6. Infinite-dimensional distributions

6.1. Introduction. Today the theory of distributions in infinite-dimensional spaces is the developed field of probability theory. It appeared as a natural generalization of the theory of distributions in Euclidean space of finite dimension. One of its branches is the theory of summation of random vectors with values in infinite-dimensional spaces, mainly, Banach ones. In the latter, in turn, one can distinguish the following two directions:

(a) Limit theorems,

(b) Probability inequalities.

In their development both of these directions passed the same stages as the theory of summation of finite-dimensional random vectors.

The first stage in the area of limit theorems was the studying conditions under which the distribution of the sum $\sum_{k=1}^{n} X_k$ of independent random vectors comes close to the Gaussian one when the number of summands increases infinitely. The statements on such approximation are versions of the Central Limit Theorem (CLT) in an appropriate space.

There exists an extensive bibliography to this subject, but we are interesting in a some other question, namely, the rate of convergence in CLT.

(A) Estimates of convergence in Central Limit Theorem in Hilbert space. I became interested in this problem in the middle of 1970 s'. That time the following problem attracted the specialists' greatest attention: what is the rate of convergence in CLT for identically distributed random variables (r.v.) in Hilbert space on the balls with the center in zero. The bound $O(n^{-1/2})$ was supposed to be the best one, the same in the order as the classic Berry – Esseen bound, where n was the number of summands. But it wasn't clear whether this bound was possible in the given case.

The first publication on this subject [1] appeared in 1965. For next ten years the bound was reduced to $O(n^{-1/6})$ [2].

While studying this problem, I decided to constrict the setting of the problem at the first stage, restricting myself by r.v.s' (random variables) with independent coordinates (keeping in mind l_2, being the realization of Hilbert space). This plan was realized in the paper [3] (see also [4]), jointly with my pupil V.I. Chebotarev. We showed that in the particular case indicated above the bound $O(n^{-1/2})$ held under the condition that $\mathbf{E}|X_1|^3 < \infty$. Here and below the symbol $|X|$ denotes the norm of X.

Historically this was the first estimate of the form $O(n^{-1/2})$ in infinite-dimensional case. Then the model which we suggested was used repeatedly by other authors to construct distinct examples and counterexamples. First of all, the work of V.V. Senatov [5] should be mentioned in this connection.

Our result announced in 1977 [4] was published in 1978 [3], but the next year (1979), the paper of F. Götze [6] appeared, in which the bound $O(n^{-1/2})$ was obtained in a general case already, however, under the condition $\mathbf{E}|X_1|^6 < \infty$. But the most principal in the work of F. Götze was the ingenious and unexpected method, permitting to estimate the characteristic function of the squared norm of the sum $\sum_{k=1}^{n} X_k$. Following this way, in 1982 V.V. Yurinsky [7] reduced the moment restriction to the minimal one: $\mathbf{E}|X_1|^3 < \infty$. The method found by F. Götze plays the essential role in all next works devoted to accuracy of normal approximation in Hilbert space (see, for instance, [8]– [20]). Both Götze and Yurinsky are interested, first of all, in dependence of the bound on the number n and moments of the distribution of the summands. In order to attain a perfect analogy with the classic Berry – Esseen bound, it was necessary to investigate the dependence of the bound on the covariance operator. The focus of further investigations shifted just to this direction.

The first work in this direction was my paper [8]. About this and next works it will be said in more details below in the Section 6.2.

Along with bounds of the Berry – Esseen type we together with Chebotarev studied asymptotic expansions in the central limit theorem in Hilbert space. The results of our investigations in this direction were stated in the papers [16] and [18]. Our predecessors in this area (asymptotic expansions are kept in mind) were F. Götze [6], V.Yu. Bentkus [10,11], V.Yu. Bentkus and B.A. Zalessky [12]. Parallel with us B.A. Zalessky, V.V. Sazonov and V.V. Ul'yanov [15] studied asymptotic expansions.

(B) Probability inequalities for sums of independent random vectors with values in Banach spaces. We start with large deviations for Gaussian measures in Banach space. Intensive studying the latter began in 1950 s' in connection with general progress in theory of stochastic processes. A great merit in stimulating interest to distributions in Banach spaces belongs to A.N. Kolmogorov and Yu.V. Prokhorov. By the words, the definition of a distribution in Banach space was given by A.N. Kolmogorov as early as 1935.

It was not known rather for a long time with which rate the probability $\mathbf{P}(|X| > r)$ decreases as $r \to \infty$, where X is a Gaussian r.v. with values in Banach space, until at last in 1970 three works [21]– [23] appeared at once in which three distinct approaches to the solution of this problem were proposed.

It was shown in the work of H. Landau, L. Shepp [21] and X. Fernique [22] that for every Gaussian r.v. X there exists a constant $c(X) > 0$ such that

$$\mathbf{P}(|X| > r) < \exp\{-c(X)r^2\}. \tag{1}$$

Not so sharp result was obtained by A.V. Skorokhod [23], namely,

$$\mathbf{P}(|X| > r) < \exp\{-c(X)r\}. \tag{2}$$

.

It was discovered later that with the help of simple arguments, basing on the infinite divisibility of the normal law, one can derive the bound of the form (1) from (2) (see, for instance, [30], p. 80).

The new approach, differ from those, used in the above-mentioned works, was suggested in my paper [31] (see also [32]). This approach brings to the bound of the form (1). It is said on the work [31] in more detail in Section 6.4.

Now we pass to the probability inequalities for sums of independent random vectors in a separable Banach space. I published works [35–37] on this subject. Upper bounds for the probability of hitting the exterior of a sphere of an arbitrary radius are deduced in two first ones, and lower bounds for the same probability are deduced in the third. The form of the bound found in [36] is new in one-dimensional case as well. One can say the same with respect to the work [37]. The upper bounds obtained in [36] are extended to dependent random vectors in Banach space in my paper [45]. In turn, the latter gives a possibility to get moment inequalities very simply (see subsection 6.6 in detail).

An alternative approach to deducing probability inequalities in Banach spaces was earlier suggested by Yurinsky [39]. This approach is based on the representation of the norm of a random vector in the form of martingale difference. Becoming acquainted with this work, I called the attention to the fact that the corresponding martingale satisfies the Fuk conditions [40]. This gives the possibility to get various probability and moment inequalities without additional efforts (see [42].)

6.2. Bounds for the convergence rate in the central limit theorem. First of all, I formulate the bound, which was obtained in my joined with Chebotarev paper [3].

Let $X_1, X_2, ..., X_n$ be a sequence of independent identically distributed random variables taking values in l_2, where $\mathbf{E}X_1 = 0$. We denote by X_{kj} the j-th coordinate of X_k. Put $\sigma_j^2 = \mathbf{E}X_{1j}^2$, $\beta_j = \mathbf{E}|X_{1j}|^3$, $\beta = \mathbf{E}|X_1|^3$. In assumption that X_{kj} are mutually independent, we got in [3] the bound

$$\sup_x \Big|\mathbf{P}\Big(\big|n^{-1/2}S_n\big| < x\Big) - \mathbf{P}\Big(|Z| < x\Big)\Big| < c\Big[n^{-1/2}\Big(\prod_1^4 \sigma_j\Big)^{-3/4} \sum_5^\infty \beta_j + \Delta_n^{(4)}\Big]. \qquad (3)$$

Here $S_n = \sum_{k=1}^n X_k$, $Z = (Z_1, Z_2, ...)$ is the normal r.v. taking values in l_2, $\mathbf{E}Z = 0$, $\mathbf{E}Z_j^2 = \sigma_j^2$,

$$\Delta_n^{(m)} = \sup_x \Big|\mathbf{P}\Big(\sum_{j=1}^m \big(n^{-1/2}S_{nj}\big)^2 < x\Big) - \mathbf{P}\Big(\sum_{j=1}^m Z_j^2 < x\Big)\Big|,$$

where $S_{nj} = \sum_{k=1}^n X_{kj}$, c is an absolute constant.

On the other hand, according to the multivariate central limit theorem,

$$\Delta_n^{(m)} < c(m)\, n^{-1/2} \sum_{j=1}^m \beta_j/\sigma_j^3.$$

Thus, only four of the infinite set of variances σ_j^2 involve in the bound (3). Without loss of generality one can consider that $\sigma_1^2 \geq \sigma_2^2 \geq \ \dots \ \geq \sigma_k^2 \geq \ \dots$. Note that variances σ_j^2 are the eigenvalues of the corresponding covariance operator.

Now we pass to the general case. We shall save the notations introduced above, but shall consider, that X_j takes values in separable Hilbert space H, not necessarily l_2.

Here I formulate the corollary from the main result of this paper. This corollary is given in my notice [24]:

$$\Delta_n(a) \leq c\beta\left(\Big(\prod_1^7 \sigma_j\Big)^{-6/7} + \frac{1}{\sigma^2\sigma_1\sigma_2\sigma_7}\right)\Big(\sigma^3 + |a|^3\Big)n^{-1/2}. \qquad (4)$$

Here

$$\Delta_n(a) := \sup_r \Big|\mathbf{P}\big(\big|n^{-1/2}S_n - a\big| < r\big) - \mathbf{P}(|Z - a| < r)\Big|,$$

$\sigma_1^2 \geq \sigma_2^2 \geq \ \dots \ \geq \sigma_j^2 \geq \ \dots$ are the eigenvalues of the covariance operator of r.v. X_1, $\sigma^2 := \mathbf{E}|X_1|^2 = \sum_{j=1}^\infty \sigma_j^2$, Z is the Gaussian r.v. with the same moment characteristic of the first and second orders as X_1. The bound (4) is very similar in form to previous one (3) with the difference that (4) contains seven eigenvalues of covariance operator instead of four. One can explain a priori taking into account that the bound (4) is designed for the balls with shifted centers in contrast to the previous one.

It was also proved in the above-mentioned work [3] that for $a = 0$ the constant in the bound of the form $\Delta_n(0) = O(n^{-1/2})$ must depend at least on three eigenvalues. A short time after appearing the paper [8] V.V. Senatov [5] had constructed the example showing that for $a \neq 0$ the constant in the bound $\Delta_n(a) = O(n^{-1/2})$ involves not less than 6 eigenvalues. More exactly, for all beforehand given $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_6^2 \geq \sigma_7^2$ there exists a distribution in $\mathbb{R}^7$, for which they are the eigenvalues of the covariance operator, such that for $\sigma_6^2 \leq |a| \leq \rho\sigma_6^2$, $\rho > 1$,

$$\limsup_{n\to\infty} \sqrt{n}\Delta_n(a) \geq c(\sigma_7^2, \rho)|a|^3/\Lambda_6^{1/2}, \tag{5}$$

where $c(\sigma_7, \rho)$ depends on σ_7 and ρ only. Here and in what follows, $\Lambda_l = \prod_{j=1}^{l} \sigma_j^2$.

Three years after the publication of my paper [8] B.A. Zalessky, V.V. Sazonov, and V.V. Ul'yanov [14, 17] obtained the bound

$$\Delta_n(a) < \frac{c\beta(\sigma^3 + |a|^3)}{\sqrt{n}\Lambda_6^{1/2}} \tag{6}$$

which depended on the first six eigenvalues.

Putting $a = 0$ in this bound, we have

$$\Delta_n(0) < \frac{c\beta\sigma^3}{\sqrt{n}\Lambda_6^{1/2}}. \tag{7}$$

The above-mentioned example of Senatov is not extended to the case $a = 0$. This suggests that actually the bound (6) is not final. Indeed, some time later I succeed to obtain the sharper bound,

$$\Delta_n(a) \ll \frac{\beta}{\sqrt{n}}\left(\frac{\sigma}{\Lambda_4^{1/2}}\left(1 + \sum_1^4 \left|\frac{a_j}{\sigma_j}\right|^{3/2}\right) + \frac{|a(5)|^3}{\Lambda_6^{1/2}}\right). \tag{8}$$

Here a_j is the j-th coordinate of the vector $a = (a_1, a_2, \ldots) \in l_2$, $a(k{+}1) = (\underbrace{0, 0, \ldots, 0}_{k}, a_{k+1}, a_{k+2}, \ldots)$. (We write $A \ll B$, if there exists an absolute constant c such that $A \leq cB$). This result was announced in [25]. Unfortunately, the proof is not published up to now.

The next bound follows from (8),

$$\Delta_n(0) < \frac{c\beta\sigma}{\sqrt{n}\,\Lambda_4^{1/2}},$$

which is sharper with respect to (7), since it depends only on four eigenvalues of the covariance operator. It is easily seen also that (8) implies (6). Indeed,

$$\frac{\sigma|a_j|^{3/2}}{\Lambda_4^{1/2}\sigma_j^{3/2}} \leq \frac{\sigma^{3/2}|a|^{3/2}}{\Lambda_6^{1/2}}, \quad j \leq 4.$$

In turn,

$$\sigma^{3/2}|a|^{3/2} \leq (\sigma^3 + |a|^3)/2.$$

Consequently,
$$\Delta_n(a) \ll \frac{\beta}{\sqrt{n}}\left(\frac{\sigma}{\Lambda_4^{1/2}} + \frac{|a|^3+\sigma^3}{\Lambda_6^{1/2}}\right) \ll \frac{\beta(|a|^3+\sigma^3)}{\sqrt{n}\Lambda_6^{1/2}}. \tag{9}$$
As early as in 1945 C.-G. Esseen [26] obtained the following bound for the special case $H=\mathbb{R}^l$,
$$\Delta(0) \le c(l)\left(\mathbf{E}|X_1|^4\right)^{3/2} n^{-l/(l+1)} \tag{10}$$
under condition that the covariance matrix is unit. Knowing this result, it was natural to assume that something had place in the general case as well. Intensive investigations in this direction were in progress in 1980 – 1990 s'. In 1982 B.A. Zalessky [9] proved that for every $\varepsilon>0$,
$$\Delta_n(0) = O(n^{-1+\varepsilon}), \tag{11}$$
if $\sigma_N^2 \ne 0$ for sufficiently large $N=N(\varepsilon)$. In 1983 V.Yu. Bentkus [27] obtained a more general result from which the bound (11) followed for
$$\Delta_{n,1}(a) = \sup_r \Big|\mathbf{P}(|n^{-1/2}S_n - a| < r) - \mathbf{P}(|Z-a|<r) - Q_{1,n}(a,r)\Big|, \tag{12}$$
where $Q_{1,n}(a,r)$ was the first term of the asymptotic expansion, in addition $Q_{1,n}(0,r)\equiv 0$.

In 1986 we together with V.I. Chebotarev [13] got a more precise bound in the case $a=0$:
$$\Delta_n(0) \le c(l,\delta)(\Gamma_{4,\,l}/n)^{l/(l+4+\delta)} + c(l)\begin{cases}(\Gamma_{3,\,l}/\sqrt{n})^{2l/13}, & 7\le l\le 12,\\ \Gamma_{3,\,l}^2/n, & l\ge 13\end{cases} \tag{13}$$
for every $\delta>0$ and integer $l\ge 7$, where $\Gamma_{4,\,l}=\beta_4\sigma^4\Lambda_l^{-4/l}$, $\beta_4=\mathbf{E}|X_1|^4$, $\Gamma_{3,\,l}=\beta\sigma^3\Lambda_l^{-3/l}$. This result is much closer to Esseen's bound (10) than that of Zalessky.

From above said one could assume that $\Delta_n(0)=O(1/n)$ holds in the essentially infinite-dimensional case, i.e. when all $\sigma_j\ne 0$. However, it became clear in further researches that this phenomenon had place for finite dimensions d as well, at least for $d\ge 9$. This result was obtained by V.Yu. Bentkus and F. Götze [28]. Their bound has the following form:
$$\Delta_n(0) \le \frac{C(T)}{n}\frac{\beta_4}{\sigma^4}, \tag{14}$$
where

$C(T)=e^{c\sigma^2/\sigma_{13}^2}$ if $13\le d\le\infty$ and $\sigma_{13}\ne 0$;

$C(T)=\frac{\sigma^4}{\sigma_d^4}\,e^{c\sigma^2/\sigma_9^2}$ if $9\le d\le\infty$ and $\sigma_d\ne 0$;

$C(T)=e^{c\sigma^2/\sigma_9^2}$ if the distribution of the element X_1 is symmetric and $9\le d\le\infty$.

Comparing the bound (14) with (13), we see that the dependence on the covariance operator in (14) appears not very natural. The more precise bound for $\Delta_n(0)$ was found in our joint with V.I. Chebotarev work [19], namely,
$$\Delta_n(0) < \frac{c}{n}\left[\Gamma_{4,13}+\Gamma_{3,13}+L_9^2\Big(\sigma^2/\Lambda_9^{1/9}\Big)^2\right],$$

where $L_l \equiv \max\limits_{1\le j\le l} \frac{\mathbf{E}|(X,e_j)|^3}{\sigma_j^3}$. Note that $\Gamma_{\mu,l}/n^{(\mu-2)/2}$ is a generalization of the Liapounoff ratio $\beta_\mu/n^{(\mu-2)}\sigma^\mu$. We obtained a similar bound for the quantity $\Delta_{n,1}(a)$ which was defined by the equality (12) (see [20]).

6.3. Asymptotic expansions. Everyone who deals with the asymptotic expansions, comes up against two problems: first, one need to construct and describe an algorithm according to which coefficients of the asymptotic expansion are calculated; secondly, one need to estimate the error which appears, when employing the asymptotic expansion.

Now we describe the algorithm which is constructed in our work [18]. It is based on the formula

$$\mathbf{E}\exp\{(2s)^{1/2}(x,\alpha)\} = \exp\{s|x|^2\}, \quad x\in H_{\mathbb{C}}, \quad s\in\mathbb{C}. \tag{15}$$

Here $H_{\mathbb{C}}$ is the complex extension of the space H, $\alpha=(\alpha_1,\alpha_2,...)$ is the sequence of independent real standard normal values. The bilinear form (x,α) is defined as $\sum_{j=1}^\infty x_j\alpha_j$, where x_j are the coordinates of the vector x in some orthonormal basis.

Let, as before, $S_n=\sum\limits_{j=1}^n X_j$, where $X_1, X_2, \dots, X_n$ is a sequence of independent identically distributed random variables with values in H and $\mathbf{E}X_1=0$. Let $\{X_j\}_{j=1}^n$ and α are independent. By (15),

$$\mathbf{E}\exp\{it|S_n|^2\} = \mathbf{E}_{S_n}\mathbf{E}_\alpha\exp\{(2it)^{1/2}(S_n,\alpha)\}.$$

Without loss of generality one can consider the random variables X_j are bounded. This allows to interchange $\mathbf{E}_{S_n}$ and $\mathbf{E}_\alpha$ on the right-hand side of the last inequality:

$$\mathbf{E}\exp\{it|S_n|^2\} = \mathbf{E}_\alpha\mathbf{E}_{S_n}\exp\{(2it)^{1/2}(S_n,\alpha)\}.$$

As a result, we have

$$(S_n,\alpha)=\sum_{j=1}^n (X_j,\alpha),$$

where the linear forms (X_j,α) are one-dimensional independent identically distributed random variables if α is fixed. Therefore, applying the classic Edgeworth expansion in $\mathbb{R}$, we can write the following formal expansion

$$\mathbf{E}_{S_n}e^{s(S_n,\alpha)} = e^{s^2\sigma^2/2}\sum_{j=0}^\infty n^{-j/2}p_j(s;(X_1,\alpha)),$$

where $p_j(s;(X_1,\alpha))$ is the polynomial with respect to s, the coefficients of which depend on semi-moments of r.v. (X_1,α), $\sigma^2=\sum_1^\infty\sigma_j^2\alpha_j^2$. Putting $s=(2it)^{1/2}$ and then averaging in α, we arrive at the formal Edgeworth expansion

$$\mathbf{E}\exp\{it|S_n|^2\} = g(t)\sum_{j=0}^\infty n^{-j/2}\mathbf{E}p_j((2it)^{1/2};(A_tX,\ \alpha)).$$

Here $g(t)=\prod\limits_{j=1}^\infty(1-2it\sigma_j^2)^{-1/2}$, and operator A_t is defined by the equality

$$A_tx=\sum_1^\infty(1-2it)^{-1/2}(x,e_j)e_j,$$

where $\{e_j\}$ is the orthonormal basis in H which is generated by the eigenvectors of the covariance operator of r.v. X_1.

The advantage of the approach described above is that it establishes direct link with the classical Edgeworth expansion. An approach different from ours is used in cited works of other authors: it goes back to the works by Götze [6] and Bentkus [11]. As to estimating the remainder term, the previous investigations were directed in main to obtaining a bound of the remainder under minimal assumptions with respect to moments of the initial distribution. However, in infinite-dimensional case the form of dependence of the remainder on the covariance operator of summands plays the important role.

Our aim was to find an explicit form of this dependence reducing, in addition, the number of eigenvalues of the covariance operator involved in the bound to the minimum.

As in one-dimensional case, except for moment restrictions, one need to impose additional ones to the characteristic functional for obtaining an acceptable bound for the remainder. Our condition is a generalization of the well-known Cramer condition. It is close to the condition (1.1) in [11].

6.4. Large deviations for the Gaussian random values in a Banach space. We start with the description of the works [21]– [23], which we mentioned said in Introduction.

It was established in the work by H. Landau and L. Shepp [21] that

$$\mathbf{P}(|X| \geq t) \leq 1 - \Phi(at/b) \tag{16}$$

for every a and b satisfying the condition

$$\mathbf{P}(|X| < b) = \Phi(a) > \frac{1}{2},$$

where Φ is the standard Gaussian law. The proof of the inequality (16) is based on the isoperimetrical inequality on the sphere in $\mathbb{R}^n$ from which the extreme property of the half-space with respect to the Gaussian measure follows in the class of convex sets in $\mathbb{R}^n$.

In the work by Fernique [22] we find the bound

$$\mathbf{P}(|X| > t) < \mathbf{P}(|X| \leq t_0) \exp\{-t^2\gamma/(24t_0)\}, \tag{17}$$

where $\gamma = \mathbf{P}(|X| \leq t_0)\mathbf{P}(|X| > t_0)$, t_0 is an arbitrary positive number. An elegant mode via which Fernique deduces (17) is based on the invariant property of the standard Gaussian distribution in $\mathbb{R}^2$ with respect to a rotation.

At last, the third method was suggested by A.V. Skorokhod [23] who obtained a bound which was less sharp, in the sense of dependence on t, than (16) and (17), namely

$$\mathbf{P}(|X| > t) < \exp\{-\eta t\}, \tag{18}$$

where $\eta > 0$ was a constant.

The starting point in the work of Skorokhod is the inequality

$$\mathbf{P}(|X| > r) < \mathbf{P}(\sup_{0 \leq t \leq 1} |X(t)| > r), \tag{19}$$

where $X(t)$ is the Wiener process satisfying the condition $X(1) = X$. On the other hand, denoting $\bar{X} = \sup_{0 \leq t \leq 1} |X(t)|$ for brevity, we have

$$\mathbf{P}(\bar{X} > nr) < \mathbf{P}^n(\bar{X} > r). \tag{20}$$

The inequality (18) follows from two latter bounds.

As to complication of the proof, the paper by Landau and Shepp exceeds much more the works of Fernique and Skorokhod, but the more precise result is obtained in it.

Later on, V.N. Sudakov and B.S. Tsirel'son, using a method, similar to one of Landau and Shepp, had shown [29] that

$$\mathbf{P}(|X| < t) = \Phi((t - d + o(1))/c), \quad t \to \infty,$$

where $c > 0$ and $d \geq 0$ are some constants.

Starting point in my paper [31] is the inequality (19), as in that of Skorokhod, but further I use a more precise bound for the probability $\mathbf{P}(\bar{X} > r)$ than (20). The proof of this bound is given below.

Let $\tau_n = \inf\{t : |X(t)| = n\lambda_0\}, \ \ n \geq 1$. In other words, τ_n is the time when the process $X(t)$ reaches the sphere $|x| = n\lambda_0$ first. Put $\tau_0 = 0$. Let further

$$\tau_n' = \inf\{t - \tau_{n-1} : t > \tau_{n-1}, \ |X(t) - X(\tau_{n-1}| = \lambda_0\}.$$

Since $X(t)$ have the strong Markov property, random variables τ_n' are mutually independent and coincide with τ_1 in distribution.

Obviously,

$$|X(\tau_n) - X(\tau_{n-1})| \geq |X(\tau_n') - X(\tau_{n-1}| = \lambda.$$

Consequently, $\tau_n' \leq \tau_n - \tau_{n-1}$. Hence,

$$\tau_n = \sum_1^n (\tau_k - \tau_{k-1}) \geq \sum_1^n \tau_k'. \tag{21}$$

Put $X_\alpha(t) = X(\alpha t)/\sqrt{\alpha}, \ \alpha > 0$. It is not difficult to see that processes $X(t)$ and $X_\alpha(t)$ generate the same measure in the space of continuous functions defined on $[0, 1]$ and taking values in B. Further,

$$\mathbf{P}(\tau_n < t) = \mathbf{P}(\sup_{0\leq s\leq t} |X(s)| \geq n\lambda_0) = \mathbf{P}(\sup_{0\leq s\leq t} |X_{1/n^2}(s)| \geq n\lambda_0) =$$

$$= \mathbf{P}(\sup_{0\leq s\leq t} |X(s/n^2)| \geq \lambda_0) = \mathbf{P}(\sup_{0\leq s\leq t/n^2} |X(s)| \geq n\lambda_0) = \mathbf{P}(\tau_1 < t/n^2).$$

Thus,

$$\mathbf{P}(\tau_n < t) = \mathbf{P}(\tau_1 < t/n^2). \tag{22}$$

In view of (21) and (22)

$$\mathbf{P}\left(\frac{1}{n^2}\sum_1^n \tau_k' > t\right) < \mathbf{P}(\tau_1 > t). \tag{23}$$

It is not difficult to obtain herefrom that

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{1}{n^2}\sum_1^n \tau_k' < t\right) = G(t), \tag{24}$$

where $G(t)$ is the stable law with the parameter $\alpha = 1/2$, logarithm of the characteristic function of which admits the representation

$$\ln f(t) = -c|t|^{\alpha}\Big(1 + i\frac{t}{|t|}\Big).$$

As it is well-known (see, for instance, [33, P. 185] or [34, P. 79]), for $c = 1$

$$G'(t) = \frac{1}{\sqrt{2\pi}\, t^{3/2}} e^{-1/(2t)} = -2\frac{\partial}{\partial t}\Phi(1/\sqrt{t}).$$

Hence, for arbitrary $c > 0$

$$G'(t) = \frac{x_0}{\sqrt{2\pi}\, t^{3/2}} e^{-x_0^2/2t} = -2\frac{\partial}{\partial t}\Phi(x_0/\sqrt{t}), \tag{25}$$

where $x_0 = c$. It follows from (22) – (25) that

$$\mathbf{P}(\tau_n < 1) < G\Big(\frac{1}{n^2}\Big) = 2\big(1 - \Phi(nx_0)\big).$$

Recollecting the definition τ_n, we can state that

$$\mathbf{P}(|X| > n\lambda_0) < \mathbf{P}(\sup_{0\le t\le 1} |X(t)| > n\lambda_0) < 2\big(1 - \Phi(nx_0)\big), \tag{26}$$

where x_0 is defined by the equality

$$2\big(1 - \Phi(x_0)\big) = \mathbf{P}(\tau_1 < 1).$$

The inequality (26) is true for the discrete series of the points $u_n = n\lambda_0$. It is obvious that for $u_n < u < u_{n+1}$

$$\mathbf{P}(|X| > u) < 2(1 - \Phi(nx_0)) < 2\bigg(1 - \Phi\Big(\Big(\frac{u}{\lambda_0} - 1\Big)x_0\Big)\bigg)$$

or

$$\mathbf{P}(|X| > u) < 2\bigg(1 - \Phi\Big(\frac{x_0}{\lambda_0}u - x_0\Big)\bigg).$$

6.5. Probability inequalities for sums of independent random variables in Banach spaces. The upper bounds for the probabilities of large deviations of the sum $S_n = \sum_{i=1}^n X_i$ of independent random variables with values in a separable Banach space are deduced in my papers [35] and [36]. Here I formulate the key result of the paper [36].

Thus, let $S_k = \sum_1^k X_i,\ M_n = \max\limits_{1\le k\le n} |S_k|$, and α is a number such that $\mathbf{P}(2M_n \ge \alpha) < 1$. Then for every $y \ge \alpha$, $1 > \delta_1 \ge \delta$

$$\mathbf{P}(M_n \ge y) \le \sum_1^{k_\alpha - 1} kP\Big(\frac{(k_\alpha - k)\alpha}{k}\Big)\delta^{k-1} + \delta^{k_\alpha} \le$$

$$\leq \delta_1^{-2} \int_0^{y/\alpha - 1} u\delta_1^u P\left(\frac{y-(u+1)\alpha}{u}\right) du + \delta_1^{y/\alpha - 2} y/\alpha, \tag{27}$$

where $k_\alpha = [y/\alpha]$, $P(y) = \min\left\{\delta, \sum_1^n (\mathbf{P}(|X_i| \geq y)\right\}$, $\sum_1^0 = 0$.

I would like to call reader's attention to the fact that bound (27) is new in one-dimensional case as well. It is distinguished by absence of any moment characteristic. If

$$\lim_{n\to\infty} \sum_{i=1}^n \mathbf{P}(|X_i| > \alpha) = 0, \tag{28}$$

then by (27) the sequence of the distribution functions $\mathbf{P}(M_n/\alpha < y)$ is compact. Using this fact, it is not difficult to show that under condition (28) the sequence $\mathbf{P}(M_n/\alpha < y)$ has limit which is a proper distribution function.

The following bound is used as a prototype of the inequality (28),

$$\mathbf{P}(M_n > lB_n + (l-1)c) \leq [\mathbf{P}(M_n > B/2)]^l,$$

where $B_n^2 = \sum_{j=1}^n \mathbf{E}|X_j|^2$, X_i $(|X_i| < c)$ are bounded r.v.'s with values in a Banach space. One can find it in the monograph of I.I. Gikhman and A.B. Skorokhod [38] (see Ch. 6, § 3, Lemma 2).

It is necessary to say that the case of unbounded summands is incomparably more difficult from point of view of proving probability inequalities. The works [35] and [36] are distinguished just with the ways of overcoming difficulties which arise here.

There are deduced many corollaries from the inequality (28) in the paper [36], in particular, the inequality of the Rosenthal type for M_n, namely, *for every* $t \geq 1$

$$\mathbf{E}M_n^t \leq c_1(t,\delta)(A_t + t\alpha^t),$$

where $c_1(t,\delta) = 2^{t-1}\gamma(t+2)\delta^{-3}(-\ln\delta)^{-t-2}$.

An alternative approach to deriving probability inequalities in Banach space originates from the work by V.V. Yurinsky [39]. It is based on the expansion

$$|S_n| - \mathbf{E}|S_n| = \sum_1^n Y_k, \tag{29}$$

where

$$Y_k = \mathbf{E}\{|S_n| \,\big|\, \mathcal{F}_k\} - \mathbf{E}\{|S_n| \,\big|\, \mathcal{F}_{k-1}\},\ 1 \leq k \leq n,$$
$$\mathbf{E}\{|S_n| \,\big|\, \mathcal{F}_0\} = \mathbf{E}|S_n|,$$

moreover, for every $t > 0$

$$(\mathbf{E}\{|Y_k|^t \,\big|\, \mathcal{F}_{k-1}\} \leq 2^t \mathbf{E}|X_k|^t). \tag{30}$$

Here $\mathcal{F}_k$ is the σ-algebra generated by the r.v.'s $X_1, X_2, ..., X_k$.

Note that r.v.'s Y_k form the martingale difference. Therefore expansion (29) combined with the bound (30) allows to apply probability inequalities for martingales (as to the latters, see, for instance, [40] and [41]). For the first time, the attention to this fact was called in [42] and, almost simultaneously, in [43].

Just the same arguments were used later in [44] later. Under approach described just now it is more natural and convenient to deduce inequalities for $\mathbf{P}\big(\big|S_n - \mathbf{E}|S_n|\big| > y\big)$ and $\mathbf{E}\big|S_n - \mathbf{E}|S_n|\big|^t$ than for $\mathbf{P}(|S_n| > y)$ and $\mathbf{E}|S_n|^t$. Just this is done in the works [42] and [44] mentioned already.

My work [37] is devoted to lower bounds for the probability $\mathbf{P}(|S_n| > u)$. The following inequality is obtained in it: *for every* $\alpha > 0$

$$\mathbf{P}(|S_n| > u) \geq \sum_{k=1}^{n} \Big[\inf_f \mathbf{P}\Big(f(S^k) \geq (1-\alpha)u\Big) - \sum_1^n \mathbf{P}(|X_j| \geq \alpha u)\Big] \mathbf{P}(|X_k| \geq \alpha u), \qquad (31)$$

where $S^k = S_n - X_k$, *and the greatest lower bound is taken in all functionals* f *belonging to space* B^* *with* $|f| = 1$.

To estimate $\mathbf{P}\big(f(S^k) \geq (1-\alpha)u\big)$ from below one can use different probability inequalities for one-dimensional r.v.'s.

Let, for example, $\mathbf{E}X_j = 0,\ j = \overline{1,n}$. Then, estimating $\mathbf{P}(f(S^k) < 1 - \alpha)$ for $\alpha > 1$ with the help of the Cantelli inequality, we obtain that

$$\mathbf{P}(|S_n| \geq u) \geq \bigg(1 - B^2u^{-2}\Big(\frac{1}{\alpha^2} + \frac{1}{B^2u^{-2} + (\alpha-1)^2}\Big)\bigg) \sum_1^n \mathbf{P}(|X_j| \geq \alpha u).$$

If r.v.'s are symmetrically distributed, then, putting $\alpha = 1$ in (31), we have

$$\mathbf{P}(|S_n| \geq u) \geq \bigg(\frac{1}{2} - \sum_1^n \mathbf{P}(|X_k| > u)\bigg) \sum_1^n \mathbf{P}(|X_k| > u).$$

Evidently, this inequality is not trivial if $\sum_1^n \mathbf{P}(|X_k| > u) < 1/2$.

6.6. Probability inequalities for sums of dependent r.v.'s in Banach spaces. The next step was extending results of my work [36] to the case of weakly dependent summands X_j. It was done in my paper [45]. Let

$$\phi(m) = \sup\bigg\{\bigg|\frac{\mathbf{P}(AB)}{\mathbf{P}(A)} - \mathbf{P}(B)\bigg| :\ 1 \leq k \leq n-m,\ A \in \mathcal{F}_1^k,\ B \in \mathcal{F}_{k+m}^n,\ \mathbf{P}(AB) \neq 0\bigg\}$$

be a uniform mixing coefficient. Here $\mathcal{F}_j^k$ denotes a σ-algebra generated by r.v.'s $X_l,\ l = \overline{j,k}$. Let $\phi(1) < 1$, and $\delta > 0$ satisfy the condition $\delta + \phi(1) < 1$. Put $\rho = \delta + \phi(1)$. Let α be a number such that $\mathbf{P}\{2M_n > \alpha\} < \delta$. Define $Q(r) = \sum_1^n \mathbf{P}\{|X_j| > r\},\ A_t = \sum_1^n \mathbf{E}|X_j|^t$.

Let us formulate the upper bound for $\mathbf{P}\{M_n > r\}$ obtained in the work [45]: *For every* $r > \alpha$ *and* $0 < \varepsilon < 1/6$

$$\mathbf{P}(M_n > r) < \frac{2}{\alpha\rho} \int_0^r Q\Big(\frac{r\alpha\varepsilon^2}{2u}\Big) \frac{du}{(1 + \varepsilon u/\alpha)^{s(\varepsilon)+1}} + \rho^{-1}\Big(1 + \frac{\varepsilon r}{\alpha}\Big)^{-s(\varepsilon)}, \qquad (32)$$

where $s(\varepsilon) = -\ln\rho/\ln(1+\varepsilon)$.

Note that the integral on the right-hand side of (32) is a convolution of two functions defined on multiplicative group of positive real numbers.

If $\phi(1) = 1$ but $\phi(2) < 1$, then one can consider two sequences $X_1, X_3, ..., X_{2k+1}, ..$ and $X_2, X_4, ..., X_{2k}, ...$. For each of them $\phi(1) < 1$. If M'_n and M''_n are defined for the first and second subsequences, respectively, then

$$\mathbf{P}(M_n > r) < \mathbf{P}\Big(M'_n > \frac{r}{2}\Big) + \mathbf{P}\Big(M''_n > \frac{r}{2}\Big).$$

Obviously, this approach can be applied in the case $\phi(k) = 1,\ 1 \le k < m,\ \phi(m) < 1$ as well. The following moment inequality is extracted from (32): *for every* $t > 0$ *and* $0 < \varepsilon < 1/6$, *such that* $s(\varepsilon) > t$,

$$\mathbf{E}M_n < c_1(t) + c_2(t)\alpha^t, \tag{33}$$

where

$$c_1(t) \ge \frac{2^{t+1}}{\varepsilon^{3t+1}\rho} B(t+1, s(\varepsilon) - t + 1), \qquad c_2(t) \ge \rho^{-1}\varepsilon^{-t} t B(t, s(\varepsilon) - t),$$

$B(\cdot,\cdot)$ *is Euler function.*

Inequality (32) principally differs from the previous probability inequalities for sums dependent r.v.'s both by form and method of proving. First of all, only one from the countable number of mixing coefficients involves in it. It does not contain any moments due to introducing the quantile α. The constants in the right-hand side of the inequality are calculated explicitly. The aforesaid also concerns moment inequality (33). Inequality (33) is universal in that sense, it allows to cover the case $0 < t < 2$, $\mathbf{E}X_j \ne 0$ as well.

A special case of Hilbert space is studied in [45] separately. If $B = H$, where H is separable Hilbert space and $\mathbf{E}X_j = 0$, then for $t > 2$ the following inequality holds:

$$\mathbf{E}M_n^t < c_1(t,\phi)A_t + c_2(t,\phi)\mathbf{E}|Y|^t\beta_t^{-1}, \tag{34}$$

where Y is a Gaussian r.v. in H with zero mean and the same covariance operator as S_n, β_t is the absolute moment of order t of one-dimensional standard Gaussian law.

It is further proved in [45] that

$$\mathbf{E}|Y|^t < (\mathbf{E}|Y|^2)^{t/2}\beta_t. \tag{35}$$

The last inequality can be considered as isoperimetrical. It shows that a maximum of absolute moment of order $t > 2$ of the norm of the Gaussian vector with fixed second moment is achieved on a one-dimensional distribution.

Using (35), we can rewrite inequality (34) in the form

$$\mathbf{E}M_n^t < c_1(t,\phi)A_t + c_2(t,\phi)(\mathbf{E}|Y|^2)^{t/2},\ t > 2.$$

The standard approach to deriving probability inequalities is based on an application of inequalities of the Rosenthal type. In order to derive these inequalities, the approach is used which is based on a representation of the even moment of the sum or the norm of the sum of r.v.'s as the sum of mixing moments. In addition, as a rule, $\sum_1^n (\mathbf{E}|X_j|^{t+\varepsilon})^{t/(t+\varepsilon)}$ is used instead of A_t. An exception is work by S.A. Utev [46]. He applies the above mentioned direct approach in the case of Hilbert space, but A_t is not replaced by $\sum_1^n (\mathbf{E}|X_j|^{t+\varepsilon})^{t/(t+\varepsilon)}$. The combinatorial arguments of Utev are very complicated in contrast to the proof of inequality (34) in [45].

The inequalities obtained in [45] permit us, in particular, to modify many results concerning an estimate of the rate of convergence in the law of large numbers for sequences of r.v.'s with uniform mixing.

The survey of probability and moment inequalities for random processes and fields with mixing and its applications is contained, for instance, in [47]. We also mention in this connection the paper by Rio [48], where the Bennet – Hoeffding and Nagaev – Fuk inequalities are extended to a sum of random variables satisfying the strong mixing condition.

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