P_2(x)\,(1 + o(1))\,. \tag{1.3}$$

Note that if $n(\chi')^2\,((1-\beta)\,x) \to 0$, then

$$P(x) = n(1 - F(x))\,(1 + o(1))\,. \tag{1.4}$$

This result has been obtained in [1] under more strong than (I)–(IV) co

Word for word repeating the arguments of § 4 of the work [2] one can show that representation (1.3) remains valid also for $x < \Lambda(n)$.

In the paper [3] asymptotic representations of such kind as (1.1)–(1.3) have been obtained in the case when $\chi(x) = -x^\alpha$, $0 < \alpha < 1$. The method of the proof in [3] is probabilistic.

In the present paper the modification of the analytic method suggested in [1] is utilized. The approach developed in [2] is also essentially used in the proof.

Come now to agreement about some notation. We shall denote by $\varepsilon$ and $\eta$ constants which can be chosen arbitrarily small. $M$ will denote arbitrarily large constant. Symbols $c$ and $C$ will be used for notation accordingly sufficiently small and sufficiently large constants. Note that the same symbol will be used for notation of different constants. After all symbol $f^{-1}(x)$ will denote the function inverse to $f(x)$.

## 2. SOME AUXILIARY RESULTS

LEMMA 2.1. *Let monotonic in segment* $[0, a]$ *function* $\psi(u)$ *satisfies the conditions*

$$\lim_{u\to 0} \psi'(u)\, u/\ln u = -\infty\,, \tag{2.1}$$

$$-\psi''(u) \geqq c\, \psi'(u)/u\,, \tag{2.2}$$

$$0 \leqq \psi'''(u) < -C\,\psi''(u)/u\,. \tag{2.3}$$

*Then there exists a function* $\omega(u)$ *tending to zero for* $u \to 0$ *such that for* $x \to \infty$

$$\int_{y'}^{y''} \exp\left\{-xu + \psi(u) + \tfrac{1}{2}\ln\left(-\psi''(u)\right) du = \sqrt{(2\pi)}\, e^{-xu_0+\psi(u_0)}\left(1 + o(1)\right)\right.$$

*uniformly with respect to* $0 \leqq y' \leqq u_0(1 - \omega(u_0))$, $a \geqq y'' \leqq u_0(1 + \omega(u_0))$. *Here* $u_0$ *is a solution of the equation* $x = \psi'(u)$.

Proof. Obviously

$$-xu + \psi(u) = -xu_0 + \psi'(u_0) + \tfrac{1}{2}\psi''(u_0)(u - u_0)^2 + \\ + O(\psi'''(\bar{u})(u - u_0)^3)\,, \quad |\bar{u} - u_0| < |u - u_0|\,.$$

It follows from (2.1) and (2.2) that $\psi''(u) < 0$.

If $\psi''(\bar{u}) < \psi''(u_0)$, then $\bar{u} < u_0$ and according to (2.3)

$$-\psi''(u_0) = -\psi''(\bar{u}) - \psi'''(\bar{\bar{u}})(u_0 - \bar{u}) = -\psi''(\bar{u})\left(1 + O\left(\frac{u_0 - \bar{u}}{\bar{u}}\right)\right), \\ |\bar{\bar{u}} - u_0| < |\bar{u} - u_0|\,.$$

Thus, in this case for $u - u_0 = o(u_0)$

(2.4) $$-xu + \psi(u) = -xu_0 + \psi'(u_0) + \tfrac{1}{2}\psi''(u_0)(u - u_0)^2 \left(1 + O\left(\frac{u_0 - u}{\bar{u}}\right)\right)$$

since $\psi'''(\bar{u}) = O(\psi''(\bar{u})/\bar{u})$.

If $\psi''(\bar{u}) > \psi''(u_0)$, then

(2.5) $$-xu + \psi(u) = -xu_0 + \psi(u_0) + \tfrac{1}{2}\psi''(u_0)(u - u_0)^2 \left(1 + O\left(\frac{u - u_0}{u_0}\right)\right).$$

By virtue of (2.1) and (2.2)

(2.6) $$\lim_{u \to 0} \psi(u)\, u^2/\ln u = \infty .$$

It follows from condition (2.3) that

(2.7) $$\ln(-\psi''(u)) < C \ln \frac{1}{u} .$$

According to (2.2)

(2.8) $$\frac{d}{du} \psi'(u)\, u^c = \psi''(u)\, u^c + c\, \psi'(u)\, u^{c-1} \leqq 0 .$$

In that way the function $\psi'(u)\, u^c$ is nonincreasing. Denote $\psi(u) + \frac{1}{2}\ln(-\psi''(u))$ by $\psi_1(u)$. Let $u_0'$ be the most remote from $u_0$ solution of the equation $x = \psi_1'(u)$.

It is clear that

(2.9) $$\psi_1'(u) = \psi'(u) + \tfrac{1}{2}\psi'''(u)/\psi''(u) .$$

It follows hence by (2.3) that

(2.10) $$\lim_{x \to \infty} \frac{\psi'(u_0')}{x} = 1$$

and

(2.11) $$\psi_1'(u) < \psi'(u)$$

i.e.

$$u_0 \leqq u_0' .$$

Using now the monotony of $\psi'(u)\, u^c$ we obtain

(2.12) $$\psi'(u_0')/\psi'(u_0) < (u_0/u_0')^c .$$

It follows from (2.10) and (2.12) that

(2.13) $$u_0/u_0' \to 1 .$$

Further

$$(2.14)\quad \ln(-\psi''(u)) - \ln(-\psi''(u_0)) = O\left(\frac{\psi'''(\bar{u})}{\psi''(\bar{u})}(u - u_0)\right) = O\left(\frac{u - u_0}{\bar{u}}\right),$$

$$|\bar{u} - u_0| < |u - u_0| .$$

It follows from (2.4), (2.5) and (2.14) that

$$(2.15)\qquad -xu + \psi_1(u) = -xu_0 + \psi(u_0)(u - u_0)^2 (1 + o(1)) + \\ + \tfrac{1}{2}\ln(-\psi''(u_0)) + o(1),$$

if

$$u_0 - u = o(u_0).$$

Choose $\omega(u)$ such that

$$\lim_{u\to 0} \omega(u) = 0, \quad \lim_{u\to 0} \psi\ (u)'' u^2\, \omega^2(u) = \infty,$$

$$u_0' \in (u_0(1 - \omega(u_0)), u_0(1 + \omega(u_0))),$$

$$(2.16)\quad \int_{u_0(1-\omega(u_0))}^{u_0(1+\omega(u_0))} e^{-xu+\psi_1(u)}\, du = e^{-xu_0+\psi_1(u_0)} \int_{-u_0\omega(u_0)}^{u_0\omega(u_0)} e^{\psi''(u_0)t^2/2}\, dt(1 + o(1)) =$$

$$= \sqrt{(2\pi)}\, e^{-xu_0+\psi(u_0)}(1 + o(1)).$$

If furthermore

$$-\psi''(u)\, u^2\, \omega^2(u) + \tfrac{1}{2}\ln(-\psi''(u)) \to \infty$$

for $u \to 0$ (it can be achieved according to (2.6) and (2.7)), then for $x \to \infty$

$$(2.17)\quad \int_0^{u_0(1-\omega(u_0))} e^{-xu+\psi_1(u)}\, du + \int_{u_0(1+\omega(u_0))} e^{-xu+\psi_1(u)}\, du = o(e^{-xu_0+\psi(u_0)})$$

because the derivative

$$\frac{d}{du}(xu - \psi_1(u)) \neq 0$$

for

$$u \in [0, u_0(1 - \omega(u_0))] \cup [u_0(1 + \omega(u_0)), a]$$

and consequently $xu - \psi_1(u)$ is monotonic in these segments.

From (2.16) and (2.17) we obtain the statement of the lemma.

Henceforth we shall suppose that

$$(2.18)\qquad \psi(u) = -\int_u^{-\chi'(0)} (-\chi')^{-1}(u)\, du + \chi(0), \quad 0 \leqq u \leqq -\chi'(0),$$

$$\psi(u) = \chi(0), \quad u > -\chi'(0).$$

LEMMA 2.2. *For* $x \to \infty$

$$e^{\chi(x)} = \frac{1}{\sqrt{2\pi}} \int_{y'}^{y''} e^{-xu+\psi_1(u)}\, du(1 + o(1))$$

*uniformly with respect to*

$$0 \leqq y' \leqq (1 - \omega(u_0))\, u_0\,, \quad 1 \geqq y'' \geqq (1 + \omega(u_0))\, u_0\,,$$

*where*

$$\psi_1(u) = \psi(u) + \tfrac{1}{2} \ln(-\psi''(u))\,,$$

$\omega(u)$ *and* $u_0$ *have the same sense as in lemma* 2.1.

Proof. Verify that $\psi(u)$ defined by (2.18) satisfies the conditions (2.1)–(
Obviously,

$$\psi'(u) = (-\chi')^{-1}(u)\,, \tag{2.19}$$

$$\psi''(u) = -1/\chi''(\psi'(u))\,. \tag{2.20}$$

Consequently for $x = \psi'(u)$

$$\psi''(u)\, u/\psi'(u) = \chi'(x)/\chi''(x)\, x\,. \tag{2.21}$$

Thus (III) ⇒ (2.2).
Further

$$\psi'''(u) = -\frac{\chi'''(\psi'(u))}{(\chi'')^3\,(\psi'(u))}\,. \tag{2.22}$$

So for $x = \psi'(u)$

$$\psi'''(u)\, u/\psi''(u) = -\chi'''(x)\, \chi'(x)/(\chi'')^2\,(x)\,. \tag{2.23}$$

Obviously,

$$\text{(IV)} \Rightarrow \chi'''(x)\, \chi'(x)/(\chi'')^2\,(x) < -L_1\, \chi'(x)/\chi''(x)\,. \tag{2.24}$$

According to the condition (III)

$$\chi''(x) \geqq 0\,. \tag{2.25}$$

It follows from (2.22), (2.25) and (IV) that

$$\psi'''(u) \geqq 0\,.$$

On the other hand we deduce from (2.22)–(2.24) and (III) that

$$\psi'''(u) < -C\,\psi''(u)/u\,.$$

Thus, the condition (2.3) also holds.

As for the condition (2.1) it follows from (I) and (2.19).

Apply now Lemma 2.1. According to (2.19)

$$-xu_0 + \psi(u_0) = x\,\chi'(x) + \psi(-\chi(x))\,. \tag{2.26}$$

Further in view of (2.18)

$$\psi(-\chi'(x)) = \int_{-\chi'(0)}^{-\chi'(x)} \psi'(u)\,\mathrm{d}u + \chi(0)\,.$$

Put

$$u = -\chi'(y)\,.$$

Obviously,

$$\mathrm{d}u = -\chi''(y)\,\mathrm{d}y\,.$$

Therefore

$$\psi(-\chi'(x)) = -\int_0^x y\,\chi''(y)\,\mathrm{d}y + \chi(0) = -\chi'(x)\,x + \int_0^x \chi(y)\,\mathrm{d}y + \chi(0) = \chi(x) - \chi'(x)\,x\,. \tag{2.27}$$

It follows from (2.26) and (2.27) that

$$-xu_0 + \psi(u_0) = \chi(x)\,. \tag{2.28}$$

The lemma is proved.

Put

$$\varphi(u) = \frac{1}{\sqrt{(2\pi)}}\,\mathrm{e}^{\psi_1(u)}\,.$$

LEMMA 2.3.

$$\varphi(u) = cu\,\varphi'(u)\,.$$

Proof. Clearly

$$\varphi'(u) = \psi_1'(u)\,\varphi(u)\,. \tag{2.29}$$

From (2.1), (2.2) and (2.10) we conclude that

$$\psi_1'(u) > c/u\,. \tag{2.30}$$

Combining (2.29) and (2.30) we obtain the statement of the lemma.

LEMMA 2.4.

$$-\chi(x) < -\chi(1)\, x^{\alpha}\,. \tag{2.31}$$

This estimate follows easily from the condition (II).

## 3. PROOF OF THE MAIN THEOREM

We shall prove only (1.1).

Put

$$\Omega(u, x) = -xu + \psi(u) + n\,K(u)\,.$$

Let

$$p(x) = \int_0^{\infty} u\,\varphi(u)\,\mathrm{e}^{-xa}\,\mathrm{d}u\,.$$

Put

$$\bar{F}_y(x) = \begin{cases} F(x), & x \leqq y\,, \\ F(y), & x > y\,, \end{cases} \quad Q_y(x) = \begin{cases} 0, & x \leqq y\,, \\ \int_y^x p(u)\,\mathrm{d}u\,, & x > y \end{cases}$$

$$F_y(x) = \bar{F}_y(x) + Q_y(x)\,.$$

If $Q(x)$ is a function of bounded variation then $Q^{(n)}(x)$ denotes its $n^{\text{th}}$ convolutio
We shall denote

$$\int_x^{\infty} \mathrm{d}Q(u) \quad \text{by} \quad \Delta Q(u)\,.$$

It is not hard to see that

$$F_y^{(n)}(x) = \bar{F}^{(n)}(x) + n\bar{F}_y^{(n-1)} * Q_y(x) + O(n^2\,Q_y^2(\infty))$$

and

$$F^{(n)}(x) = \bar{F}_y^{(n)}(x) + n\bar{F}_y^{(n-1)} * (F - \bar{F}_y)\,(x) + O(n^2(1 - F(y))^2)\,.$$

Since

$$\lim_{x\to\infty} (1 - F(x)) \Big/ \int_x^{\infty} p(u)\,\mathrm{d}u = 1$$

we have for $y \to \infty$

$$\Delta\left|(F - \bar{F}_y - Q_y) * \bar{F}_y^{(n-1)}(x)\right| \leqq \left|\Delta(F - \bar{F}_y - Q_y)\right| * \bar{F}_y^{(n-1)}(x) =$$
$$= o(\bar{F}^{(n-1)} * \Delta Q_y(x))$$

uniformly with respect to $x$.

Thus, for $y \to \infty$

$$F^{(n)}(x) = \bar{F}_y^{(n)}(x) + n\bar{F}_y^{(n-1)} * Q_y(x)\,(1 + o(1)) + O(n^2(1 - F(y))^2)\,.$$

The following inversion formula holds

$$\Delta\bar{F}_y^{(n-1)} * Q_y(x) = \frac{1}{2\pi i}\int_0^h e^{-xu} f_y^{n-1}(u)\,(g_y^+(u) - g_y^-(u))\,\frac{du}{u} + \tag{3.2}$$

$$+ \frac{1}{2\pi i}\lim_{T\to\infty}\int_{h-iT}^{h+iT} e^{-xz} f_y^{n-1}\, g_y(z)\,\frac{dz}{z}\,,$$

where

$$f_y(z) = \int_{-\infty}^{y} e^{zx}\, dF(x)\,,\quad g_y(z) = e^{zy}\int_0^{\infty}\frac{e^{-ty}t\,\varphi(t)}{t - z}\,dt\,,$$

$$g_y^{\pm}(u) = \lim_{\substack{z\to u\\ z\in D^{\pm}}} g(z)\,,$$

$D^{\pm}$ are accordingly upper and lower half-planes.

Since $u\,\varphi(u)$ satisfies the Holder condition

$$g_y^+(u) - g_y^-(u) = u\,\varphi(u) \tag{3.3}$$

(see, for example, [4], p. 37).

LEMMA 3.1.

$$|g_y(z)| < C\, e^{\mathrm{Re}\, zy}\min\left[p(y)/|\mathrm{Im}\, z|,\ \varphi'(2/y)\frac{\ln y}{y^2}\right].$$

Proof. Integrating by parts we obtain the inequality

$$\left|\int_0^{\infty}\frac{e^{(z-t)y}\,\varphi(t)\,t}{t - z}\,dt\right| \leqq \left|\int_0^{\infty}\ln|t - z|\,\frac{d}{dt}\left[\varphi(t)\,t\,e^{(z-t)y}\right]dt\right| +$$

$$+ \pi\int_0^{\infty}\left|\frac{d}{dt}\left[\varphi(t)\,t\,e^{(z-t)y}\right]\right|dt = I_1 + I_2\,.$$

Clearly

$$I_1 < e^{y\mathrm{Re}\, z}\int_0^{2/y}\left|(\varphi'(t)\,t + \varphi(t) - \varphi(t)\,ty)\ln|t - z|\right|dt +$$

$$+ \ln y\int_{2/y}^{\infty}\left|\frac{d}{dt}\left(\varphi(t)\,t\,e^{(z-t)y}\right)\right|dt = I_{11}\,e^{y\mathrm{Re}\, z} + I_{12}\,.$$

Using Lemma 2.3 we obtain the estimate

$$I_{11} < C\,\varphi'(2/y)\,\frac{\ln y}{y^2}\,.$$

Since for $t > 2/y$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,[\varphi(t)\,t\,\mathrm{e}^{-yt}] \leqq 0\,,$$

$$I_{12} < \varphi(2/y)\,\mathrm{e}^{y\operatorname{Re}z}\,\frac{\ln y}{y} < C\varphi'(2/y)\,\mathrm{e}^{y\operatorname{Re}z}\,\frac{\ln y}{y^2}\,.$$

It follows from two last estimates that

$$|g_y(z)| < C\,\mathrm{e}^{y\operatorname{Re}z}\,\varphi'(2/y)\,\frac{\ln y}{y^2}\,.$$

On the other hand

$$|g_y(z)| < \frac{\mathrm{e}^{y\operatorname{Re}z}}{|\operatorname{Im} z|}\int_0^\infty \mathrm{e}^{-ty}t\,\varphi(t)\,\mathrm{d}t = \frac{\mathrm{e}^{y\operatorname{Re}z}}{|\operatorname{Im} z|}\,p(y)\,.$$

The lemma is proved.

LEMMA 3.2. *For $y > 1/h$*

$$\left|\int_{1/h}^{y} \mathrm{e}^{zx}\,\mathrm{d}F(x)\right| < C\left\{\mathrm{e}^{y\operatorname{Re}z+\chi(y)} + h^{1/(\alpha-1)}\exp\left[\frac{1-\alpha}{1+\alpha}\,\chi(1/h)\right]\right\}.$$

To prove the lemma it is sufficient to use the estimates (4.3), (4.7), (4.9) an (4.10) of the paper [2].

Let $u_1$ be the least root of the equation

(3.4) $$x = \psi'(u) + n\,K'(u)\,,$$

$u_2$ be the root of the equation

(3.5) $$\psi''(u) + n\,K''(u) = 0\,.$$

It is not hard to see that $u_0 < u_1 \leqq u_2$.

LEMMA 3.3. *If $nu_2/x \to 0$ then $u_0/u_1 \to 1$.*

Proof. Clearly

$$nu_2/x \to 0 \Rightarrow nu_1/x \to 0\,.$$

Further $\psi'(u_1) = xa$, where

$$a = 1 - \frac{n\,K'(u_1)}{x}.$$

It is easy to see that $K'(u_1)/u_1 \to 1$, but it means that $a \to 1$. Obviously

$$u_0 = \psi'^{-1}(x) = -\chi'(x)\,, \quad u_1 = \psi'^{-1}(xa) = -\chi'(xa)\,.$$

Using now the condition (III) we obtain

$$\ln \frac{u_0}{u_1} = \ln(-\chi'(x)) - \ln(-\chi'(xa)) = O\left(\frac{\chi''(x)}{\chi'(x)}(a-1)\right) = o(1)\,.$$

The lemma is proved.

LEMMA 3.4.

$$x/n - u_1 > cu_1\,.$$

Proof. Clearly

$$x > \psi'(u_1) + nu_1\,. \tag{3.6}$$

On the other hand

$$-\psi''(u_2) > n\,.$$

Hence according to (2.21) and to condition (III)

$$\psi'(u_2)/u_2 > cn\,. \tag{3.6'}$$

Since $\psi'(u_1) > \psi'(u_2)$ and $u_1 \leqq u_2$

$$\psi'(u_1) > cnu_1\,. \tag{3.7}$$

The estimates (3.6) and (3.7) yield the statement of the lemma.

Let $u_1'$ be that of roots of the equation $x = \psi_1'(u) + n\,K'(u)$ lying on the left of $u_2$ which is the most remote from $u_2$.

LEMMA 3.5. *If $n\,\chi''(\psi'(u_1)) < \delta < 1$ and $n$ is sufficiently large, then*

$$0 < u_1' - u_1 < C/(1 - \delta/2)\,\psi'(u_1)\,.$$

Proof. Obviously,

$$\psi'(u_1) = x - n\,K'(u_1)\,, \quad \psi_1'(u_1') = x - n\,K'(u_1')\,.$$

Hence according to (2.9)

$$(3.8)\qquad (u_1' - u_1)\,\psi''(\bar{u}) + \tfrac{1}{2}\psi'''(u_1')/\psi''(u_1') = n(K'(u_1) - K'(u_1')),$$
$$|u_1 - \bar{u}| < |u_1' - u_1|.$$

Notice that by virtue of (2.11)

$$(3.9)\qquad u_1 < u_1'.$$

Therefore $\forall \varepsilon > 0$

$$(3.10)\qquad K'(u_1) - K'(u_1') > (u_1 - u_1')(1 + \varepsilon),$$

if $n$ is sufficiently large and

$$(3.11)\qquad -\psi''(\bar{u}) < -\psi''(u_1).$$

It follows from (3.8), (3.10) and (3.11) that

$$(3.12)\qquad u_1 - u_1' > \psi'''(u_1')/2(\psi''(u_1) + n(1 + \varepsilon))\,\psi''(u_1').$$

Observe that either $\psi'(u_1) > x/2$ or $n\,K'(u_1) > x/2$. If second of these inequalities holds, then according to (3.7)

$$\psi'(u_1) > cx.$$

Thus in each case

$$(3.12')\qquad \psi'(u_1) > cx.$$

On the other hand for sufficiently large $n$

$$\psi'(u_1) < \psi_1(u_1)(1 + \varepsilon).$$

Hence according to (2.8)

$$(3.13)\qquad (u_1'/u_1)^c < \psi'(u_1)/\psi'(u_1') < C.$$

We obtain from (3.12) and (3.13) that

$$(3.14)\qquad u_1' - u_1 < -C/(1 - \delta/2)\,\psi''(u_1)\,u_1.$$

The statement of the lemma follows readily from (3.14), (2.2) and (3.9).

LEMMA 3.6. *For $n \to \infty$*

$$\Omega(u_1, x) = \chi((1 - \beta)\,x) - \frac{(\beta x)^2}{2n} + \frac{(\beta x)^3}{n^2}\,\lambda_\alpha\left(\frac{\beta x}{n}\right) + o(1),$$

*where $\beta$ is the least positive root of the equation* (1.0).

Proof. Let $\beta$ be a solution of the equation

$$u_1 = -\chi'((1-\beta)x). \tag{3.15}$$

The derivative of the function $(\beta - 1)xu + \psi(u)$ at the point $u_1$ is equal to zero. Therefore $\beta x - nK'(u_1) = 0$ because the derivative of the function $-xu + \psi(u) + nK(u)$ also is equal to zero at the point $u_1$.

Hence

$$u_1 = (K')^{-1}\left(\frac{\beta x}{n}\right). \tag{3.16}$$

Thus, $\beta$ satisfies the equation (1.0).

It is easy to see that $\beta$ is the least of two positive roots of the equation (1.0). In fact otherwise $\exists \beta' < \beta$ such that $x = \psi'(\bar{u}_1) + nK'(\bar{u}_1)$, where $\bar{u}_1 = -\chi'((1-\beta')x) < u_1$ but it is impossible.

By the condition (II) and Lemma 2.4

$$-\chi'(x) < -\alpha\,\chi(1)\,x^{\alpha-1}. \tag{3.17}$$

On the other hand by virtue of (3.12′).

$$u_1 < -\chi'(cx).$$

Therefore if $x > cn^{1/(2-\alpha)}$, then

$$u_1 < Cn^{(\alpha-1)/(2-\alpha)} \tag{3.18}$$

Since $u_1 n/\beta x \to 1$ for $n \to \infty$, $\beta x/n < Cn^{(\alpha-1)/(2-\alpha)}$. If $x < Cn^{1/(2-\alpha)}$, then $\beta x/n < x/n < Cn^{(\alpha-1)/(2-\alpha)}$.

It follows from two last estimates that

$$\lim_{n\to\infty} n(\beta x/n)^{N(\alpha)} = 0.$$

Consequently for $n \to \infty$

$$\begin{aligned} -\beta\,x(K')^{-1}(\beta x/n) + n\,K((K')^{-1}(\beta x/n)) &= \\ = -\beta^2x^2/2n + \beta^3x^3\,\lambda_\alpha(\beta x/n)/n^2 + o(1). \end{aligned} \tag{3.19}$$

Finally by (2.28)

$$(\beta - 1)\,x\,\chi'((1-\beta)x) + \psi(-\chi'((1-\beta)x)) = \chi((1-\beta)x).$$

The lemma is proved.

Return now to the formula (3.2). Put $h = h(b) \equiv -\chi(bx)/b^{1-\eta}x$, where $b = 2^{-1/\alpha-1}$, $y = bx$. Here $\eta$ is a sufficiently small positive number which will be chosen later.

Evidently

$$f_y(z) = f_{1/h(b)}(z) + \int_{1/h(b)}^{y} e^{zx}\, dF(x)\,. \tag{3.20}$$

Further

$$f_{1/h}(z) = 1 + h^2/2 + \sum_{k=3}^{N(\alpha)} M\xi_k h^k + o(h^{N(\alpha)})\,. \tag{3.21}$$

Hence it follows in particular that

$$|f_{1/h}^n(z)| < e^{n(1+\varepsilon)h^2/2} \tag{3.22}$$

if $n$ is sufficiently large.

Suppose now that

$$x > cn^{1/(2-\alpha)}\,. \tag{3.23}$$

Then by Lemma 2.4

$$h(b) < Cn^{(\alpha-1)/(\alpha-2)}\,. \tag{3.24}$$

It follows from (3.21) and (3.24) that if the (3.23) holds then

$$f_{1/h}^n(u) = e^{nK(u)}(1 + o(1)) \tag{3.25}$$

uniformly with respect to $0 \leqq u \leqq h(b)$.

It follows from Lemma 3.2 according to (3.24) that

$$\int_{1/h(b)}^{y} e^{h(b)x}\, dF(x) = O(e^{(1-b^{-\eta})\chi(y)}) + o(n^{-M})\,. \tag{3.26}$$

Taking into account (3.24) we obtain

$$f_y^n(u) = f_{1/h(b)}^n(u)\,(1 + o(1)) = e^{nK(u)}(1 + o(1))\,, \quad 0 \leqq u \leqq h(b)\,. \tag{3.27}$$

By (3.3) and (3.27)

$$\int_0^{h(b)} e^{-xu} f_y^n(u)\,(g^+(u) - g^-(u))\,\frac{du}{u} = \int_0^{h(b)} e^{-xu+nK(u)}\,\varphi(u)\,du(1 + o(1)) \tag{3.28}$$

According to the condition (II)

$$u_0 = -\chi'(x) < -\alpha\,\chi(x)/x\,. \tag{3.29}$$

Hence

$$-\chi(x)/x - u_0 > (\alpha - 1)\,\chi(x)/x > \frac{1-\alpha}{\alpha}\,u_0\,. \tag{3.30}$$

Let $\gamma$ is defined by the equation $h(b) = \gamma x/n$. Then

$$-x\,h(b) + n\,h^2(b)(1+\varepsilon)/2 = -x\,h(b)\,(1 - n\,h(b)(1+\varepsilon)/2x) = \tag{3.31}$$
$$= (1 - \gamma(1+\varepsilon)/2)\,\chi(bx)/b^{1-\eta}\,.$$

Observe that if $x > vn^{1/(2-\alpha)}$ then by (2.31)

$$\gamma < \chi(1)\, b^{\alpha+\eta-1} v^{\alpha-2}\,. \tag{3.32}$$

It is not hard to obtain from the condition (II)

$$\chi(bx) < b^{\alpha}\,\chi(x)\,. \tag{3.33}$$

It follows from (3.32) and (3.33) that for $\forall\varepsilon > 0$, $\eta > 0$ $\exists v(\varepsilon, \eta)$ such that for $x > v(\varepsilon, \eta)\, n^{1/(2-\alpha)}$

$$(1 - \gamma(1+\varepsilon)/2)\,\chi(bx)/b^{1-\eta} < b^{(\alpha-1)/2+\eta}\,\chi(x) \tag{3.34}$$

if $n$ is sufficiently large.

By (2.20)

$$\chi''(\psi'(u_2)) = 1/n\,.$$

According to the conditions (II) and (III) and Lemma 2.4

$$\chi''(x) < Cx^{\alpha-2}\,. \tag{3.35}$$

Therefore

$$(\psi')^{2-\alpha}(u_2) < Cn\,,$$

i.e.

$$\psi'(u_2) < Cn^{1/(2-\alpha)}\,.$$

Using (3.6′) we obtain

$$nu_2 < Cn^{1/(2-\alpha)}\,. \tag{3.36}$$

From (3.30) and (3.36) applying Lemma 3.3 we deduce that $\forall n > 0$

$$h(b) - u_1 > \frac{1-\alpha}{2\alpha}\, u_1 \tag{3.37}$$

if $x > Cn^{1/(2-\alpha)}$ and $C$ is sufficiently large.

By virtue of (3.35) and (3.12′)

$$\chi''(\psi'(u_1)) = O((\psi')^{\alpha-2}(u_1)) = O(x^{\alpha-2})\,.$$

Therefore $\exists C$ such that for $x > Cn^{1/(2-\alpha)}$

$$-\psi''(u_1) > 2n\,. \tag{3.38}$$

Repeating the reasoning leading to Lemma 2.1 and taking into account (3.38) we come to the asymptotic representation

$$\int_0^{u_1(1+\omega(u_1))} e^{-xu+nK(u)}\,\varphi(u)\,du = \frac{e^{-xu_1+nK(u_1)+\psi_1(u_1)}}{\sqrt{(-\psi''(u_1)-n)}}\,(1+o(1)) \tag{3.39}$$

which holds for $x > Cn^{1/(2-\alpha)}$, $n \to \infty$. In addition

$$e^{-xu_3+nK(u_3)+\psi_1(u_3)} = o\left(\frac{e^{-xu_1+nK(u_1)+\psi_1(u_1)}}{\sqrt{(-\psi''(u_1)-n)}}\right), \tag{3.40}$$

where $u_3 = u_1(1+\omega(u_1))$.

It is easy to see that

$$\int_{u_1(1+\omega(u_1))}^{h(b)} e^{-xu+nK(u)}\,\varphi(u)\,du = O(h(b)\max[\exp\{-xu_3 + n\,K(u_3) + \psi_1(u_3)\}, \exp\{-x\,h(b) + n\,K(h(b)) + \psi_1(h(b))\}]) . \tag{3.41}$$

Put now $\eta = (1-\alpha)/4$. It follows from (3.31) and (3.34) that for $x > Cn^{1/(2-\alpha)}$

$$e^{-xh(b)+nK(h(b))} = o(e^{\chi(x)}) . \tag{3.42}$$

Note that

$$\psi''(u_1) = -1/\chi''(\psi'(u_1)) = -1/\chi''((1-\beta)\,x) \tag{3.43}$$

(see (3.15)).

Further

$$\int_0^{h(b)} e^{-xu+nK(u)}\,\varphi(u)\,du > \int_0^{h(b)} e^{-xu}\,\varphi(u)\,du > c\,e^{\chi(x)} . \tag{3.44}$$

It follows from (3.39)–(3.44) and Lemma 3.6 that there exists $C$ such that for $x > Cn^{1/(2-\alpha)}$

$$\int_0^{h(b)} e^{-xu+nK(u)}\,\varphi(u)\,du = P_1(x)\,(1+o(1)) . \tag{3.45}$$

Estimate now

$$J \equiv \int_{h(b)-i\infty}^{h(b)+i\infty} e^{-xz} f_y^n(z)\,g_y(z)\,dz .$$

It is not hard to see that

$$g_y(z) = g_{1/h(b)}(z) + \int_{1/h(b)}^{y} e^{zx}\,p(x)\,dx . \tag{3.46}$$

It follows from (3.46), (3.26) and Lemma 3.1 that

(3.47) $$g_y(z) = o(1)$$

uniformly with respect to $z$ with $0 \leqq \operatorname{Re} z \leqq h(b)$.

Note that by virtue of the condition (I)

(3.48) $$\lim_{x\to\infty} -\chi(x)/\ln x = \infty$$

It follows from (3.22), (3.47), (3.48), (3.31) and (3.34) that

(3.49) $$\int_{h(b)-i}^{h(b)+i} e^{-zx} f_y^n(z)\, g_y(z) \frac{dz}{z} = o\left(-\ln h(b) \exp\left\{-x\, h(b) + \frac{1+\varepsilon}{2} n\, h^2(b)\right\}\right) =$$

$$= o(\exp[C \ln x + b^{(\alpha-1)/4} \chi(x)]) = o(e^{\chi(x)}),$$

$$x > Cn^{1/(2-\alpha)}.$$

Further by Lemma 3.1

(3.50) $$\int_{\substack{\operatorname{Re} z = h(b)\\ |\operatorname{Im} z| > 1}} e^{-xz} f_y^n(z)\, g_y(z) \frac{dz}{z} = O(p(y) \exp\left[(y - x)\, h(b) + n \frac{1+\varepsilon}{2} h^2(b)\right] =$$

$$= o(\exp\{\chi(y) + (y - x)\, h(b) + n(1 + \varepsilon)\, h^2(b)/2\}).$$

It is not hard to see that

(3.51) $$\chi(y) + (y - z)\, h(b) + n(1 + \varepsilon)\, h^2(b)/2 =$$

$$= (1 + (1 - b)\, b^{\eta-1})\,(1 - \gamma'(1 + \varepsilon)/2)\, \chi(bx),$$

$$\gamma' = \gamma/(1 + b^{1-\eta} - b).$$

From (3.50) and (3.51) we obtain using (3.34) that for $x > Cn^{1/(2-\alpha)}$

(3.52) $$\int_{\substack{\operatorname{Re} z = h(b)\\ |\operatorname{Im} z| > 1}} e^{-xzi} f_y^n(z)\, g_y(z) \frac{dz}{z} = o(e^{\chi(x)}).$$

It follows from (3.49) and (3.52) that

(3.53) $$J = o(e^{\chi(x)}), \quad x > Cn^{1/(2-\alpha)}.$$

From (3.2), (3.28), (3.45) and the estimate (3.53) we obtain the asymptotic representation

(3.54) $$\Delta \bar{F}_y^{(n-1)} * Q_y(x) = P_1(x)\,(1 + o(1))$$

valid for $x > Cn^{1/(2-\alpha)}$.

By virtue of (3.27), (3.31), (3.34) and (3.22)

$$\Delta \bar{F}_y^{(n)}(x) = O(\exp\{-x\,h(b) + n(1+\varepsilon)\,h^2(b)/2\}) = o(e^{\chi(x)}), \quad x < Cn^{1/(2-\alpha)}. \tag{3.55}$$

After all

$$n^2(1 - F(y))^2 = O(\exp\{2\ln n + 2\chi(y)\}) = o(e^{\chi(x)}), \quad x > Cn^{\varepsilon}. \tag{3.56}$$

From (3.1) and (3.54)–(3.56) we obtain (1.1) for $x > Cn^{1/(2-\alpha)}$.

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