1. Markov chains

1.1. Introduction. In my paper [1] published in 1957 (see references at the end of this essay) *the spectral theory for linear operators in Banach spaces* was first applied to an asymptotic analysis of Markov chains. This approach is applied successfully till now.

The paper [2] was a continuation of [1]. Edgeworth expansions and Cramer-type theorems an large deviations are proved in [2]. In [3] the spectral method was first extended to Harris Markov Chains. In the paper [4] published in 1965 ergodic theorems were first proved for Harris Markov chains full volume.

In [5] – [7] the analytical approach to Harris Chains alternative to the Ney – Athreya – Nummelin splitting method has been developed. There is no anything similar in the literature worldwide.

1.2. Spectral method. The beginning of my research work is connected with Markov chains. By tradition researches in Markov Chains held a honorable place in Tashkent University. It was T.A. Sarymsakov who attracted my attention to Markov chains. He was the supervisor of my graduation thesis and then of my candidate thesis. To the middle of 1950s the theory of finite Markov chains was primarily completed. S.Kh. Sirazhdinov [8] succeeded even to get asymptotic expansions in a multi-dimensional CLT.

At the same time the asymptotic analysis of Markov chains with a general and even countable state space was restrained because of lack of the adequate analytical approach.

At that time I was interested very much in this problem. In spring of 1956, while reading up for my candidate examination and becoming acquainted with the spectral theory of Banach Algebras in this connection, it occurred to me that the latter can serve as the basis of the required analytical method. I realized this idea in paper [1]. It should be noticed almost none of specialists in probability, at least in Soviet Union, were not familiar with the spectral theory of Banach Algebras, so my paper turned out unexpected. The analytical approach suggested by me was used in [1] for proving CLT for sums of random variables defined on Markov chains under minimal moment assumptions. The research which I started in [1] was continued in [2], where asymptotic expansions in CLT and Cramer-type theorems on large deviations are deduced. The above-mentioned analytical approach is applicable directly only to chains satisfying the condition

$$\lim_{n\to\infty} \sup_{x\in X} \sup_{A\in S} |p^{(n)}(x) - p(A)| = 0. \qquad (1)$$

Here X is a state space, S is σ-algebra on which the transition probability $p^{(n)}(x, A)$ and limit distribution p are defined. Notice that the rate of convergence in (1) is necessarily exponential. From the standpoint of conception of a weak dependence relation (1) means that the uniform mixing coefficient goes to 0 as $n \to \infty$.

1.3. Extension of the spectral method to Harris Markov chains. The analytical method used in [1] was extended in my paper [3] to chains satisfying the conditions:

(A) *There exists the set $A_0 \in S$, nonnegative measure ϕ with $\phi(A_0) > 0$ and the positive integer $n_0 \geq 1$ such that*

$$p_0^{(n_0)}(x, y) > \delta > 0$$

for $x \in A_0$, $y \in A_0$, where $p_0^{(k)}(x, \cdot)$ is the density of the absolutely continuous with respect to ϕ component of $p^{(k)}(x, \cdot)$.

(B) *For any* $x \in X - A_0$, $\mathbf{P}(X_n \in A_0 \text{ for at least one } n > 0 \,|\, X_0 = x) = 1$ *where* $\{X_n\}$, $n = 0, 1, ...,$ *is the Markov chain with the state space* X.

The possibility of such extension was not a priori evident. The special construction was worked out, which allowed to reduce this case to that considered in [1] and [2].

1.4. Local limit theorems. Local limit theorems for countable Markov chains have been the subject of my paper [9] and [10]. In [9] the local limit theorem is proved for chains satisfying so called Markov condition: $p_{j1} > \lambda > 0$ for all j. This condition implies an uniform ergodicity in the sense of (1). The general case is considered in [10]. The positive recurrent non-periodic Markov chains with a countable state space e_j, $j = 1, 2, ..., \infty$ is the subject of a study in [10]. Denote transition probabilities by p_{ij}. Let q_n be the probability that the return time to e_1 equals n.

Put

$$\mu_1 = \sum_{n=1}^{\infty} n q_n, \quad \mu_2 = \sum_{n=1}^{\infty} n^2 q_n.$$

Let ω_n be a sequence of states $e_1, e_{i_1}, ..., e_{i_n}$ such that $i_k \neq 1$, $k = 1, 2, ..., n$. If $i_k \neq 1$ for $k = 1, 2, ..., n-1$ and $i_n = 1$, then we denote this sequence by $\bar{\omega}_n$. Let $\mathbf{P}(\omega_n)$ and $\mathbf{P}(\bar{\omega}_n)$ be respectively the probabilities of ω_n and $\bar{\omega}_n$.

Consider the real function $f(e_j) = a + k(e_j)h$, where $k(e_j)$ takes integer values, a and h are real numbers. Define random variable $\bar{f}(\bar{\omega}_n)$ letting

$$\bar{f}(\bar{\omega}_n) = \sum_{k=1}^{n} f(e_{i_k}).$$

Let Z be the of two–dimensional vectors with integer coordinates representable as a linear combination of vectors of the type $\left(\frac{\bar{f}(\bar{\omega}_n) - na}{h},\ n\right)$ with integer coefficients under condition $\mathbf{P}(\bar{\omega}_n) > 0$.

Consider the sum

$$S_n = \sum_{k=1}^{n} f(E_k) - na,$$

where E_k is the state of the chain in the moment k.

Put

$$z_{nk} = \frac{an + kh - A_n}{\sigma\sqrt{n}},$$

where $A_n = \frac{1}{\mu_1} n \mathbf{E}\bar{f}$, $\sigma^2 = \frac{1}{\mu_1}\mathbf{E}\bar{f}_1^2$,

$$\bar{f}_1(\bar{\omega}_n) = \bar{f}(\bar{\omega}_n) - \frac{n}{\mu_1}\mathbf{E}\bar{f}.$$

With this notation the local limit theorem is formulated as follows: *If* Z *coincides with the set of all two–dimensional vectors with integer coefficients,* $\mu_2 < \infty$ *and* $0 < \sigma < \infty$, *then uniformly with respect to* k

$$\frac{\sigma\sqrt{n}}{h}\mathbf{P}(S_n = kh) - \frac{1}{\sqrt{2\pi}} e^{-z_{nk}^2} \to 0.$$

Previously local limit theorems have been proving only for finite Markov chains.

1.5. Ergodic theorems. Side by side with limit theorem for sums of random variables defined on Markov chains I had been engaged in studying ergodic properties of Markov chains satisfying conditions **(A)** and **(B)**. Notice that these conditions are much more effective than the recurrence condition by Harris.

In [4] I proved that *under conditions* **(A)** *and* **(B)** *there exists the finite or* σ *- finite invariant measure* q *for the chain* $\{X_n\},\ n = 0, 1, \ldots$.

Moreover

$$\lim_{n\to\infty} p^{(n)}(x, B) = q(B), \tag{2}$$

if $q(X) < \infty$, *and*

$$\lim_{n\to\infty} p^{(n)}(x, B) = 0, \tag{3}$$

if $q(X) = \infty$, *but* $q(B) < \infty$.

Researches in Markov chains performed in 1957 – 1962 have been the subject of my candidate [11] and doctorate [12] theses.

In papers by Athreya and Ney [13] and by Nummelin [14] published almost simultaneously in 1978 the method of proving ergodic theorems was suggested, which is based on the regenerating extension of Markov chains. Along with condition **(A)** they use slightly modified condition **(B)**:

There exist $A_0 \in S$, *non-negative measure* $\phi(A_0) > 0$ *defined on* $A_0 S$ *with* $\phi(A) > 0$, *and* n_0 *such that for any* $x \in A_0$ *and* $B \in A_0 S$

$$p^{(n_0)}(x, B) \geq \phi(B).$$

In the framework of the suggested approach authors of [13] anb [14] prove the existence of an invariant measure and statements (2) and (3) i.e. the same results as in my paper [4]. There are no references to my paper [2] in [13] and [14].

1.6. The alternative for method by Athreya–Ney–Nummelin. In 1980 I returned to Markov chains in connection with papers [13] and [14]. In [5] I suggested the new analytical method which alternate with the pure probabilistic approach by Athreya–Ney–Nummelin.

This method is based on the representation of the transition function $p^{(n_0)}(\cdot\,,\,\cdot)$ as the sum of two substochastic transition functions

$$p^{(n_0)}(\cdot\,,\,\cdot) = \omega(\cdot\,,\,\cdot) + \phi(\cdot\,,\,\cdot),$$

where $\omega(\cdot\,,\,\cdot) = p^{(n_0)}(\cdot\,,\,\cdot) - \phi(\cdot\,,\,\cdot)$. Denote by P, Ω, Φ Markov operators defined relatively by the transition functions $p^{(n_0)}(\cdot\,,\,\cdot),\ \omega(\cdot\,,\,\cdot),\ \phi(\cdot\,,\,\cdot)$.

Suppose for simplicity that $n_0 = 1$. It easily seen that

$$(a+b)^n = a^n + \sum_1^n a^{n-k} b (a+b)^{k-1},\ n \geq 1,$$

where a and b are elements of any ring.

Letting $a = \Omega,\ b = \Phi$, we obtain

$$P^n = \Omega^n + \sum_1^n \Omega^{n-k} \Phi P^{k-1}. \tag{4}$$

Since for any k, j

$$\int_X \omega^{(k)}(x, dy) \int_X p^{(j)}(z, B)\, \phi(y, dz) = \omega^{(k)}(x, A_0) \int_{A_0} p^{(j)}(z, B)\, \phi(dz),$$

it follows from (4) that

$$p^{(n)}(x, B) = \omega^{(n)}(x, B) + \sum_1^n \omega^{(n-k)}(x, A_0)\, p_{k-1}(B), \tag{5}$$

where

$$p_k(B) = \int_{A_0} p^{(k)}(z, B)\, \phi(dz), \quad k \geq 1, \quad p_0(B) = \phi(A_0 B).$$

Integrating both sides of (5) with respect to measure ϕ we have

$$p_n(B) = \omega_n(B) + \sum_1^n \omega_{n-k}(A_0)\, p_{k-1}(B),$$

where $\omega_k(B) = \int_B \omega^{(k)}(x, B)\, \phi(dx)$. Thus, the recurrent formula $p_n(B)$ is valid, which similar to that for local renewal probabilities. Hence, under condition $\sum_0^\infty \omega_k(B) < \infty$

$$\lim_{n\to\infty} p_n(B) = \frac{1}{\mu} \sum_0^\infty \omega_k(B), \tag{6}$$

where $\mu = \sum_0^\infty (k+1)\, \omega_k(A_0)$, and $1/\mu = 0$ if $\mu = \infty$.

It is proved in [5] that $q(B) := \sum_0^\infty \omega_k(B)$ is an invariant measure. As to μ, it is connected with q by the formula

$$\mu = q(X)/\phi(A_0).$$

As a result we obtain that

$$\lim_{n\to\infty} p_n(B) = \frac{q(B)\, \phi(A_0)}{q(X)},$$

if $q(x) < \infty$, and

$$\lim_{n\to\infty} p_n(B) = 0,$$

if $q(x) = \infty, \ \ q(B) < \infty$.

Returning now to (5) and taking into account that

$$\sum_o^\infty \omega^{(k)}(x, A_0) = \frac{1}{\phi(A_0)},$$

we get assertions (2) and (3). In [5] non-periodic chains are considered. In [6] the approach offered in [5] is extended to periodic chains.

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