

5. Branching processes

5.1. Introduction. The development of the theory of branching processes in SSSR was pioneered the second half of 1940s. by A.N. Kolmogorov. It was S.Kh. Sirazhdinov who advocated this line of inquiry in Tashkent. In the middle of 1950s. he organized the seminar on the theory of branching processes in Tashkent University.

It was then that my steady interest in branching processes arose which I retained up to now.

My papers in branching processes may be grouped in the following way.

(A) Transient phenomena. While studying branching processes started in Moscow, originally two simplest models were considered: the Galton –Watson process and Markov branching process with continuous time (we use the abbreviation m.b.p.c.t. for the latter in what follows).

At that time, two questions were, the focus of attention: what is an asymptotic behaviour of the probability that a process will not extinct to the time t as $t \rightarrow \infty$, and whether there exists the limit distribution of the number of particles as $t \rightarrow \infty$ under the condition that a process did not extinct to the time t , and if yes what is it.

This questions were answered in the short run, in course of a few years, being dependent on the expectation A of the number of direct offspring. In particular, if $A = 1$ (such a process is called critical), then the non-extinction probability decreases as c/t , where c is some constant depending on parameters of a process, and conditional distribution of a number of particles at the instant t normalized in the appropriate way converges to the exponential law as $t \rightarrow \infty$.

Since we are not able to guarantee for real processes that $A = 1$, the question arises how a process behaves for A close to 1, more exactly for $A \rightarrow 1$ as $t \rightarrow \infty$. B.A. Sevast'yanov [1] proved that for m.b.p.c.t. the convergence to the exponential law holds, no matter with what rate A is going to 1 as $t \rightarrow \infty$.

The same problem was an the agenda for the Galton–Watson process. This problem was solved in Joint with my student R.Kh. Mukhamedhanova paper [2]. From this point on my studies in branching process began.

The result turned out the same as for m.b.p.c.t., however, the special modification of the difference method was required for this.

Phenomena described in [1,2] are called *transient*. Immediately after completing paper [2] I decided to study transient phenomena for the comparatively new at that time object: the Bellman–Harris process.

This time the problem turned out much more complicated: one needed to find a new approach to analyzing the integral equation which the offspring generating function satisfied. The approach that I have found did not allow to solve the problem with the maximal generality. Therefore, I has separately considered two cases:

- (a) the distribution of lifetime of particle is arithmetic,
- (b) the distribution of lifetime has an absolute continuous component.

The cases of lattice, but non arithmetic, and singular distributions remained uncovered.

It is my paper [3] which is devoted to transient phenomena in case (a). In contrast with the Galton –Watson process and m.b.p.c.t. the stability of the exponential distribution is

assured only under some additional assumption of the closeness of A to 1 (for more detail, see section 5.2).

Paper [3] is not lucky: though I wrote it as early as 1965, I could publish the latter only on 1974. Fortunately, it did not get out of date per 9 years. Moreover, it remains the only paper on transient phenomena for the Bellman–Harris process.

In the second half of 1980 I performed paper [5], joint with my post-graduate student A.V. Karpenko, in limit theorems for the total progeny of the Galton–Watson process. In particular, transient phenomena were studied for the total progeny therein. It turned out that the entire family of distribution depending on the parameter $r \leq 1$ may be limit ones if $n|1 - A| \rightarrow -\ln r$ as $n \rightarrow \infty$ (for more detail see 5.2).

We presented our results in preprint [4]. The paper under the same was submitted to journal "Teor. Veroyatn. i Primen." and published in 1993 [5], in five years after submission.

(B) Large deviations in branching processes.

The problem of large deviations in branching processes did not attracted the attention of researchers for a long time.

The first work in this direction was joint with my student N.N. Vakhrushev paper [6] in which the probability inequality for the critical Galton–Watson process was deduced under Cramer–type condition.

Following in footsteps of this paper G.D. Makarov [7] proved theorem on Cramer–type large deviations of the critical Galton–Watson process in the narrow zone, where the approximation by the exponential law is valid.

The theorem on large deviations in wider zone, with non-degenerating to 1 correction factor, was obtained in paper [8] written jointly with my post-graduate student V.I. Vakhtel.

In other joint with Vakhtel paper [9] two types of probability inequalities for maximum of the critical Galton–Watson process on the finite time interval are deduced. These are inequalities of type [6], but sharper, and the Nagaev–Fuk–type inequalities [10] (for more detail see 5.4).

Proving these inequalities, the classical martingale inequalities going back to Doob are used, as well as the inequality for supermartingales generalizing one of Fuk’s probability inequalities for martingales [11].

The importance of inequality obtained in [9] is comparable with the role of Nagaev–Fuk inequalities in the theory of summing independent random variables [10].

The point is that these any qualities are very suitable for proofing limit theorems for the critical Galton–Watson processes (see, e.g., [12, 13]).

(C) Local theorems for branching processes. The interest in local limit theorems for branching processes arose as early as 1950. It is apparently V.M. Zolotarev [14] to whom belongs the first local limit theorem for branching processes. In his paper the asymptotic behaviour of the probability $\mathbf{P}(Z_t = k)$ for given k is studied for the critical m.b.p.c.t. Z_t . Next V.D. Chist’yakov [15] found the asymptotics of $\mathbf{P}(Z_t = k)$ for $t, k \rightarrow \infty$ under assumption that fourth moment of the number of direct offspring is finite.

Every limit theorem for the Galton–Watson process has its m.b.p.c.t. analog, i.e. two theories duplicate each other.

However, different methods are used while proving: differential equation in the theory of m.b.p.c.t., and difference methods for Galton–Watson processes. True, one can reduce limit theorems for m.b.p.c.t. to corresponding results for Galton–Watson processes, using

the fact that any m.b.p.c.t. considered for $t=1,2,3,\dots$ is the Galton–Watson process but I think nobody was involved in it.

It is said in Chist'yakov's paper [15] that N.V. Smirnov proved the local limit for the critical Galton–Watson process.

However, neither the statement nor the proof of this result were not published.

In the beginning of 1960s I performed with Mukhamedkhanova the research in limit theorems for the critical Galton–Watson process $Z_n, n = 0, 1, 2, \dots$, the results of which were presented in [16].

In particular, the local limit theorems was proved in this paper under condition $\mathbf{E}Z_1^4 < \infty$.

At the same time paper [17] by H.Kesten, P. Ney, F. Spitzer appeared in which the local limit theorem was proved under condition

$$\mathbf{E}Z_1^2 \ln(1 + Z_1) < \infty.$$

Under the assumption $\mathbf{E}Z_1^2 < \infty$ minimal for the convergence to the exponential law the local limit theorem was proved only after 40 years in my joint with Wakhtel paper [18].

(D) Branching processes with migration.

Several my works, mainly in co-authorship with my student M.Kh. Asadullin, are devoted to branching processes with immigration [19–23].

The development herein was going in the direction of weakening the conditions on both components: a process of immigration and a branching process. The most general approach to the description of the a branching process with immigration is contained in [23], where the latter is interpreted as the double sum

$$\sum_{i=1}^{v_n} \sum_{j=1}^{v_{ni}} \xi_{ij}^{(n)}.$$

Here random variables $\xi_{ij}^{(n)}$ are mutually independent for any n , and v_n, v_{ni} do not depend on the family $\{\xi_{ij}^{(n)}\}$.

In joint with my student L.V. Khan paper [24] the model of the a Galton–Watson process with migration is suggested. In this model the process of immigration is supplemented with process of emigration.

Historically, is was the first model of such a kind. Up to then the models either only with immigration or only emigration were considered.

5.2. Limit theorems for Bellman–Harris processes.

Let $Z(t)$ be the Bellman–Harris process, $f(s)$ be the generating function of direct offspring, G be the life–time distribution function continuous from the right. Put

$$M(t) = \mathbf{E}\left\{Z(t) \mid Z(t) > 0\right\}, \quad F(t, s) = \mathbf{E}s^{Z(t)}, \quad A = f'(1), \quad B = f''(1).$$

The problem is to prove the convergence of distribution of $Z(t)/M(t)$ to the exponential law under the condition $Z(t) > 0$ as simultaneously $t \rightarrow \infty, A \rightarrow 1$.

In what follows I explain the essence of the approach which I apply, along with the nature of arising difficulties.

The starting point is the equation

$$F(t, s) = \int_{0-}^{t+} f(F(t - \tau, s)) dG(\tau) + s(1 - G(\tau)). \quad (1)$$

We restrict ourselves, for simplicity, to the case $s = 0$.

Letting $Q(t) = 1 - F(t, 0)$ and using the Taylor expansion of f in the neighborhood of 1, we arrive at the equation

$$Q(t) = A \int_{0-}^{t+} Q(t - \tau) dG(\tau) - \frac{B}{2} \int_{0-}^{t+} Q^2(t - \tau) dG(\tau) + \int_{0-}^{t+} R(t - \tau) dG(\tau) + 1 - G(\tau),$$

where R is a correction term. Hence,

$$Q(t) = \frac{B}{2} \int_{0-}^{t+} Q^2(t - \tau) dH(\tau, A) + \int_{0-}^{t+} R(t - \tau) dH(\tau, A) + H_1(t, A),$$

where

$$H(t, A) = \sum_{m=1}^{\infty} A^{m-1} G^{*m}(t), \quad H_1(t, A) = (1 - G(t)) * \sum_{m=0}^{\infty} A^m G^{*m}(t).$$

Suppose now the $G(t)$ is concentrated on natural numbers (it is precisely these distributions that are considered in my paper [3]). Let, for simplicity $A = 1$. Then by the renewal theorem

$$u_k := H(k, 1) - H(k - 1, 1) = \frac{1}{\mu} + r_k,$$

where $\mu = \int_0^{\infty} u dG(\mu)$, and the rate of decay of r_k depends on the rate of decay of $1 - G(t)$ as $t \rightarrow \infty$.

If we neglect R , then

$$Q(n) = \frac{B}{2\mu} \sum_{k=1}^n Q^2(n - k) u_k + \sum_{k=0}^n (1 - G(n - k)) u_k.$$

Hence,

$$Q(n - 1) - Q(n) = \frac{B}{2\mu} Q^2(n) + \Omega(n), \quad (2)$$

where

$$\Omega(n) = \sum_{k=1}^n [Q^2(n - k) - Q^2(n - k - 1)] r_k + G(n) - 1 - \sum_{k=0}^n g_{n-k} r_k,$$

$g_k = \mathbf{P}(\tau = k)$. Further, it is shown that $\lim_{n \rightarrow \infty} Q(n) = 0$ with

$$\Omega(n) = o(Q^2(n)).$$

Returning to (2), we ensure that

$$Q(n - 1) - Q(n) \sim \frac{B}{2\mu} Q^2(n).$$

The same relation with $\mu = 1$ holds for the critical Galton–Watson process (see [2]). It means that for $n \rightarrow \infty$

$$Q(n) \sim \frac{2\mu}{Bn}.$$

The added complication in the case $A \neq 1$ was for lack of renewal theorems for $H(t, A)$. This gap is filled in my paper [25] in which local renewal theorems for $H(t, A)$ are proved on the case when $A \rightarrow 1$, $t \rightarrow \infty$ simultaneously.

One may summarize the main result of [3] as follows: *the convergence to the exponential law holds if either $A \downarrow 1$ or $A \uparrow 1$ in such a way that $\psi(t) < A < 1$, the rate of the convergence of $\psi(t)$ to 1 being dependent on the rate of the decay of $1 - G(t)$ as $t \rightarrow \infty$. In the case $A \uparrow 1$, $A > \psi(t)$ the issue remains open.*

The attention of specialists was called mainly simpler case of fixed A (see [26–30]). For $A = 1$ the most complete result has been obtained by M. Goldstein [30]. The key point in his paper is lemma (1.6) which links the generating functions of the Bellman–Harris process and the embedded Galton–Watson process together, namely: for $t \geq 0$, $m = 1, 2, \dots$, $0 \leq s \leq 1$

$$1 - f_m(s) - (1 - s)G_m(t) \leq 1 - F(t, s) \leq 1 - f_m(s) + (1 - s)(1 - G_m(t)),$$

where $f_m(s)$ is the m -th iteration of the generating function $f(s)$ of the number of direct offspring, $G_m(t) = G^{*m}(t)$ is m -th convolution of $G(t)$.

If it is assumed that $1 - G(t) = o(1/t^2)$, then $t < m(\mu - \varepsilon)$

$$G_m(t) = o(1/t).$$

Respectively, if $t < m(\mu + \varepsilon)$, then

$$1 - G_m(t) = o(1/t).$$

On the other hand, it is known that

$$1 - f_m(s) = \frac{1 - s}{1 + (g''(1)/2)m(1 - s)}(1 + \delta(m)),$$

where $\lim_{m \rightarrow \infty} \delta(m) = 0$ uniformly with respect to s , $0 \leq s \leq 1$. Hence, it follows that for $m = \left\lceil \frac{1}{\mu t} \right\rceil$

$$1 - F(t, s) \sim 1 - f_m(s)$$

as $t \rightarrow \infty$ which is the main result in Goldsteins paper [30]. The convergence to the exponential law is deduced here from without difficulty.

Notice that the reduction to the Galton–Watson process presents implicitly in my paper [3]. The point is that model difference equation (2) is typical for Galton–Watson processes.

5.3. Limit theorems for the total progeny of the Galton–Watson process. Let Z_k be the the Galton–Watson process initiated from one particle, $A_n = \mathbf{E}\left\{\sum_{k=1}^n Z_k \mid N = n\right\}$ be the extinction time of Z_k .

In 1985 we with my student A.V. Karpenko proceeded to studying the conditional distributions

$$\mathbf{P}\left(\frac{1}{A_n} \sum_{k=1}^n Z_k < x \mid N = n\right)$$

for all spectrum of values $A = \mathbf{E}Z_1$.

This definition of the problem was new even for fixed A . We arrived at it in connection with constructing the sequential test for estimating the parameter A , which end in the random time N .

Formerly the distributions of $\sum_{k=1}^n Z_k$ were being studied under the conditions $N > n$ [31] or $N < n$ [32].

Results of our studying in this direction are presented in [5].

The main result of this paper looks as follows:

1) Let $\lim_{A \rightarrow 1} n|1 - A| = -\ln r < \infty$. Then

$$\lim_{A \rightarrow 1} \mathbf{P}(S_n/m_N < x | N = n) = G(x, r),$$

where

$$\int_0^\infty e^{-tx} d_x G(x, r) = g(t, r) := \begin{cases} 3t / \sinh^2 \sqrt{3t}, & r = 1 \\ \frac{(1-r)^2 r^{d(t,r)} (d(t,r))^2}{r(1-r^{d(t,r)})^2}, & r < 1, \end{cases}$$

$$t \geq 0, \quad d(t, x) = \sqrt{1 + 2t / (1 - x)^2 h(\ln x)},$$

$$h(x) = \frac{x(1 + e^x) + 2(1 - e^x)}{(e^x - 1)^3}.$$

2) If $\lim_{A \rightarrow 1} n|1 - A| = \infty$, then

$$\lim_{A \rightarrow 1} \mathbf{P}(|S_n/m_N - 1| > \epsilon | N = n) = 0.$$

Thus the stability of the limit distribution $G(x, 1)$ holds only if $|A - 1| = o(n^{-1})$.

5.4. Large deviations in branching processes. I start with describing my joint with Vakhrushev paper [6].

Let Z_n , $n = 0, 1, 2, \dots$, be the critical Galton–Watson process initiating from one particle. Put $f(x) = \sum_{k=0}^\infty p_k x^k$, where p_k is the probability that one particle generates k particles. Let $B(x) = f''(x)$, $B = B(1)$. Denote by R the radius of convergence of the series $\sum_{k=0}^\infty p_k x^k$. In these terms the following inequality holds: if $B > 0$ and $R > 1$, then for $0 < y_0 < R - 1$

$$\mathbf{P}(Z_n \geq k) \leq (1 + y_0) \left(1 + \frac{1}{B_0 n / 2 + 1 / y_0} \right)^{-k}, \quad (3)$$

where $B_0 = B(1 + y_0)$.

The function $B(1 + y)n/2 + 1/y$ achieves minimum at $y = y_1$, where y_1 is the solution of the equation $B'(1 + y)y^2 = 2/n$.

If $f(x)$ is fixed, then $y_1^2 = O(1/n)$. Consequently, $B'(1 + y_1) = B'(1) + O(1/\sqrt{n})$, i.e. $y_1 = \sqrt{\frac{2}{nB'(1)}} + O(n^{-3/2})$.

If we put now $y_0 = y_1$ and $k = \lfloor Bun/2 \rfloor$, then according (3)

$$\lim_{n \rightarrow \infty} \mathbf{P}(Z_n \geq k) \leq e^{-u}.$$

On the other hand, by the integral limit theorem for the critical Galton–Watson process

$$\lim_{n \rightarrow \infty} \frac{Bn}{2} \mathbf{P}(Z_n \geq k) \leq e^{-u}.$$

One may consider this comparison as evidence that bound (3) is sharp enough for values k large relative to n .

Under influence of our work [6] G.D. Makarov [7] proved the limit theorem on large deviations of Cramer’s type, namely,

$$\lim_{n \rightarrow \infty} \sup_{u \leq u_n} e^u P_n(u) = 1, \quad (4)$$

where $P_n(u) = \mathbf{P}(2Z_n/Bn > u \mid Z_n > 0)$, $u_n = o\left(n/(\ln n \ln_{(N)} n)\right)$. Here $\ln_{(N)}$ is the N –th iteration of logarithm, and $N \geq 2$.

Based on the analogy to summing i.i.d., we should expect that for $R > 1$ there exists the zone of values of u in which

$$P_n(u) = e^{-u} \Omega_n(u) (1 + o(1)), \quad (5)$$

where $\Omega_n(u)$ is an explicitly calculated correction factor.

Indeed, it was shown in joint with my student Wakhtel paper [8] that *if $k/n \rightarrow \infty$ and $k = o(n^2)$, then*

$$\mathbf{P}(Z_n \geq k) = \frac{2}{Bn} \exp\left\{\frac{2k}{Bn} - \frac{2\gamma}{Bn^2} \ln\left(\frac{k}{n}\right)\right\} \left(1 + O\left(\frac{k}{n^2} + \frac{\ln^2 n}{n}\right)\right),$$

where $\gamma = 1 - 2C/(3B^2)$.

It means that for $u = o(n)$

$$P_n(u) \sim e^{-u} \exp\left\{-\frac{\gamma}{n} u \ln u\right\},$$

i.e. the correction factor $\Omega(u)$ in (5) is given by the equality

$$\Omega_n(u) = \exp\left\{-\frac{\gamma}{n} u \ln u\right\}.$$

It is easily seen that for $\gamma \neq 0$

$$\lim_{n \rightarrow \infty} \Omega(v_n) = 1 \quad (6)$$

iff

$$v_n = o\left(\frac{n}{\ln n}\right).$$

If $\gamma = 0$, then (6) holds in wider zone $v_n = o(n)$.

Thus, the approximation with the exponential law $P_n(u) = e^{-u}$ holds for $u = o(n/\ln n)$ if $\gamma \neq 0$. Otherwise, this zone extends to $u = o(n)$. The bound $u = o(n)$ corresponds to that $x = o(\sqrt{n})$ in the classical Cramer theorem on large deviations for sums of i.i.d.

The direct continuation of work [6] is my joint with Vakhtel paper [9] in which the inequality of type (3) is deduced for the random variable $M_n = \max_{k \leq n} Z_k$, namely, if $R > 1$, then for every $0 < y_0 < R - 1$

$$\mathbf{P}(M_n \geq k) \leq y_0 \left[\left(1 + \frac{1}{1/y_0 + B_0/2} \right)^k - 1 \right]^{-1}. \quad (7)$$

where $B_0 = f''(1 + y_0)$.

It follows, in particular, from bound (7) that for any $1 < \rho < R$

$$\mathbf{P}(M_n \geq k) \leq \frac{4}{n\Omega_\rho} \exp\left\{-\frac{k}{n\Omega_\rho + 1}\right\}, \quad (8)$$

if simultaneously $n > \frac{2}{(\rho-1)\Omega_\rho}$, $k > n\Omega_\rho + 1$, where $\Omega_\rho = f''(\rho)$.

Obviously, the factor $1/n$ is essential if the ratio k/n is not very large. Inequality (8) is an analog of Petrov's inequality [33] (see p. 81, th. 16).

The starting point for deducing bound (7) is the inequality

$$\mathbf{P}(M_n \geq k) \leq \frac{f_n(e^h) - 1}{e^{hk} - 1} \quad (9)$$

which in turn follows from the well-known Doob inequality for submartingales.

Indeed ,

$$\mathbf{P}(M_n \geq k) = \mathbf{P}\left(\max_{i \leq n} Y_i(h) \geq e^{hk} - 1\right),$$

where $Y_n(h)$ is submartingal $e^{hZ_n} - 1$. On the other hand, by the Doob inequality

$$\mathbf{P}\left(\max_{i \leq n} Y_i(h) \geq e^{hk} - 1\right) \leq \frac{\mathbf{E}Y_n(h)}{e^{hk} - 1} = \frac{f_n(e^h) - 1}{e^{hk} - 1}.$$

The same value $h = h(n)$ is chosen in (9) as in paper [6]. The difference consists in the fact that the bound

$$\mathbf{P}(Z_n \geq k) < \frac{f_n(e^h)}{e^{hk}}$$

is applied in the latter. It is due the fact that the unit is subtracted from $f_n(e^h)$ in (9) the factor y_0 appears instead of $1 + y_0$ in (3). In turn, it allows to obtain the additional relative to (8) factor $1/n$ in (8), letting $y_0 = 2/n\Omega_\rho$.

In the paper [9] the case is also considered when the Cramer condition does not hold, i.e. $R = 1$. Corresponding probability inequalities are formulated in terms both truncated and total moments of the random variable ξ which is equal to Z_1 in distribution. Denote $B_r = \mathbf{E}\xi^r$, $r > 1$. For every $N > 0$ put $\bar{B} = \mathbf{E}\left\{\xi(\xi - 1); \xi \leq N\right\}$, $\bar{B}_r = \mathbf{E}\left\{\xi^{r-1}(\xi - 1); \xi \leq N\right\}/2$. If $r \geq 2$, $N \geq 1$ and $y_0 > 0$, then inequality

$$\mathbf{P}(M_n \geq k) \leq \left(y_0 + \frac{1}{N}\right) \left[\left(1 + \frac{1}{1/y_0 + e^r \bar{B}n/2 + n\bar{B}_r e^{y_0 N}/N^{r-2}} \right)^k - 1 \right]^{-1} + n\mathbf{P}(\xi > N) \quad (10)$$

holds.

Deducing this inequality, truncation of the random variable ξ on the level N is used. The summand $n\mathbf{P}(\xi > N)$ estimates the error arising as a result. This approach was used previously in my joint with Fuk paper [10], by deducing probability inequalities for sums of independent random variables. The parameters y_0 and N are free. How to select them, depends on concrete situation: at times we are interested in accuracy, above all and at other time simplicity and clearness are of interest. The inequality

$$\mathbf{P}(M_n \geq k) \leq \exp\left\{-\frac{k}{l(r)Bn}\right\} + \frac{G(r, B, B_r)n}{k^r} \quad (11)$$

which is valid for $r > 2$ and $k \geq Bn$ is the example of an illustrative bound. Here $l(r)$ and $C(r, B, B_r)$ are constants depending on parameters which are indicated in parentheses.

Inequality (11) is quite predictable taking into account that the process $W_n := \sqrt{Z_n}$ is supermartingale with uniformly bounded moments of the order $t \geq 2$

$$\mathbf{E}\left\{|W_{n+1} - W_n|^t \mid Z_n = k\right\} < c\mathbf{E}\xi^{t/2}.$$

Fuk's result [11] are applicable to such supermartingales. In particular,

$$\mathbf{P}(W_n > x) < e^{-c_1x^2/nB^2} + \frac{c_2n\mathbf{E}\xi^{t/2}}{x^t}.$$

Returning now to the original process Z_n , we obtain the inequality of type (11) for $r = t/2$.

5.5. Local limit theorems for branching processes. As it has already been said in section (C) of Introduction two works [16, 17] appeared in 1966 in which the local limit theorem is proved for the critical Galton–Watson process.

The asymptotic formula

$$\frac{B^2n^2}{4}\mathbf{P}(Z_n = k) = \exp\left\{-\frac{2k}{Bn}\right\} + \alpha_{kn} + O(k^{-1}\ln n) \quad (12)$$

is deduced in [16], where $\alpha_{kn} \rightarrow 0$ as $n \rightarrow \infty$ uniformly with respect to k , under condition that $\mathbf{E}Z_1^4 < \infty$ and the greatest common divisor d of the set $\{k : \mathbf{P}(Z_1 = k) > 0\}$ equals 1. Here $B = f''(1)$, f is the generating function of the random variable of Z_1 .

At the same time paper [17] by Kesten H., Ney P., Spitzer F. appeared, in which the next result was stated: *if k and n tend to infinity so that the ratio k/n remains bounded, then*

$$\lim_{n \rightarrow \infty} n^2 \exp\left\{\frac{2kd}{Bn}\right\} \mathbf{P}(Z_n = kd) = \frac{4d}{B^2}. \quad (13)$$

Comparing (12) and (13) we see that (13) follows from (12) only under condition that $k^{-1}\ln n$ tends to zero. On the other hand, it follows from (12) that relation (13) remains valid if k/n tends to infinity sufficiently slowly.

Autors of [17] point that (13) is valid without any excessive moment restrictions, i.e. only the condition $B < \infty$ is sufficient, however, they have performed proving only under more restrictive condition

$$\mathbf{E}Z_1^2 \ln(1 + Z_1) < \infty. \quad (14)$$

They write also that this assumption is made for simplicity. However, it is said in Atreya's and Ney's book [34] that the proof of the local limit theorem under condition $B < \infty$ has not published anywhere to this point.

I returned to the local theorem after many years. In 2005 my joint with Wakhel paper was published [9] in which we have managed to remove the restriction (14), namely, we have obtained the next result: *if $B < \infty$, k and n tend to infinity so that the ratio k/n remains bounded, that*

$$\lim_{n \rightarrow \infty} \frac{B^2 n^2}{4d} (1 + 2d/Bn)^{k+1} \mathbf{P}(Z_n = kd) = 1. \quad (15)$$

It is clear that replacing the factor $(1 + 2d/Bn)^k$ with the equivalent expression $\exp\{2kd/Bn\}$ we obtain exactly (13).

Thus, we approximate the distribution of Z_n by the geometric one with the parameter $2d/B_n$ instead of the exponential law. This approach is more natural because the distribution of the Z_n is discrete. In addition, the approximation by the geometrical distribution is generally speaking more precise. For example, for the critical process with a bilinear generating function

$$\mathbf{P}(Z_n = k) = \frac{4}{B^2 n^2} (1 + 2/Bn)^{-k-1}$$

for $k \geq 1$. Thus, relation (15) is fulfilled for all k , and (13) holds only for $k = o(n^2)$.

The proof of (15) is based on the next statement which is of an independent interest as well.

Proposition. *If $B < \infty$, then there exists the constant $C = C(f)$ such that*

$$\sup_{n, k \geq 1} n^2 \mathbf{P}(Z_n = k) \leq C.$$

Our method of proving the local limit theorem differs from that of Kesten, Ney, Spitzer though we use some of their results.

References

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