7. Martingales and supermartingales

7.1. Introduction. A variety of inequalities have a significant place in the theory of martingales and supermartingales with the discrete time. The first inequalities were deduced by the founder of the theory of martingales J.L. Doob (see [1]).

In what follows, we denote by S_k, $k \geq 1$, a supermartingale defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_k)_{k\geq 0}, \mathbf{P})$ with $S_0 = 0$, $\mathcal{F}_0 = \{\varnothing, \Omega\}$, i.e.

$$\mathbf{E}\{S_k \,|\, \mathcal{F}_{k-1}\} \leq S_{k-1}.$$

Put $X_k = S_k - S_{k-1}$, $k \geq 0$. Define the random variables σ_k^2 by the equalities

$$\sigma_k^2 = \mathbf{E}\{X_k^2 \,|\, \mathcal{F}_{k-1}\}.$$

Denote

$$B_k^2 = \sum_1^k \sigma_j^2, \quad \overline{S}_n = \max_{1\leq k\leq n} S_k, \quad \overline{X}_n = \max_{1\leq k\leq n} X_k, \quad A_t = \sum_1^n \mathbf{E}|X_j|^t.$$

Define

$$Q(x) = \mathbf{P}(\overline{X}_n > x) + \mathbf{P}(B_n > x).$$

As in the case of independent summands one can distinguish two types of inequalities: **a)** moment inequalities, i.e. inequalities for $\mathbf{E}f(S_n)$, where f is a function satisfying some restrictions, **b)** probability inequalities, i.e. bounds for $\mathbf{P}\big(f(S_n) > x\big)$. The simplest case is that of $f(y) = |y|^t$, $t > 0$.

(A) Moment inequalities. We start with so called comparison inequalities for martingales obtained in 1966 . by D.L. Burkholder [2],

$$c_t\Big(\sum_1^n \mathbf{E}X_k^2\Big)^{t/2} < \mathbf{E}|S_n|^t < C_t\Big(\sum_1^n \mathbf{E}X_k^2\Big)^{t/2}, \tag{1}$$

where c_t and C_t are some constants. Of course, one can write these inequalities in the form

$$C_t^{-1}\mathbf{E}|S_n|^t < \mathbf{E}\Big(\sum_1^n X_k^2\Big)^{t/2} < c_t^{-1}\mathbf{E}|S_n|^t. \tag{2}$$

Inequality (1) extends to martingales the well-known inequalities due to Marcinkiewicz – Zygmund [3] for independent random variables. In 1973 Burkholder [4] obtained for martingales the next extension of the Rosenthal inequality [5],

$$k_t\Big(D_t^{1/t} + \mathbf{E}^{1/t}B_n^t\Big) < \mathbf{E}^{1/t}\big|\widehat{S}_n\big|^t < K_t\Big(D_t^{1/t} + \mathbf{E}^{1/t}B_n^t\Big), \tag{3}$$

where $\widehat{S}_n = \max_{1\leq k\leq n} S_k$. The variable B_n in (3) is random in contrast to the Rosenthal inequality. Thus, the special problem of estimating the expectation $\mathbf{E}B_n^t$ arises. If the conditional variances σ_k^2 admit the uniform bound

$$\sigma_k^2 < b_k^2, \tag{4}$$

where b_k^2 is some sequence of constants then

$$\mathbf{E}B_n^t < \Big(\sum_1^n b_j^2\Big)^{t/2}.$$

In Burkholder's paper the constants k_t and K_t are not estimated. A step forward in this direction was made by P. Hitchenko [6] who proved that

$$K_t < K\,\frac{t}{\ln t}, \tag{5}$$

where K is an absolute constant. In my paper [7] the upper bound of the Burkholder type is deduced for the moments $\mathbf{E}\Big\{S_n^t; S_n \geq 0\Big\}$ of supermartingales S_n with the constant K_t satisfying inequality (5). This bound is discussed comprehensively in Section 7.3. Moreover, the numerical bound for the constant K is obtained in my next paper [8]. A short time later, the latter was sharpened by E.L. Presman [9].

The detailed survey of moment inequalities is contained in [10].

(B) Probability inequalities. As to probability inequalities for martingales the case of bounded martingale differences $X_k < L$ or $|X_k| < L$ satisfying in addition condition (4) is studied for the most part. The point is that in this case the generating function of moments $\mathbf{E}e^{hS_n}$ admits, in essence, the same bounds as in the case of independent summands X_k. This allows to get for $\mathbf{P}(S_n > x)$ the bounds of the Hoeffding and Bernstein type. The papers [11,12] were the firsts in this direction. The papers [13–16] are devoted to generalizing the Hoeffding and Bernstein inequalities. The probabilities of large deviations of S_n are studied in [17] under condition $\max\limits_{1\leq k\leq n} \mathbf{E}|X_k|^t < \infty$.

In Bentkus's paper [18] the probabilities $\mathbf{E}(S_n > x)$ are estimated in terms $\mathbf{E}(Z_n > x)$, where Z_n is a sum of independent identically distributed Bernoulli random variables chosen in a proper way. The bounds obtained are compared with the Hoeffding inequalities.

In my paper [7] the bound of the new type

$$\mathbf{P}(\overline{S}_n > x) < c(t)x^{-t}\int_0^x Q(\varepsilon_t u)u^{t-1}\,du$$

was obtained, where $c(t)$, ε_t are constants which are defined below in Section 7.2. In the next sections I comprehensively describe the probability and moment inequalities obtained in my papers [7, 8, 19].

7.2. Probability inequalities. After appearance in 1971 the Nagaev – Fuk inequalities (see [20]) the problem arose to generalize these inequalities to martingales. The first step to this direct was made by D.Kh. Fuk [21] in 1973 under assumption that for some sequence $\{y_k\}_{k\geq 1}$, $y_k > 0$,

$$\mathbf{E}\{X_k^2(y_k)\,|\,\mathcal{F}_{k-1}\} < d_k^2, \qquad \mathbf{E}\{(X_k^+)^t(y_k)\,|\,\mathcal{F}_{k-1}\} < a_k, \tag{6}$$

where d_k^2 and a_k are constants, $t > 2$,

$$X_k(y) = \begin{cases} X_k, & X_k \leq y, \\ 0, & X_k > y, \end{cases}, \quad X_k^+(y) = \max\{0,\, X_k(y)\}.$$

These restrictions can seem too strong. It turned out, however, that they are fulfilled, in particular, for the martingale

$$\mathbf{E}\Big\{\Big\|\sum\nolimits_{j=1}^{n} X_j\Big\|\,\Big|\,\mathcal{F}_k\Big\},$$

where X_j are independent random variables, taking values in a separable Banach space, $\mathcal{F}_k$ being σ-algebra generated by random variables $X_1, X_2, \ldots, X_k$, provided

$$\mathbf{E}\|X_j\|^t < \infty, \qquad j \in \overline{1,n}$$

(see, in this connection, [22–24]).

If the martingale S_n does not satisfy Fuk's conditions, one can attain this under some restrictions by means of appropriate transformation $f(S_n)$. As applied to Galton – Watson process, this is made in [25]

In the work [26] one of Fuk's inequalities, which contains normal component, is generalized to Banach space under assumption that

$$\mathbf{E}\|X_j\|^3 < \infty, \qquad j \in \overline{1,n}.$$

In addition a restriction of the same type as Fuk's one is imposed upon the conditional second moments.

Haesler [27] generalized one of Fuk's inequalities as follows: *for any* $x,\ u,\ v > 0$

$$\mathbf{P}(\overline{S}_n > x) < \sum_{i=1}^{n} \mathbf{P}(X_i > u) + \mathbf{P}(B_n > v) + P_0(x,u,v), \tag{7}$$

where

$$P_0(x,u,v) = \exp\Big\{\frac{x}{u}\Big(1 - \ln\Big(\frac{xu}{v^2}\Big)\Big)\Big\}.$$

In [28] this result is extended to continuous-time martingales. In [29] P_0 is replaced by for

$$P_1(x,u,v) = \exp\Big\{\frac{x}{u} - \Big(\frac{v^2}{u^2} + \frac{x}{u}\Big)\ \ln\Big(\frac{xu}{v^2} + 1\Big)\Big\}. \tag{8}$$

The bound (7) is applied in [27] for estimating the rate of convergence in the functional CLT for discrete-time martingales. The probability inequalities in [28] and [29] are used in similar way.

The following inequalities were obtained in my work [7].

Theorem 1. *Let* $0 < \gamma \le 1$ *and* t *satisfy the condition* $t \ge \max(e^6, e^4/\gamma^2)$. *Then for every* $y > 0$

$$\mathbf{P}(\overline{S}_n > y) < c(t,\gamma) y^{-t} \int_0^y Q(\varepsilon_t u) u^{t-1} du, \tag{9}$$

where

$$\varepsilon_t = \frac{\ln t - 2\ln\ln t}{2t},\ \ c(t,\gamma) = \frac{2e^{6\gamma t}}{\gamma}.$$

If $\varepsilon_t = \eta/t,\ \eta > 0$, *then the inequality*

$$\mathbf{P}(\overline{S}_n > x) < c_1(t,\eta)\, x^{-t} \int_0^x Q(\varepsilon_t u)\, u^{t-1}\, du \tag{10}$$

holds for every $t > 0$, where $c_1(t,\eta) = te^{3\eta\alpha(\eta)}/\eta\alpha(\eta),\ \ \alpha(\eta) = e^{\eta+1}$.

Note that

$$\frac{1}{x^t}\int_0^x Q(\varepsilon_t u)\, u^{t-1}\, du = \int_0^1 Q(\varepsilon_t sx)\, s^{t-1}\, ds$$

which means that right-hand side of (9) decreases in x.

The bound (9) extends the inequality from Theorem 4 of the paper [20] (see also [30, Theorem 1.10]). The inequality (9) is closed in form to the main inequality of the paper [31]. The method of the proof is similar to that which used in the papers [31–33].

Naturally the question arises, how Theorem 1 associates with Haesler's inequality (7). Since the inequalities (7) and (9) strongly differ in form it is not so simply to compare them. It is shown in [7] that (9) is not a corollary of (7). Analogues considerations show that it is impossible to derive Burkholder's inequality (15) (which is a generalization of Rosenthal's one) via that of Haesler. The proof of probability inequalities obtained in [7], was later modified in my paper [8], namely, two statements were selected from the former proof, which makes the presentation more transparent. However they are of interest independently.

Proposition 1. *For every $x > 0,\ \ y > 0,\ \ \alpha > 1$ the following inequality holds,*

$$f(x+\alpha y) < f(x)\, e^{\alpha(1-\ln\alpha)} + Q(y), \tag{11}$$

where $f(x) = \mathbf{P}(\overline{S}_n \ge x)$.

Proposition 2. *If the function $f(x)$ does not increase and for every $x > 0,\ \ y > 0,\ \ \alpha > 1$ satisfies inequality* (11), *then for every $\alpha > 1\ \ \varepsilon > 0$*

$$f(x) < \frac{\omega(\alpha,\varepsilon)}{x^{s(\alpha,\varepsilon)}}\int_0^x Q(\varepsilon u)\, u^{s(\alpha,\varepsilon)-1}\, du,$$

where $s(\alpha,\varepsilon) = \alpha(\ln\alpha - 1)/\ln(1+\alpha\varepsilon)$, $\omega(\alpha,\varepsilon) = (\alpha\varepsilon)^{-1}e^{3\alpha(\ln\alpha-1)}$.

7.3. Moment inequalities. Moment inequalities are deduced easily from inequality (9) in the same way as for independent random variables (see [25, 30–32, 34] in this connection).

Indeed, by multiplying both sides of (9) for $t+1$ by tx^{t-1}, and integrating with respect to x from 0 to ∞, we obtain the following inequalities.

For every t and γ such that $t > \max(e^6, e^2/\gamma^2) - 1,\ \ 0 < \gamma \le 1$,

$$\mathbf{E}\{\overline{S}_n^{\,t};\ \overline{S}_n \ge 0\} < c(t+1,\gamma)\, \varepsilon_{t+1}^{-t}(\overline{D}_t + \mathbf{E}B_n^t), \tag{12}$$

where

$$\overline{D}_t = \mathbf{E}\{\overline{X}_n^{\,t}\ \overline{X}_n \ge 0\}.$$

Similarly, if $\varepsilon_t = \eta/t$, then for every $t > 0$

$$\mathbf{E}\{\overline{S}_n^{\,t};\ \overline{S}_n \ge 0\} < c_1(t+1,\eta)\, \varepsilon_{t+1}^{-t}(\overline{D}_t + \mathbf{E}B_n^t). \tag{13}$$

If $\{S_k\}_{k\ge1}$ being a martingale, the inequalities (12) and (13) remain valid for

$$\mathbf{E}\{|\widetilde{S}_n|^t;\ \widetilde{S}_n \le 0\},$$

where $\widetilde{S}_n = \min\limits_{1\le k\le n} S_k$, with replacement of $\overline{D}_t$ by

$$\widetilde{D}_t = \mathbf{E}\big\{|X_n|^t;\ \min_{1\le k\le n} X_k < 0\big\}.$$

By summing the inequalities for $\mathbf{E}\{\overline{S}_n^{\,t};\ \overline{S}_n \ge 0\}$ and $\mathbf{E}\{|\widetilde{S}_n|^t;\ \widetilde{S}_n < 0\}$, we conclude the following: *if S_n being a martingale, then for every $t > 0$ and $\eta > 0$,*

$$\mathbf{E}\widehat{S}_n^{\,t} < c_t(\eta)(D_t + 2\mathbf{E}B_n^t), \tag{14}$$

where $\widehat{S}_n = \max_{1\le k\le n} |S_k|$, $D_t = \mathbf{E}(\max_{1\le k\le n} |X_k|^t)$, $c_t(\eta) = c_1(t+1,\, \eta)((t+1)/\eta)^t$.

If $t > \max(e^6,\, e^4/\gamma^2) - 1$, $0 < \gamma \le 1$, *then inequality* (14) *holds with the constant* $c_t'(\gamma) = c(t+1, \gamma)\, \varepsilon_{t+1}^{-t}$.

By raising both sides of inequality (14) to the power $1/t$, we have for $t \ge 1$,

$$\mathbf{E}^{1/t}\widehat{S}_n^{\,t} < \widehat{c}_t(D_t^{1/t} + 2^{1/t}\mathbf{E}^{1/t}B_n^t), \tag{15}$$

where $\widehat{c}_t = c_t^{1/t}(\eta)$. If $t > e^6 \vee e^4/\gamma^2$, then one can take $(c_t'(\gamma))^{1/t}$ as $\widehat{c}_t$. As it was said above, the latter inequality was obtained by Burkholder [4] without any explicit expression for the constant $\widehat{c}_t$.

Since

$$\mathbf{P}(\overline{X}_n > x) \le \sum_1^n \mathbf{P}(X_j > x), \tag{16}$$

one may replace $\overline{D}_t$ in the inequality (12) by

$$A_t^+ = \sum_1^n \mathbf{E}\{X_j^t;\ X_j \ge 0\}.$$

Respectively in the inequality (14) one may replace D_t by A_t, making it more close in form to the Rosenthal inequality [5]. Sharp bounds in the Rosenthal inequality for $\mathbf{E}S_n^{2k}$, where S_n is the sum independent random variables with zero mean, are given in [35].

The inequalities (9) and (10) allow to estimate $\mathbf{E}\{g(\overline{S}_n); \overline{S}_n \ge 0\}$ for more extensive class of functions than power ones. We formulate one of such type possible results: *if a differentiable function $g(x)$ with $g(0) = 0$ satisfies the condition*

$$\frac{g'(x)}{x^{t-2}} < \alpha \frac{g'(y)}{y^{t-2}},\ t > 0,\ \alpha > 0,$$

then for every $\eta > 0$

$$\mathbf{E}\{g(\overline{S}_n); \overline{S}_n \ge 0\} < \alpha c_1(t,\eta)\Big(\mathbf{E}\{g(\varepsilon_t^{-1}\overline{X}_n); \overline{X}_n \ge 0\} + \mathbf{E}g(\varepsilon_t^{-1}B_n)\Big), \tag{17}$$

where $\varepsilon_t = \frac{\eta}{t}$. Indeed, in view of the second assertion of Theorem 1,

$$\mathbf{E}\{g(\overline{S}_n); \overline{S}_n \ge 0\} < c_2(t,\eta) \int_0^\infty \Big(x^{-t} \int_0^x Q(\varepsilon_t u)u^{t-1}du\Big) g_1'(u)dx \equiv c_2(t,\eta) I_t.$$

Changing the order of integration, we find

$$I_t = \int_0^\infty u^{t-1} Q(\varepsilon_t u)\Big(\int_u^\infty x^{-t} g'(x)\, dx\Big)\, du < \alpha \int_0^\infty Q(\varepsilon_t u) g'(u)\, du$$

$$= -\alpha \int_0^\infty g(u)\, dQ(\varepsilon_t u) = \alpha\Big(\mathbf{E}\{g(\varepsilon_t^{-1}\overline{X}_n); \overline{X}_n \geq 0\} + \mathbf{E} g(\varepsilon_t^{-1} B_n)\Big).$$

The desired result follows immediately from two last relations. The inequality of type (17) is deduced in [4] under slightly weaker restrictions on the function g.

Denote $\beta_t(n) = \mathbf{E}|X_n|^t$. The following bound is obtained in [19].

Let $\{S_k\}_0^\infty$ *be a martingale. Then for every* $t \geq 2$

$$\mathbf{E}|S_n|^t \leq c_t \Big(\sum_{j=1} n\beta_t^{2/t}(j)\Big)^{t/2}, \tag{18}$$

where

$$c_t = \Big(\frac{t(t-1)}{2}\Big)^{t/2}. \tag{19}$$

Note that the bound in (18) is achieved under $t = 2$. For independent random variables the inequality (18) was obtained by Whittle [36] with the constant

$$c_t' = \frac{2^{3t/2}}{\sqrt{\pi}} \Gamma\Big(\frac{t+1}{2}\Big).$$

Note that

$$c_4' \approx 36.11\, \Gamma\Big(\frac{5}{2}\Big) > c_4 = 36$$

but

$$c_5' \approx 204.31 < c_5 = 10^{2.5} \approx 316.22.$$

The following bound extends to martingales the inequality of Dharmadhikari and Jogdeo [37] (see, also, [38, P. 98]).

Let $\beta_t(j) = \beta_t$, $1 \leq j \leq n$. *Then for* $t > 2$

$$\mathbf{E}|S_n|^t < c_t n^{t/2} \beta_t,$$

c_t *being from* (19).

Using the well-known inequality

$$\mathbf{E}\widehat{S}_n^t < \Big(\frac{t}{t-1}\Big)^t \mathbf{E}|S_n|^t$$

(see, for example, [39, P. 526, Theorem 2]),we get the following statement: *for every* $t \geq 2$

$$\mathbf{E}\widehat{S}_n^t < c_t \Big(\sum_{j=1}^n \beta_t(j)\Big)^{t/2}, \tag{20}$$

where $c_t = \left(\frac{t^3}{t-1}\right)^{t/2}$.

If $1 \le t < 2$, then the following bound for martingale S_n holds,

$$\mathbf{E}|S_n|^t < c_t \sum_{j=1}^{n} \beta_t(j)$$

(see [40]).

Note that it is not difficult to get inequalities of type (20) for $\mathbf{E}|\widehat{S}_n|^{2k}$ from (15). Indeed, using the inequality

$$\mathbf{E}\prod_{l=1}^{m} \sigma_l^{2i_l} \le \prod_{l=1}^{m} \beta_{2k}^{i_l/2k}(l),$$

we have

$$\mathbf{E}B_n^{2k} < \left(\sum_{j=1}^{n} \beta_{2k}^{1/k}(j)\right)^k.$$

Hence,

$$\mathbf{E}|\widehat{S}_n|^{2k} < c_{2k}\left(A_{2k} + 2\left(\sum_{j=1}^{n} \beta_{2k}^{1/k}(j)\right)^k\right),$$

where $c_{2k} = \min\limits_{\eta} c_{2k}(\eta)$. Making this estimate crude, we can write

$$\mathbf{E}\widehat{S}_n^{2k} < 3c_{2k}\left(\sum_{j=1}^{n} \beta_{2k}^{1/k}(j)\right)^k.$$

7.4. Numerical estimates. P. Hitchenko [6] had shown that

$$\mathbf{E}\widehat{S}_n^{\,t} < K\,\frac{t}{\ln t}\,(D_t^{1/t} + \mathbf{E}^{1/t}B_n^t), \tag{21}$$

where K is an absolute constant (see also [41]).

In connection with the inequality (21) the bound of the quantity

$$K_t = \frac{\mathbf{E}^{1/t}\{\overline{S}_n^{\,t};\ S_n \ge 0\}}{\overline{D}_t^{1/t} + \mathbf{E}^{1/t}B_n^t}\,\frac{\ln t}{t}$$

is of interest.

Putting $\widehat{c}_t = (c_t'(\gamma))^{1/t}$ in (15), we arrive to the bound

$$\limsup_{t\to\infty} K_t \le 2e^{6\gamma}.$$

Since γ can be made arbitrarily small, we have the right to state that

$$\limsup_{t\to\infty} K_t \le 2. \tag{22}$$

It was shown in [42] that

$$\lim_{t\to\infty} \widehat{c}_t \frac{\ln t}{t} = \frac{1}{e}$$

if X_k are independent symmetrically distributed random variables.

Estimates for K_t under $t > 2$ are given in my work [8].

Let us introduce the definitions,

$$g_0(t) = \Big(1 + \frac{1}{t}\Big) \ln(t+1) - \ln t + \ln\ln t,$$
$$g_1(t) = g_0(t) + \frac{2.76}{t} + 1.39,$$
$$g_2(t) = g_0(t) + \frac{3.694}{t} + 1.064,$$
$$g_3(t) = g_0(t) + \frac{1.74}{\ln t - 1.1} - \Big(1 + \frac{1}{t}\Big) \ln(\ln t - 2.86) + 0.57\Big(1 + \frac{1}{t}\Big) + \frac{0.57}{t}\Big(2.86 - \ln t\Big) \quad \text{if} \quad \ln t > 2.86,$$
$$g_3(t) = \infty \quad \text{if} \quad \ln t \le 2.86.$$

The following result was obtained in [8]: *for every* $t > 2$

$$K_t < g(t) := \min_{1\le j\le 3} e^{g_j(t)}. \tag{23}$$

Analysis of behavior of the functions $g_i(t)$, $i = \overline{1,3}$, leads to the following deduction (see also Fig. 1 and 2). First of all, $g_1(t) < g_2(t)$ for $2 < t < t_0 = 2.865\ldots$, $g_1(t) > g_2(t)$ for $t > t_0$.

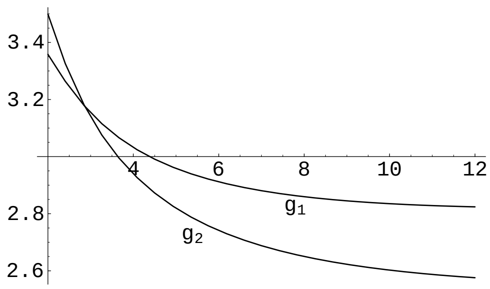

Fig. 1. $g_1(t) < g_2(t)$, $2 < t < t_0$; $g_1(t) > g_2(t)$, $t > t_0$

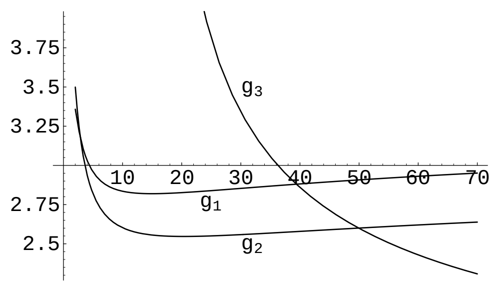

Fig. 2. $g_3(t) < g_2(t)$, $t > t_2$

As to the function g_3, it decreases, where $g_3(t) > g_2(t)$ for $e^{2.86} \equiv 17.46\ldots < t < t_2 = 49.936\ldots$, $g_3(t) < g_2(t)$ for $t > t_2$, t_2 being the root of the equation $g_2(t) = g_3(t)$. Thus,

$$K_t < \begin{cases} e^{g_1(t)}, & 2 < t < 3, \\ e^{g_2(t)}, & 3 \le t < t_2, \\ e^{g_3(t)}, & t > t_2. \end{cases}$$

In particular, the next result follows from the aforesaid,

$$\sup_{2<t<3} K_t \le e^{g_1(2)} \approx 29, \ \sup_{3\le t<4} K_t \le e^{g_2(3)} \approx 23.1, \ \sup_{t\ge 4} K_t \le e^{g_2(4)} \approx 18.9. \tag{24}$$

It is easily seen that $g_3(t) \to 0.57$ for $t \to \infty$. Consequently,

$$\limsup_{t\to\infty} K_t \le e^{0.57} < 1.77.$$

Similar bounds for independent X_i were obtained in [43] as well as in [44].

Recently Presman [9] proved that

$$\limsup_{t\to\infty} K_t \le 1, \quad K_t \le 9.46.$$

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