4. Boundary problems

4.1. Introduction. One of the problems going back to A.N. Kolmogorov is estimating the rate of convergence in so called boundary problems for sums of independent random variables. More precisely let $S_n = \sum_{i=1}^{n} X_i$, where X_i are i.i.d. with the distribution function $F(x)$, $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$, $\beta_3 = \mathbf{E}|X_1|^3$, $g_2(x) < g_1(x)$ are functions defined on $[0,1]$.

As early as 1930s Kolmogorov and Petrovsky proved that the probability

$$P_n = \mathbf{P}\Big(g_2(k/n) < S_k/n^{1/2} < g_1(k/n), \ k = 1,...,n\Big)$$

$n \to \infty$ converges to $v_0(0,0)$ as $n \to \infty$, where $v_0(t,x)$ is the solution of the equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\frac{\partial^2 v}{\partial x^2} = 0$$

in the domain

$$\Omega = \Big\{t,x : 0 \le t \le 1, \ g_2(t) < x < g_1(t)\Big\},$$

satisfying the boundary conditions

$$v_0(1,x) = 1, \ v_0(t, g_i(t)) = 0, \ 0 \le t \le 1, \ i = 1,2.$$

(In reality Kolmogorov and Petrovsky considered more general case of non–identically distributed random variables).

On the other hand,

$$v_0(0,0) := Q_0 = \mathbf{P}\Big(g_2(t) < W(t) < g_1(t), \ 0 \le t \le 1\Big),$$

where $W(t)$ is the standard Winer process with $W(0) = 0$.

Naturally, this raises the question of the estimating the rate of convergence P_n to Q_0.

Beginning from the second half of 1940s this problem attracted the attention of many mathematicians. Without question it stimulated the development of the general theory of random processes and promoted the appearance of new methods both analytical and probabilistic (one may mention, for example, the method of one probability space.)

The conjecture, from the very beginning, was that the error

$$|P_n - Q_0| = O(n^{-1/2}). \tag{1}$$

under the condition $\beta_3 < \infty$.

In the end this conjecture turned out valid, however, proving required large efforts.

The best bounds to the beginning of 1960s were obtained by Yu. V. Prochorov [1] and Skorochod [2]. Prochorov's bound was obtained in the more general case of non–identically distributed summands. It follows, in particular, from the latter that

$$P_n - Q_0 = O(\ln^2 n/n^{1/8}),$$

if $\beta_3 < \infty$.

A.V. Skorochod brought the bound for $P_n - Q_0$ to $O(\ln n/n^{1/2})$ under the additional assumption that X_i are bounded.

Prochorov, as well as Skorochod, used the direct probabilistic methods especially elaborated by them, which fall in the category of methods of one probability space.

The particular case of the boundary problem formulated above is estimating the of convergence for the distribution $\bar{F}_n(x) = \mathbf{P}(\bar{S}_n < x)$ of the maxima of cumulative sums $\bar{S}_n = \max\limits_{1 \le k \le n} S_k$. 1962 . .. Borovkov [3] obtained the asymptotic expansion for the distribution $\bar{F}_n(xn^{1/2})$ in powers of $n^{-1/2}$ including large deviations. In so doing he supposed that the Cramer condition

$$\int_{-\infty}^{\infty} e^{hx}\,dF(x) < \infty, \quad 0 < h < h_0,$$

is fulfilled, and $F(x)$ has an absolutely continuous component. Borovkov's approach is pure analytical. It is based on the modified Wiener–Hopf method with a subsequent application of the saddle – point method.

The deciding step in proving conjecture (1) was done in my papers performed in the end of 1960 s.

In papers [4,5] the bound $O(1/\sqrt{n})$ for the rate of convergence of the distribution $\bar{F}(x)$ is deduced for the first time under the minimal condition $\beta_3 < \infty$.

The classical Kolmogorov–Petrovsky problem is considered in paper [6]. In the latter the bound

$$|P_n - Q_0| = O(n^{-1/2})$$

is deduced under condition $\beta_3 < \infty$, and the boundary functions g_1 and g_2 satisfy the Lipschitz condition with some constant K, dependence on β_3 and g_1, g_2 being written in explicit form. Thereby the initial conjecture (1) was proved, and even with some excess. It should be noticed also that the methods which were elaborated in [4,6] differ radically from those used in [1]– [3].

After the publication of my paper [6] attempts were being making to obtain the bound $O(L_n)$ for non–identically distributed random variables, where L_n is the Lyapunov ratio. However, shortly the counterexamples were constructed which show that one can not obtain the bound sharper than $O(L_n^{1/3})$. One example of this type which belongs to me was published in Borovkov's paper [7], another one is contained in paper [8] by T.V. Arak and V.B. Nevsorov.

4.2. The distribution of the maximum of cumulative sums. I started my research on boundary problem in 1965 once. I have moved from Tashkent to Novosibirsk by the invitation of Borovkov. As many young people, I was undertaking readily and with gusto to solve difficult problem.

First, I have looked to the distribution of the maximum of cumulative sums. In my paper [4] the absolutely new approach to studying this distribution. It is based on the representation

$$\phi_n(t) = f^n(t) + \sum_{k=0}^{n-1} f^k(t)\bar{\phi}_{n-k}(t). \tag{2}$$

Here $f(t) = \int_{-\infty}^{\infty} e^{itx}\,dF(x)$, $\phi_n(t)$ is the characteristic function of $\max\{0, \bar{S}_n\}$, $\bar{\phi}_n(t) = \mathbf{P}(\bar{S}_n < 0) - \int_{-\infty}^{0} e^{itx}d\bar{F}_n(x)$.

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My first purpose was to obtain the Berry–Esseen bound with the help of representation (2). It was done in [4,5]. The bound looks as follows:

$$\left| \bar{F}_n(xn^{1/2}) - (2/\pi)^{1/2} \int_0^x e^{-u^2/2} du \right| < c \frac{\beta_3^2}{n^{1/2}}, \tag{3}$$

where c is an absolute constant. The only thing which somewhat spoils the impression from this bound is that it contains β_3^2 instead of β_3 in the classical Berry–Esseen bound.

However, it should be noticed that historically bound (3) was the first in which the dependence on moments was taken into consideration in the explicit form. I recollect the first incentive to clarify this question was to apply the asymptotic expansion for $\bar{F}_n(xn^{1/2})$ by Borovkov in [3]. But it turned out impossible or, at least, very difficult since coefficients, even in the first term, are not calculated explicitly, so that their association with moment is absolutely unclear.

However, .. Aleskeviciene [9] succeeded to replace β_3^2 with β_3, by sharpening lemma 3 in my paper [4].

4.3. The problem of Kolmogorov–Petrovsky. Once I have obtained bound (3), I turned to the boundary problem with curvilinear bounds. In this case I had also to invent a new approach. Now I proceed to describing this approach (it was suggested in my paper [6]).

Let

$$P_n(t,x) = \mathbf{P}\Big(g_2(t) < S_k/n^{1/2} < g_1(t), \ t + \frac{k}{n} \le 1 \Big).$$

Obviously, $P_n(0;0) = P_n$.

Let $q_n(t,x)$ is defined by the two–dimensional Fouier transform

$$\int \int e^{i(\theta_1 t + \theta_2 x)} q_n(t,x)\, dtdx = \frac{1 - e^{-i\theta_1/n} \bar{f}(\theta_2/n^{1/2})}{i\theta_1 + \theta_2^2/2},$$

where $\bar{f}$ is the complex–conjugate with f function. It is easily seen that the function $\bar{P}(t,x) := P_n * q_n(t,x)$ satisfies the equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} \right) \bar{P}_n(t,x) = 0$$

in the domain $\Omega_n = \Big\{ t,x : 0 < t < 1 - 1/n, \ g_2(t) < x < g_1(t) \Big\}$.

It still remains, for a first glance, a little: to estimate the deviation $\bar{P}(t,x)$ from $P(t,x)$ in the point $(0,0)$.

However, in reality this problem turned out very complicated technically. The approach suggested in [4] found out very useful here (it was already mentioned above.)

I describe the starting premises on which my paper [6] is based so carefully because they are obscured in reader's perception by the complicated analytical apparatus which is used in this paper. The main result obtained in [6] is the Berry–Esseen bound

$$\left| P_n - Q_n \right| < \frac{c\beta_3^2(K+1)}{n^{1/2}}, \tag{4}$$

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where K is the constant in the Lipshitz condition to which the bounds $g_1(x)$ and $g_2(x)$ satisfy.

The reason of appearing β_3^2 instead β_3 here the same as in bound (3): the insufficiently sharp estimate in Lemma 2 which is a generalization of Lemma 3 from [4].

A few years later A.V. Sakhanenko [10] replaced β_3^2 with β_3 in much the same way as Aleskeviciene in bound (3).

4.4.The asymptotic expansion. I used representation (2) also for deducing expansion the asymptotic expansion for $\bar{F}_n(xn^{1/2})$ (see [11, 12]). In contrast to Borovkov [3] the existence of the finite number of moments is required only. What is more, the condition that there exists an absolutely continuous component of $F(x)$ which Borovkov does impose is replaced by the Cramer condition $\lim\sup_{|t|\to\infty} |f(t)| < 1$.

In addition, the relation between moments of the initial distribution and coefficients of an expansion is clarified (I have already mentioned this problem above.) Basing on representation (2), Aleskeviciene proved the theorem on large deviations of Cramer's type for $\bar{S}_n$ [13], nd different versions of a local limit theorem (see [14, 15]).

4.5. Ruin problem. In addition to those boundary problems which were discussed above I have been engaged in the classical ruin (or absorption) problem. Recall how this problem is stated. As above, the random walk is considered, which is generated by a sequence of i.i.d random variables $X_j, j = 1, 2, ..., n, ...$.

Let n_x be the first hitting time of the complement of the interval (a, b) for a random walk starting from the point x, i..

$$n_x = \min\Big\{n : S_n + x \notin (a, b)\Big\}.$$

Assume that the distribution function $F_\lambda(x)$ of the random variable X_1 depends on some parameter λ, and $\mathbf{E}X_1^2 = \lambda^2$. Further, let $P_\lambda(x) = \mathbf{P}\Big(S_{n_x} + x \geq b\Big)$.

(A) The case of zero expectation. As early as 1930s (see, e.g. the well–known A.Ya. Khinchin book [16]) it was proved that in the case $\mathbf{E}X_1 = 0$

$$\lim_{\lambda\to 0} P_\lambda(x) = \frac{x - a}{b - a}, \ x \in (a, b), \tag{5}$$

if for every $\varepsilon > 0$

$$\lim_{\lambda\to 0} \frac{1}{\lambda^2} \int_{|x|>\varepsilon} x^2 dF_\lambda(x) = 0 \tag{6}$$

(Lindberg condition). Obviously, for $a < 0 < b$

$$\lim_{\lambda\to 0} P_\lambda(0) = \frac{a}{a - b}. \tag{7}$$

Conversely, if (7) holds for every $a < 0 < b$, then (5) is valid. One can interpret $P_\lambda(0)$ as the probability that the prize of a gambler will exceed the sum b before his loss achieves the admissible for him sum $-a$.

Notice that

$$\frac{x - a}{b - a} = \mathbf{P}\Big(W_x(\tau) \geq b\Big),$$

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where τ is the first exit time from the interval (a, b) for the Winer process $W_x(t)$ starting from the point x.

Thus, the invariance principle takes place in ruin problem.

Naturally, the question about the rate of convergence in (5) under minimal conditions on moments of $F(x)$ arises similar to those in CLT.

This problem was solved in my paper [17] in which the Berry–Esseen bound was obtained

$$\sup_{a<x<b} \left| P_\lambda(x) - \frac{x-a}{b-a} \right| < L \frac{\beta(\lambda)}{b-a},$$

where $\beta(\lambda) = \mathbf{E}|X_1|^3$.

(B) The case of nonzero expectation. Abandon now the condition $\mathbf{E}X_1 = 0$. Denote $m = m(\lambda) = \mathbf{E}X_1$. Suppose that $\lim_{\lambda \to 0} m\lambda^{-2} = \alpha$ and $\int_{|x|>\varepsilon} x^2 \, dF(x) = o(\lambda^2)$ for any $\varepsilon > 0$. In this case (see [16])

$$\lim_{\lambda \to 0} P_\lambda(x) = v(x), \quad x \in (a, b), \tag{8}$$

where $v(x)$ is the solution of the equation

$$v'' + \alpha v' = 0 \tag{9}$$

satisfying the boundary conditions $v(a) = 0$, $v(b) = 1$.

As to an estimate of the rate of convergence in (8), the latter is given in my paper [18] and appears as

$$\sup_{a<x<b} |P_\lambda(x) - v(x)| < \frac{Lc_3}{(b-a)\lambda^2} \left(1 + \frac{|m|}{\lambda^2}(b-a) \right), \tag{10}$$

where $v(x)$ is the solution of equation (9) for $\alpha = m/\lambda^2$ satisfying the boundary conditions $v(a) = 0$ $v(b) = 1$. It was shown in the course of deducing bound (10) that $L \leq 30$.

References

[1] *Prochorov Yu.V.* Convergence of random processes and limit theorems in probability// Theory Probab. Appl., 1956, **1**, No 2, 157 - 214.

[2] *Skorochod A.V.* Studies in the theory of random processes// MatStats. library, 1965.

[3] *Borovkov A.A* New limit theorems in boundary problems for sums of independent terms// Selected Transl. in Math. Stat. and Probab.,**5**, , 315–372.

[4] *Nagaev S.V.* Estimating the rate of convergence for the distribution of the maximum sums of independent random quantities// Sib. Math. J., 1969, **10**, No 3, 443 - 458. **PDF** Original Russian Text@ Sib. Mat. Zh., 1969, **10**, No 3, 614–633.

[5] *Nagaev S.V.* On the speed of convergence of the distribution function of the maximum sum of independent identically distributed random quantities// Theory Probab. Appl., 1970, **15**, No 2, 309 - 314. **PDF** Original Russian Text@Teor. Verojatn. i Primen., 1970, **15**, No 2, 320–326.

[6] *Nagaev S.V.* On the speed of convergence in a boundary problem// I,II. Theory Probab. Appl., 1970, **15**, No 2, 163 - 186, No 3, 403 - 429. **PDF** Original Russian Text@ Teor. Verojatn. i Primen., 1970, **15**, No 2, 179 - 199, **PDF**; No 3, 419–441.

[7] *Borovkov A.A.* On the rate of convergence for the invariance principle// Theory Probab. Appl., 1973, **28**, No 2, 207–225. Original Russian Text@ Teor. Verojatn. i Primen., 1973, **28**, No 2, 217–234.

[8] *Arak T.V., Nevzorov V.B.* Some estimates foe the maximum cumulative sum of independent random variables// Theory Probab. Appl., 1973, **28**, No 2, 384–386. Original Russian Text@ Teor. Verojatn. i Primen., 1973, **28**, No 2, 402–405.

[9] *Aleskeviciene A.K.* Nonuniform estimate of the distribution of the maximum of cumulative sums of independent random variables// Litov. Mat. Sb.,, 1973, **13**, No 2, 15–43. In Russian.

[10] *Sakhanenko A.I.* On the speed of convergence in a boundary problem// Theory Probab. Appl., 1974, **19**, No 2, 399–402. Original Russian Text@ Teor. Verojatn. i Primen., 1974, **19**, No 2, 416–421.

[11] *Nagaev S.V.* Asymptotic expansions for the maximum of sums of independent random variables. Theory Probab. Appl., 1971, **15**, No 3, 514-515. **PDF** Original Russian Text@ Teor. Verojatn. i Primen., 1970, **15**, No 3, 527–529.

[12] *Nagaev S.V.* Asymptotic expansions for the distribution of the maximum of sums of independent identically distributed variables. Sib. Math. J., 1970, **11**, No 2, 288 - 309. **PDF** Original Russian Text@ Sib. Mat. Zh., 1970, **11**, No 2, 381–406. **PDF**

[13] *Aleskeviciene A.K.* On the probabilites of large deviations for the maximum of sums of independent random variables// Theory Probab. Appl., 1979, **24**, No 1, 16–33. Original Russian Text@ Teor. Verojatn. i Primen., 1979, **24**, No 1, 322–337.

[14] *Aleskeviciene A.K.* Lokal limit theorems for densities of the maximum of sums of independent random variables// Litov. Mat. Sb., 1973, **13**, No 2, 5–21.

[15] *Aleskeviciene A.K.* Local limit theorems for the density function of the maximum sum of independent variables// Theory Probab. Appl., 1976, **21**, No 3, 449–469. Original Russian Text@ Teor. Verojatn. i Primen., 1976, **21**, No 3, 465–485.

[16] Khinchin A.Ya. Asymptotic laws of the probability rheory// .– L., NTI, 1936.

[17] *Nagaev S.V.* Estimates of the rate of convergence for the absorption probability in the case of zero mathematical expectation// Theory Probab. Appl. 1968, **13**, No 1, 160 - 163. Original Russian Text@ Teor. Verojatn. i Primen., 1968, **13**, No 1, 160–164. **PDF**

[18] *Nagaev S.V.* An estimate for the convergence rate for the absorption probability// Theory Probab. Appl., 1971, **16**, No 1, 140-148. **PDF** Original Russian Text@ Teor. Verojatn. i Primen., 1971, **16**, No 1, 147 - 154.

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