MINKOWSKI DUALITY AND ITS APPLICATIONS



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*Dedicated to*
*Leonid Vital'evich KANTOROVICH*
*on his sixtieth birthday*

# MINKOWSKI DUALITY AND ITS APPLICATIONS

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This article is an account of problems grouped around the concept of Minkowski duality – one of the central constructions in convex analysis. The article consists of an introduction, four sections, and a commentary.

In §1 we set out the main facts about $H$-convex elements and introduce the Minkowski–Fenchel and the Minkowski–Moreau schemes; we consider the space of $H$-convex sets. Here we collect together the main examples, namely the convex and sublinear functions, and the stable, normal, and convex sets in the sense of Fan, amongst others.

§2 is concerned mainly with representations of positive functionals over continuous $H$-convex functions and sets. Here we also establish the links between such constructions and the Choquet theory.

In §3 we introduce various characterizations of $H$-convexity in the form of theorems on supremal generators. In particular, we consider in detail theorems on the definability of convergence of sequences of operators in terms of their convergence on a cone. Other applications of supremal generators are also given.

In §4 problems of isoperimetric type (with an arbitrary number of constraints) in the geometry of convex surfaces are analyzed as problems of programming in a space of convex sets. We examine as particular examples exterior and interior isoperimetric problems, the Uryson problem, and others.

## Contents



## Introduction

Within the framework of modern functional analysis a new branch, convex analysis, has been added, a discipline that studies various problems relating to convex functions, convex sets, extremal problems, and so on [45]. There are standard traditional applications of convex analysis in mathematical programming, the constructive theory of functions, the calculus of variations, linear inequalities, and so on. A basic technique for working in convex analysis is the systematic use of various notions in duality.

The apparatus of Lagrange duality for extremal problems is set out, for instance, in [27], [96]. The theory of adjoint convex functions is treated in detail in [41], [91]. These articles shed light on traditional applications of convex analysis. A group of ideas in duality theory connected with mathematical models in economic dynamics is in [77]. For recent applications of adjoint functions to the theory of differential inequalities, the reader is referred to [72].

However, the theory of Minkowski duality, in spite of its great clarity and its many mathematical applications, has not been adequately sorted out.

This paper is an account of the fundamental methods and applications involving this basic simple algebraic object, namely Minkowski duality. In § 1 we explain the general Minkowski–Fenchel scheme relating to $H$-convex sets and $H$-convex functions, and we introduce various examples and realizations of this scheme. In § 2 we give the basic techniques of duality methods for representing $H$-convex functions. The central ideas are the decomposition theorem and the operator principle of preservation of inequalities.

The fields of application of the second method are discussed in § 3, where the principle of preservation of inequalities is deduced in the form of theorems on supremal generators (that is, theorems on the definability of a sequence of operators in terms of their convergence on a cone). The detailed account of dual local characterizations of $H$-convexity is due to the fact that a considerable literature is devoted precisely to this form of the question. § 4 concerns the application of the results obtained to what was really the objective that gave rise to Minkowski duality theory, namely to extremal problems in the geometry of convex sets. In particular, we give a method of analysis and solutions of a number of problems of isoperimetric type with an arbitrary number of constraints for which the symmetrization technique is not applicable on principle.

Our aim has been to give a fairly broad account of the problem of the relationship of Minkowski duality with other constructions of analysis, without cluttering the text with technical details. With this in mind, we have illustrated the theory with a large number of examples and have selected theorems so as to explain the main ideas using the minimal equipment, but providing interesting results.

Comments of a historical and bibliographical nature have on the whole been collected in the final section. In the paper we use an independent numbering within the parts of each section.

In cross-references we quote in addition the number of the corresponding section. The notation and results of the theory of partially ordered spaces is taken for granted (see [23], [46]).

The symbol $\stackrel{\Delta}{=}$ means "equal by definition".

## § 1. $H$-convex functions and sets

**1.0. Preliminary remarks.** The first account of convex functions as an independent object of study was apparently given by Jensen in [39], although the defining inequality for convex functions (now called Jensen's inequality) had previously been studied by Hölder [25]. The first mention of convex functions in a text-book is in [120] by Stolz.

At about the same time, various dual relationships between convex functions and convex sets were being discovered, the latter objects had been known long before the introduction of convex functions. The fundamental discovery was the elucidation of the special role of sublinear (that is, convex, homogeneous) functions, for the so-called gauge or support functions (the Minkowski functionals). The bulk of this discovery is due to Minkowski [80], who with his geometric insight and versatility successfully applied this class of functions in his numerous and diverse researches. The main work of Minkowski is gathered together in the two volumes "Gesammelte Abhandlungen von Hermann Minkowski", 1911, under Hilbert's editorship.

An important step forward in the development of Minkowski's views was Fenchel's theorem on the recovery of a convex surface from its support function. This result, nowadays called the Minkowski–Fenchel theorem, serves as a fundamental device for applying the theory of convex functions to the study of convex sets and, conversely, for using geometrical constructions in a number of problems in analysis.

More recently, especially in the context of Hörmander's work [113], it has become clear that the classes of convex (or sublinear) functions and convex sets are in a certain sense indistinguishable. Their common property, which plays a decisive role in a number of problems, is the fact that both classes are $H$-convex functions, that is, upper envelopes over a cone $H$ (for appropriate $H$). It turns out that one can construct in the same way other (sometimes very wide) classes of functions. Thus, the continuous functions of one variable do not differ in this sense from the convex functions of two variables. Therefore, to study certain properties, for instance, of continuous functions, one can apply the tools of convex analysis. The investigation of various classes of $H$-convex functions (and sets) by methods of convex analysis is, in fact, the essence of the theory of Minkowski duality.

In this section we collect together the basic facts about $H$-convex elements and give the most typical examples and constructions connected with the Minkowsky–Fenchel scheme.

**1.1. Minkowski Duality.** Consider a complete lattice $Y$ with minimum element $-\infty$. Let $X$ be a subset of $Y$ and $H$ a subset of $X$ such that

$-\infty \notin H$. We say that $p \in X$ is *H-convex* if there exists a set $U$ in $H$ such that $p = \sup U$. The collection of all $H$-convex elements of $X$ is denoted by $P(H, X, Y)$. For $p \in Y$ we write $U_p \stackrel{\Delta}{=} \{h \in H: h \leqslant p\}$. If $U \subset H$ and $p \stackrel{\Delta}{=} \sup U$, then $U \subset U_p$, and in addition, $p = \sup U_p$. An element $h \in U_p$ is called a *support element* to $p$ and the set $U_p$ itself is called the set of supports (or *support set*) to $p$.

A set $U$ in $H$ is called $H$-convex if $U = \{h \in H: h \leqslant \sup U\}$, that is, if $U$ is the support set to $p \stackrel{\Delta}{=} \sup U$. The collection of all $H$-convex sets which support an element of $X$ is denoted by $\mathfrak{B}(H, X, Y)$. (Note that $\mathfrak{B}(H, Y, Y)$ is the family of all $H$-convex sets.) We omit one of the letters $X$, $Y$ in the notation $P(H, X, Y)$ and $\mathfrak{B}(H, X, Y)$, whenever its role is clear from the context. We order $\mathfrak{B}(H)$ by inclusion. We endow the set $P(H)$ with the ordering induced by $Y$. It is clear that the mapping $\varphi: P(H) \to \mathfrak{B}(H)$ defined by

$$\varphi: p \mapsto U_p, \tag{1.1}$$

is an order isomorphism between $P(H)$ and $\mathfrak{B}(H)$. This isomorphism $\varphi$ is the so-called *Minkowski duality*.

Let $U \subset H$, and set $p_U \stackrel{\Delta}{=} \sup U$. The set $U_{p_U} \stackrel{\Delta}{=} \{h \in H: h \leqslant p_U\}$ is called the *H-convex hull* of $U$ and is denoted by $\mathrm{co}_H(U)$. Note that $\mathrm{co}_H(U) = U$ if and only if $U$ is $H$-convex. If $p_U \in X$ (where $H \subset X \subset Y$), then $\mathrm{co}_H(U) \in \mathfrak{B}(H, X, Y)$. The mapping $U \mapsto \mathrm{co}_H(U)$ defined on the family of all subsets of $H$ is a closure in the sense of Moore [8], that is, $\mathrm{co}_H(U) \supset U$, $\mathrm{co}_H(\mathrm{co}_H(U)) = \mathrm{co}_H(U)$, $\mathrm{co}_H(U_1) \supset \mathrm{co}_H(U_2)$ if $U_1 \supset U_2$. It follows from Moore's theorem that for the family $(U_\alpha)_{\alpha \in A}$ of elements of $\mathfrak{B}(H, Y, Y)$ we have $\bigwedge_{\alpha \in A} U_\alpha = \bigcap_{\alpha \in A} U_\alpha$, $\bigvee_{\alpha \in A} U_\alpha = \mathrm{co}_H(\bigcup_{\alpha \in A} U_\alpha)$.

The Minkowski duality shows that the family $P(H, Y, Y)$ of all $H$-convex elements of $Y$ is a complete lattice isomorphic to the lattice $\mathfrak{B}(H, Y, Y)$. Furthermore, the supremum of any family $(p_\alpha)_{\alpha \in A}$ of elements of the lattice $P(H, Y, Y)$ is the same as the supremum of the same family of points computed in $Y$. (The corresponding statements for the infimum is not true, in general.) If $X$ $(H \subset X \subset Y)$ is an upper sub-lattice (relative to the ordering induced by $Y$) such that the supremum of two elements of $X$ relative to $X$ is the same as the supremum relative to $Y$, then the set $P(H, X, Y)$ is also an upper sub-lattice; furthermore, the suprema of two elements of $P(H, X, Y)$, considered in $X$ and in $P(H, X, Y)$, respectively, coincide.

Before giving some examples we note that the main interest of the standard work on Minkowski duality lies in the description of $H$-convex elements and sets in terms of $H$ itself (and not of $X$). As a rule, $X$ is realized as a vector lattice of functions defined on a convex set $Q$, and $H$

is a vector subspace consisting of affine functions. In this case, $Y$ is taken to be the set $\tilde{R}^Q$ of all functions $f\colon Q \to [-\infty, +\infty]$, with the natural ordering. The required description for this situation is obtained by means of various separation theorems. We now explain the connection between $H$-convexity and separation properties for this case.

We denote the complete lattice $\tilde{R}^Q$ by $Y$, where $Q$ is a set. Let $X$ be a conditionally complete upper sub-lattice of $Y$ such that the supremum (in $X$) of any bounded set in $X$ is the same as the pointwise supremum (that is, the supremum in $Y$). The following is then immediate from the definitions:

PROPOSITION 1.1. *Let $H \subset X$. A set $U \subset H$ is in $\mathfrak{B}$ $(H, X, Y)$ if and only if it is bounded above in $X$ and for any $h' \in H$, $h' \notin U$, there exists an $x \in Q$ such that $h'(x) > \sup_{h \in U} h(x)$.*

Henceforth, whenever $X$ is a function space, $H$-convex elements of $X$ will be called $H$-convex functions.

We now give some further definitions. The real line is denoted by $R$, and the half-line of non-negative reals by $R_+$. The extended real line is denoted by $\tilde{R}$. If $Q$ and $T$ are sets, then $T^Q$ denotes, as usual, the set of all functions $f\colon Q \to T$. The ordering on $\tilde{R}^Q$ and $R^Q$ is the natural one. We denote by $X_Q$ the subset of $\tilde{R}^Q$ consisting of all functions $f\colon Q \to (-\infty, +\infty]$ together with the function $-\boldsymbol{\infty}\colon x \mapsto -\infty$ $(x \in Q)$. It is clear that $X_Q$ is a complete lattice relative to the ordering induced by $\tilde{R}^Q$. Given a function $f$ in $\tilde{R}^Q$, we call the set $\det f \triangleq \{(x, \lambda) \in Q \times R\colon \lambda \geqslant f(x)\}$, the supergraph of $f$. If $Q$ is a topological space, the supergraph $\det f$ of a function in $\tilde{R}^Q$ is closed if and only if $f$ is lower semi-continuous. In this context we call lower semi-continuous functions closed from below (or, if there is no fear of ambiguity, merely closed). Consider a locally convex space $V$. A function $f$ defined on $V$ is called *convex* if $f \in X_V$ and the supergraph $\det f$ is closed and convex. In other words, $f$ is convex if it is lower semi-continuous, its range is not the set $\{-\infty, +\infty\}$, and it satisfies the "convexity (or Jensen's) inequality"

$$(1.2) \qquad f(\alpha x + \beta y) \leqslant \alpha f(x) + \beta f(y) \qquad (x, y \in V;\ \alpha, \beta \geqslant 0,\ \alpha + \beta = 1).$$

If of a function $f$ in $\tilde{R}^V$ it is merely known that is satisfies (1.2) (and that (1.2) is meaningful), then it is called *weakly convex.*[1] A convex function $p$ is called a *sublinear functional* if it is *positively homogeneous* (that is, $p(\lambda x) = \lambda p(x)$ for $\lambda > 0$ and $x \in V$). A sublinear functional is *subadditive* (that is, $p(x + y) \leqslant p(x) + p(y)$). A function in $X_V$ is a sublinear

[1] In the literature, such a function is usually called convex.

functional if and only if its supergraph is a closed cone (where by a *cone* we mean a convex set $K$ such that $\lambda K \subset K$ for all $\lambda > 0$). A convex finite function is called *affine* if (1.2) holds with the equality sign. An affine function is continuous and is of the form $f\colon x \mapsto l(x) + c$, where $l \in V'$, $c \in R$. (Here $V'$ denotes the dual space of $V$.) Let $\xi$ be a convex subset of $V$. A function $f\colon \xi \to \widetilde{R}$ is said to be convex on $\xi$ if either it is identically $-\infty$, or it is the trace on $\xi$ of a convex function $\widetilde{f}$ on $V$ such that $\operatorname{dom}\widetilde{f} \triangleq \{x \in V\colon f(x) < +\infty\} \subset \xi$. If $\xi$ is a cone, then we define a sublinear functional on $\xi$ in similar fashion.

1.2. **Examples of $H$-convex functions and sets.** EXAMPLE 2.1. Let $V$ be a locally convex space, and set $Y \triangleq \widetilde{R}^V$, $X \triangleq X_V$, $H \triangleq V'$. Then the family $P(H, X, Y)$ of $H$-convex functions consists of all sublinear functionals defined on $V$; the family $\mathfrak{B}(H, X, Y)$ of $H$-convex sets consists of all convex subsets of $V'$ that are closed in the weak topology $\sigma(V', V)$. These statements are proved by Hörmander in [113]. (Note that the assertion about $\mathfrak{B}(H, X, Y)$ follows immediately from Proposition 1.1.) In the present case $P(H, X, Y) = P(H, Y, Y)$ (and therefore $\mathfrak{B}(H, X, Y) = \mathfrak{B}(H, Y, Y)$). Furthermore, $P(H, Y)$ and $\mathfrak{B}(H, Y)$ are complete lattices. We note also that if $U \subset H$, then $\mathrm{co}_H(U)$ is the same as the closed convex hull $\overline{\mathrm{co}}(U)$ of $U$.

EXAMPLE 2.2. Let $V$, $Y$ and $H$ be as in the previous example, but with $X \triangleq R^V$. Then $P(H, R^V)$ is the set of all finite-valued sublinear functionals, and $\mathfrak{B}(H, R^V)$ is the set of all non-empty bounded weakly closed subsets (with respect to $\sigma(V', V)$). In addition, $P(H, R^V)$ and $\mathfrak{B}(H, R^V)$ are conditionally complete upper sublattices.

EXAMPLE 2.3. This time we let $X$ be the space $C(V)$ of continuous functions[1] on $V$ (with $V$, $Y$ and $H$ as before). Then it is clear that $P(H, C(V))$ is the family of all continuous sublinear functionals on $V$. A sublinear functional on $V$ is continuous if and only if it is bounded, that is, for any neighbourhood $W$ of the origin we have $\sup_{w \in W} |p(w)| < +\infty$. It follows easily that $\mathfrak{B}(H, C(V))$ is the family of all convex closed (with respect to $\sigma(V', V)$) equicontinuous sets. Note that $P(H, C(V))$ and $\mathfrak{B}(H, C(V))$ are upper sublattices.

EXAMPLE 2.4. Let $K$ be a closed cone in a locally convex space $V$. Assume for the sake of simplicity that $K$ is generating, that is $V = K - K$. Let $Y \triangleq \widetilde{R}^K$, and let $H$ be the subset of $X_K$ consisting of all traces on $K$ of linear functionals on $V$ (that is, elements of $V'$). The members of $P(H, X_K, Y)$ can be identified with the sublinear functionals on $K$. To describe $\mathfrak{B}(H, X_K)$, we make the following definition. If $L$ is a cone in a vector space $Z$, then a subset $\Omega$ of $Z$ such that $\Omega + L = \Omega$ is called

[1] Continuous functions are always considered to be finite.

*L-stable.* It can be shown (see [94]) that $\mathfrak{V}(H, X_K)$ is the set of all convex, $\sigma(V', V)$-closed, $(-K)^*$-stable sets (where $K^*$ denotes the dual cone to $K$). It is not difficult to check that the $H$-convex hull $\mathrm{co}_H(U)$ of a convex subset $U$ of $H$ is $\overline{U - K^*}$ (where the bar denotes the closure in $\sigma(V', V)$).

REMARK. Convex closed $K$-stable sets have various applications in mathematics. We mention, in particular, the theory of the growth of entire functions of several complex variables [22], [92] and the theory of models in economic dynamics [77], [93].

EXAMPLE 2.5. Let $K$ and $L$ be closed cones in a locally convex space $V$, with $K \supset L$ and $L$ generating. Consider the dual cone $K^*$ and let $H_{K, L}$ denote the family of traces of functionals of $K^*$ on $L$. We are interested in the upper sub-lattices $P(H_{K, L}) \triangleq P(H_{K, L}, R^L, \widetilde{R}^L)$ and $\mathfrak{V}(H_{K, L}) \triangleq \mathfrak{V}(H_{K, L}, R^L, \widetilde{R}^L)$. It will be convenient to identify $H_{K, L}$ with $K^*$. A subset $U$ of $K^*$ (or, what is the same, of $H_{K, L}$) is called *L-normal* if $U = (\overline{U - L^*}) \cap K^*$ (where the bar denotes the closure in $\sigma(V', V)$). It can be shown that $\mathfrak{V}(H_{K, L})$ is the family of all weakly bounded convex $L$-normal subsets of $V'$. If $U \subset H_{K, L}$, then $\mathrm{co}_{H_{K,L}}(U) = (\overline{\mathrm{co}\, U - L^*}) \cap K^*$. The set $P(H_{K, L})$ consists of all finite sublinear functionals $p$ defined on $L$ and having the following property: there exists a sublinear functional $\widetilde{p}: V \to R$ such that $\widetilde{p}(v) = p(v)$ $(v \in L)$; $\widetilde{p}(v_1) \geqslant \widetilde{p}(v_2)$ $(v_1 v_2 \in V;\ v_1 \in v_2 + K)$.

REMARK. $L$-normal sets play an important role, for instance, in the theory of economic dynamics [77].

We now go on to consider proper convex functions.

EXAMPLE 2.6. Let $V$ be a locally convex space with $Y \triangleq \widetilde{R}^V$, $X \triangleq X_V^{\cdot}$, $H$ denoting the space $A(V)$ of all affine functions on $V$. (Henceforth we identify $A(V)$ with $V' \times R$; if $(f, c) \in V' \times R$, $(x, \lambda) \in V \times R$, then we set $(f, c)((x, \lambda)) \triangleq f(x) - \lambda c$.) It can be shown, using Hörmander's results [113], for example, that $P(H) \triangleq P(H, X, Y)$ coincides with the family of all convex functions on $V$. Furthermore, $P(H)$ is a complete lattice. A subset $U$ of $H = V' \times R$ $(U \neq \emptyset,\ U \neq H)$ is $H$-convex if and only if it is convex, weakly closed $(\{0\} \times R_+)$-stable and not $(\{0\} \times R)$-stable. (See Example 2.4 for the definition of stability.) Also, the empty set and the whole of $H$ are $H$-convex.

REMARK. A set $U$ is $H$-convex if and only if it is the supergraph of some convex function defined on $V'$. If $f$ is a convex function on $V$ then the function $f^*$ whose supergraph is the set $U_f$ of all supports to $f$ is determined thus: $f^*\colon h \mapsto \sup\limits_{v \in V} (h(v) - f(v))$. The function $f^*$ is called the *adjoint* of $f$ (see § 1.5).

In the following examples we only consider either $H$-convex functions or $H$-convex sets.

EXAMPLE 2.7. Let $\xi$ be a convex subset of a locally convex space $V$, and let $A_\xi$ be the family of traces on $\xi$ of the affine functions on $V$. Then it follows from the previous example that $P(A_\xi, X_\xi, \widetilde{R}^\xi)$ is the set of all convex functions on $\xi$.

EXAMPLE 2.8. Let $\xi$ be a (not necessarily convex) subset of $V$ and let $A_\xi$ be the family of traces on $\xi$ of affine functions on $V$. It is natural to call the members of $P(A_\xi, X_\xi, \widetilde{R}^\xi)$ convex functions on $\xi$. In fact, each $A$-convex function admits an extension to a convex function defined on $V$.

The above examples use, in one way or another, the idea of a convex function. We now mention examples of an essentially different kind.

EXAMPLE 2.9. Let $H$ be the cone in $C([a, b])$ consisting of the trinomials $x \mapsto kx^2 + lx + m$ (where $k \leqslant 0$, $l \geqslant 0$, $m \leqslant 0$); here $a$ is assumed to be positive. Then $P(H, C([a, b]), \widetilde{R}^{[a,b]}) = C([a, b])$. (A more general statement is given in §3.2.) It should be further noted that $P(H, X_{[a,b]}, \widetilde{R}^{[a,b]})$ consists of all lower semi-continuous functions on $[a, b]$. This example shows that even such an extensive set as the family of all lower semi-continuous functions can be generated by means of the operation of taking the upper envelope (pointwise supremum) from subsets of a very meagre set – namely a cone spanned by three generators.

EXAMPLE 2.10. Consider again the space $C([a, b])$, and denote by $\hat{C}([a, b])$ its $K$-completion (the Dedekind completion) [23]. We denote by $Y$ the complete lattice obtained by adjoining to $\hat{C}([a, b])$ a greatest and a least element. Let $C_{\text{per}} \triangleq \{f \in C([a, b]); f(a) = f(b)\}$. Then $P(C_{\text{per}}, C([a, b]), Y) = C([a, b])$. Furthermore, $P(C_{\text{per}}, C([a, b]), \widetilde{R}^{[a,b]}) = C_{\text{per}}$.

In this example we have an instance where the lattice $Y$ is not $\widetilde{R}^Q$. Other examples of this kind are in §3.

EXAMPLE 2.11. Let us touch upon the connection between $H$-convexity and Fan convexity. First we give the relevant definition. Let $Q$ be a set and let $\Phi$ be a family of finite real-valued functions on $Q$ that separates the points of $Q$. (This means that for any distinct points $x, y \in Q$ there is a function $\varphi \in \Phi$ such that $\varphi(x) \neq \varphi(y)$.) Then a subset $A$ of $Q$ is called $\Phi$-convex (in the sense of Fan [28], [102]) if for each $x \in Q \setminus A$ there exists a function $\varphi \in \Phi$ such that $\varphi(x) > \sup_{y \in A} \varphi(y)$.

We consider $Q$ embedded in the lattice $R^\Phi$ as follows: each $x \in Q$ is identified with the evaluation map $\hat{x}$: $\varphi \mapsto \varphi(x)$ ($\varphi \in \Phi$). The collection of all functions $\hat{x}$, where $x \in Q$, is denoted by $Q_\Phi$. We also use the symbol ^ to denote the image of a subset of $Q$ under the evaluation map. From

Proposition 1.1 it follows that a subset $A$ of $Q$ is $\Phi$-convex if and only if the subset $\hat{A}$ of $R^{\Phi}$ is $Q_{\Phi}$-convex.

1.3. **The Minkowski–Fenchel scheme.** The Minkowski duality is of special interest in the case when the sets $H$ and $X$ considered in § 1.1 are equipped with algebraic operations. In this connection we make the following definition. A set $S$ is called a *semilinear space* if there is defined on it a binary operation + under which $S$ is a commutative semigroup, and an operation of multiplication by positive numbers such that

$$1\cdot s = s \ (s \in S), \quad \lambda s + \mu s = (\lambda + \mu)s, \quad (\lambda\mu)s = \lambda(\mu s),$$
$$\lambda(s_1 + s_2) = \lambda s_1 + \lambda s_2 \quad (\lambda > 0,\ \mu > 0;\ s_1, s_2 \in S).$$

Convex subsets of a semilinear space are defined in the obvious way. The simplest example of a semilinear space is a cone in a vector space.

A semilinear space $S$ which is at the same time an upper semilattice is called a *K-semilineal* if the following conditions hold: (a) $x \geqslant y$ implies that $x + z \geqslant y + z$ for all $z \in S$; (b) $x \geqslant y$ implies that $\lambda x \geqslant \lambda y$ $(\lambda > 0)$; (c) if $A \subset S$ has a supremum, then for each $z \in S$ we have $\sup (z + A) = z + \sup A$. Isomorphism of $K$-semilineals is defined in the natural way.

As in § 1.1, let $Y$ be a complete lattice, $X$ a subset of $Y$, and $H$ a subset of $X$. Suppose that $X$ is a $K$-semilineal (where the ordering on $X$ is the induced ordering from $Y$, and the supremum in $X$ of two elements of $X$ is the same as their supremum in $Y$); we suppose that $H$ is a semilinear subspace of $X$. Then $P(H, X)$ is a $K$-semilineal (with respect to the algebraic operations and ordering induced by $X$); furthermore, the supremum of two elements of $P(H)$ is the same whether regarded in $P(H)$ or in $X$. Consider now the set $\mathfrak{B}(H, X)$. It is easy to verify that its members are convex sets. We define in $\mathfrak{B}(H, X)$ the operation of multiplication by a positive number in the obvious fashion, and we define a binary operation (Minkowski sum) $\oplus$, as follows:

$$U_1 \oplus U_2 \triangleq \mathrm{co}_H(U_1 + U_2) \qquad (U_1, U_2 \in \mathfrak{B}(H, X)).$$

It can be shown that under these operations and the ordering determined by inclusion $\mathfrak{B}(H)$ is a $K$-semilineal. The following is an immediate consequence of the definitions:

MINKOWSKI–FENCHEL THEOREM. *The Minkowski duality $\varphi$: $p \mapsto U_p$ is an isomorphism of the K-semilineals $P(H, X)$ and $\mathfrak{B}(H, X)$.*

The Minkowski–Fenchel theorem shows that the Minkowski duality is analogous to the relationship between convex sets and sublinear functionals that is established in the classical Minkowski–Fenchel scheme (that is, in the conditions of Example 2.2 when $V$ is $R^n$). Note that the conditions of the Minkowski–Fenchel theorem are satisfied in each of the Examples 2.1–2.10. (We suppose here that in the lattice $X_Q$, where $Q$ is a set, the algebraic operations are defined thus: in the set $X_Q \setminus \{-\infty\}$ these operations are the natural ones; in addition, $x + (-\infty) \triangleq -\infty$ for any $x \in X_Q$, and $\lambda(-\infty) \triangleq (-\infty)$ for any $\lambda > 0$. It is easy to check that this makes $X_Q$ a $K$-semilineal.) By virtue of the Minkowski–Fenchel theorem we can consider the sets $P(H, X)$ and $\mathfrak{B}(H, X)$ as different realizations of the same $K$-semilineal. It is useful in many applications to endow these sets with topologies so that the Minkowski duality is a homeomorphism (see, for example, [26]). In this connection it is often convenient to identify the $K$-semilineals $P(H, X)$ and $\mathfrak{B}(H, X)$; more precisely, we identify the $H$-convex element $p$ with the $H$-convex set $U_p$ corresponding to it under the Minkowski duality. We make use of this identification in the next subsection.

**1.4. The space of $H$-convex sets.** Let $Y$ be a complete lattice and let $X$ be a $K$-semilineal contained in $Y$ (where the ordering on $X$ is induced by $Y$ and is such that the supremum of two elements of $X$ is the same whether taken in $X$ or in $Y$). Let $H$ be a semilinear subspace of $X$. Suppose that $X$ is a $K$-semilineal with cancellation (that is, $x + z = y + z$ implies that $x = y$ $(x, y, z \in X)$). Then $P(H) \triangleq P(H, X, Y)$ is also a $K$-semilineal with cancellation, and since $P(H)$ and $\mathfrak{B}(H)$ are algebraically and order isomorphic, $\mathfrak{B}(H)$ is also a semilineal with cancellation. This fact allows us, by the usual method of embedding a semigroup with cancellation in a group, to construct vector spaces $[P(H)]$ and $[\mathfrak{B}(H)]$ in which $P(H)$ and $\mathfrak{B}(H)$ are, respectively, embedded (to within isomorphism) as generating cones. Then $[P(H)]$ and $[\mathfrak{B}(H)]$ prove to be isomorphic. Furthermore, we can endow $[P(H)]$ with an ordering relation, inducing the original ordering on $P(H)$, so that $[P(H)]$ becomes a $K$-lineal. Clearly, we can deal with $[\mathfrak{B}(H)]$ in similar fashion. (In a typical special case this construction was carried out by Pinsker [86].) The space $[\mathfrak{B}(H)]$ (and its isomorph $[P(H)]$) is appropriately called the *space of convex sets*. If the original $K$-semilineal $X$ is a vector space, then the space of convex sets coincides (to within isomorphism) with the vector subspace of $X$ spanned by the cone $P(H)$. This fact greatly simplifies the study of the space of convex sets. We consider in detail just one important example. Let $(V, \|\cdot\|)$ be a Banach space with $Y \triangleq \widetilde{R}$, $X \triangleq R^V$, $H \triangleq V'$. Then, as in Example 2.2, $P(H, R^V)$ consists of all finite-valued sublinear functionals on $V$, while $\mathfrak{B}(H, R^V)$ consists of all non-empty convex closed and bounded (in $\sigma(V', V)$) subsets of $V'$. Since $V$ is complete, the elements of $\mathfrak{B}(H, R^V)$ are bounded in norm. It therefore follows easily (see Example 2.3) that the elements of $P(H, R^V)$ are continuous functionals. In what follows we identify a continuous sublinear functional with the set of its support linear functionals, and denote these two objects by the same letter.

In the present instance, the space of convex sets $[\mathfrak{V}(H)]$ coincides with the subspace of the $K$-lineal $R^V$ consisting of all functionals that can be expressed as the difference of two sublinear functionals; $[\mathfrak{V}(H)]$ is a $K$-lineal relative to the ordering induced by $R^V$. (Note that the infimum of two elements of $[\mathfrak{V}(H)]$ considered in $R^V$ is, in general, different from the infimum considered in $[\mathfrak{V}(H)]$.) Furthermore, $[\mathfrak{V}(H)]$ is an Archimedean $K$-lineal of bounded elements (we can take as the unit element in $[\mathfrak{V}(H)]$ the sublinear functional $x \mapsto \| x \|$). We endow $[\mathfrak{V}(H)]$ with the standard norm, setting $\| p \| \triangleq \sup_{\| v \| = 1} | p(v) |$. (The corresponding topology is called the Hausdorff topology [113]). According to the Krein–Kakutani theorem (see, for instance, [23]), there exists a compactum $Q$ such that the completion of $[\mathfrak{V}(H)]$ is isometrically isomorphic to the space $C(Q)$ of continuous functions on $Q$. The compactum $Q$ has a particularly simple desciption in the case when $V = R^n$; we can then take for $Q$ the unit sphere $Z_n$ ($Z_n \triangleq \{x \in R^n : |x| = 1\}$; here and from now on, $|x|$ denotes the Euclidean length of $x$); thus, $[\mathfrak{V}(H)] \triangleq [\mathfrak{V}(R^n)]$ is realized as a dense subspace of $C(Z_n)$. This follows from the fact that each element of $[\mathfrak{V}(R^n)]$ (being a positively homogeneous functional) can be identified with its restriction to the sphere $Z_n$. In the same way, an element $\mathfrak{x}$ in $\mathfrak{V}(R^n)$ can be identified with the support function $u \mapsto \max_{x \in \mathfrak{x}} (x, u)$. Henceforth we emphasize this identification by denoting an element of $[\mathfrak{V}(R^n)]$ and its restriction to $Z_n$ by the same symbol. We denote $[\mathfrak{V}(R^n)]$ by $[\mathfrak{V}_n]$ and the cone $\mathfrak{V}(R^n)$ by $\mathfrak{V}_n$. The elements of $\mathfrak{V}_n$ are called convex figures. The solid elements of $\mathfrak{V}_n$ are called convex bodies or convex surfaces. Strictly convex smooth bodies are called regular (surfaces).

Note that the equation $[\overline{\mathfrak{V}}_n] = C(Z_n)$ implies the well-known geometrical fact that any continuous function on the unit sphere can be uniformly approximated by linear combinations of restrictions of support functions on $Z_n$ [17]. This same equation shows that $[\mathfrak{V}_n]'$ is the same as $C'(Z_n)$, that is, the space of all Borel measures on $Z_n$.

**1.5. Adjoint functions (the Fenchel–Moreau scheme).** As we remarked in Example 2.6, the set of all supports to a convex function $f$ defined on a locally convex space $V$ can be regarded as the supergraph of the adjoint function $f^*$. The theory of adjoint convex functions, developed by Fenchel and Moreau and further evolved in papers by Rockafellar, Brøndsted and others, plays an important role in convex analysis [91]. It turns out that a number of (algebraic) results in this theory are founded on $H$-convexity. We illustrate this with the example of the Fenchel–Moreau theorem, which it is natural to state for $H$-convex functions.

Consider a set $Q$ and a subset $H$ of $R^Q$. Suppose for the sake of simplicity that $H$ separates the points of $Q$. As we have already remarked in Example 2.11, the elements of $Q$ can be regarded as functions in $R^H$ (if $x \in Q$, then the corresponding function is of the form $h \mapsto h(x)$); note that distinct elements give rise to distinct functions. If $L \subset R^Q$, then we denote by $L \odot 1$ the algebraic sum in $R^Q$ of the line $(\lambda 1)_{\lambda \in R}$ and $L$ (here $1$ is the evaluation map $1: x \to 1$ $(x \in Q)$). Recall that $X_Q$ denotes the set $(-\infty, +\infty]^Q \cup \{-\infty\}$.

Let $f \in X_Q$. The function $f^*$ defined on $H$ by the formula $f^*: h \mapsto \sup\limits_{x \in Q} (h(x) - f(x))$ is called the *adjoint* (or, more precisely, the *H-adjoint*) function to $f$ or the Young $H$-transform of $f$. It follows immediately from the definition that $f^* \in H_\mu$. Furthermore, for $x \in Q$ and $h \in H$ Young's inequality holds: $f^*(h) \geqslant h(x) - f(x)$.

PROPOSITION 5.1. *For any $f \in X_Q$ the function $f^*$ is $Q \odot 1$-convex (that is, $f^* \in P(Q \odot 1, X_H, \widetilde{R}^H)$.*

PROOF. Set $U \overset{\Delta}{=} \{y \in Q \odot 1: y = x - f(x)1,\ x \in \operatorname{dom} f\}$.[1] Then $f^*(h) = \sup\limits_{x \in Q} (h(x) - f(x)) = \sup\limits_{x \in \operatorname{dom} f} (x(h) - f(x)) = \sup\limits_{x \in \operatorname{dom} f} (x - f(x)1)(h) = \sup\limits_{y \in U} y(h)$ for any $h \in H$. This completes the proof.

Since $f^* \in X_H$, the adjoint $f^{**}$ to $f^*$ is well defined. It follows from the definition that $f^{**}(x) = \sup\limits_{h \in H} (h(x) - f^*(h))$ $(x \in Q)$. Note that for $f \in X_Q$ we have $f \geqslant f^{**}$. For by Young's inequality $f(x) \geqslant h(x) - f^*(h)$, so that $f(x) \geqslant \sup\limits_{h \in H} (h(x) - f^*(h)) = f^{**}(x)$.

THE FENCHEL–MOREAU THEOREM. *A function $f$ in $X_Q$ is $H \odot 1$-convex if and only if $f = f^{**}$.*

PROOF. If $f = f^{**}$, then by Proposition 5.1 $f$ is $H \odot 1$-convex. Suppose, conversely, that $f$ is $H \odot 1$-convex. Then there is a subset $U$ of $H \odot 1$ such that $f(x) = \sup\limits_{g \in U} g(x)$ $(x \in Q)$. Let $g \in U$, $g = h + \alpha 1$. Then $f(x) \geqslant h(x) + \alpha$ for all $x \in Q$, and therefore $f^*(h) = \sup\limits_{x \in Q} (h(x) - f(x)) \leqslant -\alpha$, and it follows that $f^{**}(x) \geqslant h(x) - f^*(h) \geqslant h(x) + \alpha = g(x)$. Finally, $f^{**}(x) \geqslant \sup\limits_{g \in U} g(x) = f(x)$, that is, $f^{**} \geqslant f$; the reverse inequality always holds.

COROLLARY. *If $f \in X_Q$, then $f^{**}$ is the greatest $H \odot 1$-convex function minorizing $f$.*

[1] Recall that $\operatorname{dom} f = \{x \in Q: f(x) < \infty\}$.

We note, in conclusion, that along with $H$-convex functionals and sets it is appropriate to consider $H$-concave functions and sets (interchanging sup and inf in the relevant definitions and results).

## §2. Dual methods of representing $H$-convex functions

**2.0. Preliminaries.** In this section we shall, as a rule, be concerned with continuous $H$-convex functions, or more precisely, elements of $P(H, C(Q), \widetilde{R}^Q)$, where $Q$ is a compact topological space, $C(Q)$ is the space of continuous functions on $Q$, and $H$ is a cone in $C(Q)$. In what follows, we denote $P(H, C(Q), \widetilde{R}^Q)$ by $P(H)$, so that an $H$-convex function is an element of the cone $P(H)$. Thus, a (continuous) function $f$ is $H$-convex if

$$f(x) = \sup_{h \leqslant f,\, h \in H} h(x)$$

for all $x \in Q$. In particular, the concept of $H$-convexity is in the present situation of a local character; in other words, it makes sense to talk about a function being $H$-convex at a point.

Below we shall introduce two fundamental ways of representing $H$-convex functions. The first method is tied up with the study of positive measures on $P(H)$. The main field of application of this first method is to extremal problems over $\mathfrak{B}(H)$. In §4 we discuss a number of geometric problems of this sort.

The second method is tied up with the connection between the behaviour of operators on the cone $H$ and their properties on $P(H)$. We leave such techniques for the moment and resume the discussion in §3. It is interesting to note that for a number of cones one can link the structure of the adjoint cone with properties of positive operators.

First of all we explain the intuitive idea leading to the required representations. Consider the case of continuous functions on a compact convex set $\Omega$. The convexity of $f$, by definition, implies that for any representations of the form $z = \sum_{k=1}^{n} \alpha_k z_k$, where $\alpha_k \geqslant 0$, $\sum_{k=1}^{n} \alpha_k = 1$, $z, z_k \in \Omega$, we have

$$\sum_{k=1}^{n} \alpha_k f(z_k) \geqslant f(z). \tag{0.1}$$

Denote by $N$ the set of measures of the form $\sum_{k=1}^{n} \alpha_k \varepsilon_{z_k} - \varepsilon_z$ (where $\varepsilon_x$ is the Dirac measure $\varepsilon_x\colon f \mapsto f(x)$). Let $K(N)$ be the weakly closed conical hull of $N$. The set $K(N)$ can be represented as a collection of consequences of Jensen inequalities. It is clear that the polar $W^*$ of the cone of convex functions coincides with $K(N)$. Thus, the integral inequalities

$$\int_{\Omega} f\,d\mu \geqslant \int_{\Omega} f\,d\nu, \tag{0.2}$$

where $\mu$, $\nu \geqslant 0$ and $f$ is convex, are determined by the element $\mu - \nu$ $\mu - \nu \in K(N)$.

The first natural attempt to describe the measures satisfying (0.2) consists in decomposing the measures $\mu$ and $\nu$ into parts analogous to the Jensen inequalities. The formalization of this idea leads to the so-called Reshetnyak–Loomis decomposition. The second approach to (0.2) is based on an idea closely related to the work of Choquet. Namely, the measure $\sum_{k=1}^{n} \alpha_k \varepsilon_{z_k}$, figuring in (0.1) is, roughly speaking, the measure $\varepsilon_z$ (or the unit mass at $z$) distributed at the points $z_1, \ldots, z_n$. It is natural therefore to represent $\mu$ as the result of spreading $\nu$ over $\Omega$. The formalization of this leads to the concept of $H$-distributions of measures.

**2.1. The decomposition theorem.** We turn to the following situation. Let $X$ be a locally convex space which is at the same time a $K$-lineal; $H_1, \ldots, H_n$ are closed cones in $X$. Suppose that the topology and ordering are such that: (a) the conical slice $\langle 0, f\rangle \overset{\Delta}{=} \{g \in K^*\colon g \leqslant f\}$ is $\sigma(X', X)$ compact for any $f \in K^*$ (where $K \overset{\Delta}{=} \{x \in X\colon x \geqslant 0\}$); (b) for any $f \in K^*$ and arbitrary $h_1 \in H_1, \ldots, h_n \in H_n$ there is a partition $\{f_1, \ldots, f_n\}$ of $f$ such that $f(h_1 \vee \ldots \vee h_n) = \sum_{k=1}^{n} f_k(h_k)$. (By a *partition* $\{f_1, \ldots, f_n\}$ we mean a collection of functionals $f_k \geqslant 0$ $(k = 1, \ldots, n)$ such that $\sum_{k=1}^{n} f_k = f$.)

Under these hypotheses we have:

THE DECOMPOSITION THEOREM. *Let $f, g \in K^*$. Then the inequality $f(h_1 \vee \ldots \vee h_n) \geqslant g(h_1 \vee \ldots \vee h_n)$ holds for any $h_k \in H_k$ $(k = 1, \ldots, n)$ if and only if for any partition $\{g_1, \ldots, g_n\}$ of $g$ there is a partition $\{f_1, \ldots, f_n\}$ of $f$ such that $f_k - g_k \in H_k^*$ $(k = 1, \ldots, n)$.*

SUFFICIENCY. Let $h_k \in H_k$ $(k = 1, \ldots, n)$. We can find a partition of $g$ for which $g(h_1 \vee \ldots \vee h_n) = \sum_{k=1}^{n} g_k(h_k)$. For this partition we choose a partition of $f$ such that $f_k - g_k \in H_k^*$. It is clear that

$$g(h_1 \vee \ldots \vee h_n) = \sum_{k=1}^{n} g_k(h_k) \leqslant \sum_{k=1}^{n} f_k(h_k) \leqslant \sum_{k=1}^{n} f_k(h_1 \vee \ldots \vee h_n) = f(h_1 \vee \ldots \vee h_n).$$

NECESSITY. Let $S \stackrel{\Delta}{=} \{(f_1, \ldots, f_n) \in (K^*)^n\colon \sum_{k=1}^{n} f_k = f\}$. Set $\widetilde{S} \stackrel{\Delta}{=} S - H_1^* \times \ldots \times H_n^*$. It is clear that $\widetilde{S}$ is a non-empty weakly closed convex set. Suppose that $(g_1, \ldots, g_n) \notin \widetilde{S}$, where $g_1, \ldots, g_n$ is a partition of $g$. By the separation theorem there exist $h_k \in H_k$ $(k = 1, \ldots, n)$ such that $\sum_{k=1}^{n} f_k(\widetilde{h}_k) < \sum_{k=1}^{n} g_k(\widetilde{h}_k)$ for $(f_1, \ldots, f_n) \in \widetilde{S}$. Let $\{\widetilde{f}_1, \ldots, \widetilde{f}_n\}$ be a partition of $f$ such that $\sum_{k=1}^{n} \widetilde{f}_k(\widetilde{h}_k) = f(\widetilde{h}_1 \vee \ldots \vee \widetilde{h}_n)$. Then

$$f(\widetilde{h}_1 \vee \ldots \vee \widetilde{h}_n) < \sum_{k=1}^{n} g_k(\widetilde{h}_k) \leqslant g(\widetilde{h}_1 \vee \ldots \vee \widetilde{h}_n) \leqslant f(\widetilde{h}_1 \vee \ldots \vee \widetilde{h}_n).$$

This contradiction completes the proof.

REMARK. The decomposition theorem holds in each *KN*-lineal of bounded elements.

Next we give some further definitions. Let $H$ be a cone in $C(Q)$, and let $\mu$ and $\nu$ be positive (Radon) measures on $Q$. We say that $\mu$ is an *H-successor* of $\nu$ if $\mu(f) \geqslant \nu(f)$ for all $f \in P(H)$; this will be written $\mu \underset{H}{\succ} \nu$.

We say that $\mu$ is *H-stronger* than $\nu$ if for all partitions $\{\nu_1, \ldots, \nu_n\}$ of $\nu$ there is a partition $\{\mu_1, \ldots, \mu_n\}$ of $\mu$ such that $\mu_k - \nu_k \in H^*$, and we write $\mu \underset{H}{\gg} \nu$. We say that the cone $H$ has the *Reshetnyak–Loomis property* if $\mu \underset{H}{\succ} \nu \Rightarrow \mu \underset{H}{\gg} \nu$ (clearly the converse always holds). From the decomposition theorem we have the corollary:

THEOREM 1.1. *A closed cone $H$ in $C(Q)$ has the Reshetnyak–Loomis property.*

Suppose now that $T_1, T_2 \in \mathscr{L}^+(C(Q), B(Q))$, where $B(Q)$ is the space of bounded functions on $Q$, and $\mathscr{L}^+(X, Y)$ denotes the set of positive operators[1] between the partially ordered spaces $X$ and $Y$. We say that $T_1$ is *H-stronger* than $T_2$ ($T_1 \underset{H}{\gg} T_2$) if for any partition $T_2^1, \ldots, T_2^n\}$ of $T_2$ (that is, $\sum_{k=1}^{n} T_2^k = T_2$, $T_2^k \geqslant 0$) there is a partition $\{T_1^1, \ldots, T_1^n\}$ of $T_1$ such that $T_1^k h \geqslant T_2^k h$ $(h \in H, k = 1, \ldots, n)$. The set $\{T'\colon T' \underset{H}{\gg} T\}$ is called the *decompositional germ* of $T$ over the cone $H$ and is denoted by Dpr $(T, H)$.

[1] Throughout, unless explicitly stated to the contrary, an operator is understood to be additive and homogeneous.

THEOREM 1.2. *Let $H$, $P(H)$ be closed cones in $C(Q)$. Then $f \in C(Q)$ is $H$-convex if and only if for any $T \in \mathcal{L}^+ (C(Q), B(Q))$ and $T' \in \mathrm{Dpr}(T, H)$ the condition $T'f \geqslant Tf$ holds.*

This is a slightly modified version of Theorem 1.1.

2.2. **The operator principle of preservation of inequalities.** It is of interest to narrow down the set of operators whose decompositional germs define $H$-convexity. Consider the triple $H \subset X \subset Y$, where $Y$ is a $K$-space, $X$ is a vector subspace and $H$ is a cone in $X$. We say that $H$ is *minorant* if for any $x \in X$ the set of supports $\{h \in H: h \leqslant x\}$ is non-empty. The $K$-space $Y$ becomes a complete lattice if a maximal and a minimal element are adjoined. Because of this it makes sense to talk about $H$-convex elements.

We denote by $\mathrm{Spr}(T, H)$ the *positive germ of the operator $T$ over the cone $H$*; $\mathrm{Spr}(T, H) \triangleq \{T' \in \mathcal{L}^+(X, Y): T'h \geqslant Th (h \in H)\}$; $E: X \to Y$ is the inclusion map. For the proof of the operator principle of preservation of inequalities we need the following:

THE HAHN–BANACH–KANTOROVICH THEOREM ([46]). *Let $V$ be a vector space, $Y$ a $K$-space and $q: V \to Y$ a superadditive ($q(x + y) \geqslant q(x) + q(y)$), positively homogeneous ($q(\lambda x) = \lambda q(x)$, $\lambda > 0$) operator. Suppose that the operator $T_1$ with values in $Y$ is defined on a vector subspace $V_1$ of $V$, is additive and homogeneous, and satisfies the inequality $T_1 x \geqslant q(x)$ ($x \in V_1$). Then there is an additive homogeneous operator $T: V \to Y$ extending $T_1$ such that $Tx \geqslant q(x)$ for all $x \in V$.*

THEOREM 2.1 (THE OPERATOR PRINCIPLE OF PRESERVATION OF INEQUALITIES). *Let $H$ be a minorant cone in a subspace $X$ of a $K$-space $Y$. Then an element $x \in X$ is $H$-convex if and only if $Tx \geqslant x$ for all $T \in \mathrm{Spr}(E, H)$.*

PROOF. Suppose that $x \in P(H, X, Y)$. Then for $h \in U_x$ we have $Tx \geqslant Th \geqslant h$, that is, $Tx \geqslant \sup U_x = x$. Conversely, suppose that $x > \sup U_x$. Consider the operator $q_H: x' \longmapsto \sup U_{x'}$, $q_H: X \to Y$. Then clearly $q_H$ is superadditive and positively homogeneous. Let $X_1 \triangleq \{\alpha x\}_{\alpha \in R}$ and let $A_1: X_1 \to Y$ be defined by $A_1(\alpha x) = \alpha q_H(x)$. Then $A_1$ majorizes $q_H$ on the subspace $X_1$. By the Hahn–Banach–Kantorovich theorem there is an extension $A: X \to Y$ of $A_1$ such that $Ax' \geqslant q_H(x')$ ($x' \in X$). Clearly $A \in \mathrm{Spr}(E, H)$. On the other hand, $Ax < x$.

2.3. ***H*-distributions.** It is of interest to connect the operator principle of preservation of inequalities with the properties of polars of cones of $H$-convex functions in $C(Q)$. (For the sake of convenience we take $Q$ to be a compact metric space.)

To begin with, let $H$ be a closed cone in $C(Q)$, and let $\nu \triangleq \sum_{k=1}^{n} \alpha_k \varepsilon_{x_k}$ be a discrete measure (it is always understood that a discrete measure has finitely many carriers), and let $\mu \underset{H}{\succ} \nu$. For the partition $\{\alpha_1 \varepsilon_{x_1}, \ldots, \alpha_n \varepsilon_{x_n}\}$ of $\nu$ we find a partition $\{\mu_1, \ldots, \mu_n\}$ of $\mu$ such that $\mu_k(h) \geqslant \alpha_k \varepsilon_{x_k}(h)$ $(h \in H)$. We now define $T_{x_k} \triangleq \mu_k/\alpha_k$ $(k = 1, \ldots, n)$ and $T_x(h) \triangleq \varepsilon_x$ $(x \neq x_k)$. Note that: (a) $T_x$ is a positive Radon measure; (b) $T_x(h) \geqslant h(x)$ $(h \in H)$; (c) for any $f \in C(Q)$ the function $x \mapsto T_x(f)$ is bounded and Borel measurable; (d) $\mu(f) = \int\limits_Q T_x(f) d\nu$ for $f \in C(Q)$.

If for the positive measures $\mu$ and $\nu$ there exists a function $x \mapsto T_x$ such that conditions (a)–(d) hold, then we say that $\mu$ is a (weakly measurable) *H-distribution* of $\nu$ and write $\mu \underset{H}{\supset} \nu$.

The following implications are easily verified:

$$\begin{array}{ccc} & \mu \underset{H}{\succ} \nu & \\ \nearrow & & \nwarrow \\ \mu \underset{H}{\supset} \nu & \Longrightarrow & \mu \underset{H}{\gg} \nu \end{array} \tag{3.1}$$

The problem of replacing the implications by equivalences in (3.1) has so far not been completely solved. A fairly detailed study has been made of the case when for any $\mu, \nu \geqslant 0$ the equivalence $\mu \underset{H}{\succ} \nu \Longleftrightarrow \mu \underset{H}{\supset} \nu$ holds. In this situation we say that the cone $H$ has the *Hardy–Littlewood–Pólya property.* Such a cone automatically satisfies the Reshetnyak–Loomis property; on the other hand, elementary counterexamples show that the converse is false.

The study of $H$-distributions is based on Strassen's theorem.

Let $X$ be a separable Banach space, and let $(\Omega, S, \mu)$ be a probability space. Let $\omega \mapsto h_\omega$ be a weakly measurable mapping from $\Omega$ into $P(X', R^X, \widetilde{R}^X)$, (that is, for any $x \in X$ the mapping $\omega \mapsto h_\omega(x)$ is $S$-measurable). It is clear that the function $\omega \mapsto \| h_\omega \|$ (where $\| h \| \triangleq \sup\limits_{\| x \| = 1} | h(x) |$) is also $S$-measurable. Suppose that $\int\limits_\Omega \| h_\omega \| d\mu < +\infty$. Then the integral $x \mapsto h(x) \triangleq \int\limits_\Omega h_\omega(x) d\mu$ is a sublinear functional on $X$.

STRASSEN'S THEOREM ([121]). *Let $l$ be a support linear functional to $h$. Then there exists a weakly measurable mapping $\omega \mapsto l_\omega$ from $\Omega$ to $X'$ such that $l_\omega$ is a support to $h_\omega$, and $l(x) = \int\limits_\Omega l_\omega(x) d\mu$ for all $x \in X$.*

From this we can derive the following fact.

THEOREM 3.1. *A minorant cone $H$ in $C(Q)$ (where $Q$ is a metrizable compact space) has the Hardy–Littlewood–Pólya property.*

Here is an outline of the proof. If $\mu \underset{H}{\succ} \nu$, then we set

$\widetilde{p}(f) \triangleq \int_Q (\mathrm{co}_H f)(\cdot)\,d\nu$ (where $(\mathrm{co}_H f)(x) \triangleq \sup\{h(x) \colon h \in H,\ h \leqslant f\} =$

$= \sup\{\varphi(x) \colon \varphi \in P(H),\ \varphi \leqslant f\}$). Now for positive measures $\mu'$ we have $\mu'(\mathrm{co}_H f) = \sup\ \mu'(\varphi) \colon \varphi \in P(H),\ \varphi \leqslant f$ , so that $-\mu$ is a support to the functional $-\widetilde{p}$. The existence of the mapping $x \longmapsto T_x$ in the definition of the relation $\underset{H}{\supset}$ now follows from Strassen's theorem.

REMARK. It was established above that any closed cone $H$ in $C(Q)$ has the Reshetnyak–Loomis property. Theorem 3.1 shows that the condition on $H$ to be closed can be replaced by the condition that it is minorant. Note also that $P(H)$ being closed does not, in general, imply that $H$ is closed; on the other hand, if $H$ is minorant, then $P(H)$ is closed.

2.4. **Examples.** We now give typical applications of the fundamental technique of decompositions. For details about $H$-distributions see [105] and also §3.

EXAMPLE 4.1. Let $Q$ be a compact convex subset of the locally convex space $V$, and let $A(Q)$ be the set of continuous affine functions on $Q$ (that is, $A(Q) \triangleq \overline{V'\,|_Q + R}$). The cone $A(Q)$ has the Reshetnyak–Loomis property (this is the Cartier–Fell–Meyer theorem [47]). If $Q$ is metrizable, then $A(Q)$ has the Hardy–Littlewood–Pólya property. This last result is called the Hardy–Littlewood–Pólya–Blackwell–Stein–Sherman–Cartier theorem [105].

EXAMPLE 4.2. We retain the symbol $R^n$ for the subspace of traces on the sphere of directions $Z_n$ of linear functionals on $R^n$. Note that $P(R^n)$ (or $\mathfrak{V}_n$ ) is the cone of convex compact sets in $R^n$. The fact that $R^n$ has the Reshetnyak–Loomis property provides a description of positive Minkowski linear functionals on convex surfaces. In particular:

THEOREM 4.1. *Let $\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}, \mathfrak{y}_1, \ldots, \mathfrak{y}_{n-1}$ be convex surfaces in $R^n$. Then for any convex figure $\mathfrak{z}$ the inequality $V(\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}, \mathfrak{z}) \geqslant V(\mathfrak{y}_1, \ldots, \mathfrak{y}_{n-1}, \mathfrak{z})$ holds if and only if $\mu(\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}) \underset{R^n}{\geqslant} \mu(\mathfrak{y}_1, \ldots, \mathfrak{y}_{n-1})$ (here $V(\cdot, \ldots, \cdot)$ and $\mu(\cdot, \ldots, \cdot)$ are the mixed volume and mixed surface functions, respectively* ([1], [17])).

To prove this it suffices to note that

$$V(\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}, \mathfrak{z}) = \frac{1}{n} \int_{Z_n} \max_{x \in \mathfrak{z}} (x, \cdot)\, d\mu(\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}),$$

$$V(\mathfrak{y}_1, \ldots, \mathfrak{y}_{n-1}, \mathfrak{z}) = \frac{1}{n} \int\limits_{Z_n} \max_{x \in \mathfrak{z}} (x, \cdot)\, d\mu(\mathfrak{y}_1, \ldots, \mathfrak{y}_{n-1}),$$

and then apply Theorem 1.1.

EXAMPLE 4.3. Let $K$ be a generating cone in $R^n$ and $N$ the collection of traces on $K^* \cap Z_n$ of elements of $K$ regarded as functionals on $R^n$. By example 1.2.5 the $N$-convex sets can be identified in the present case with the convex $K$-normal subsets of $K$. The cone $N$ does not, in general, have the Hardy–Littlewood–Pólya property, but it does have the Reshetnyak–Loomis property.

EXAMPLE 4.4. Consider $R^{n-1}$ as the hyperplane $x_n = 0$ of $R^n$. We can find functionals which are linear with respect to the Minkowski operations and positive on the set of conical pyramids with bases in $R^{n-1}$ and vertices on the (orthogonal) ray $L \triangleq \{\alpha e_n : \alpha \geqslant 0\}$. By definition, a conical pyramid is the convex hull of an element $y \in \mathfrak{V}_{n-1}$ and a point of the ray $L$ (here $\widetilde{\mathfrak{V}}_{n-1}$ is the cone of compact convex subsets of $R^{n-1}$). It is clear that the present situation falls under the decomposition theorem (with the standard identifications); in other words, the following holds:

THEOREM 4.2. *A positive functional on the conical pyramids is the difference of non-negative measures $\mu$, $\nu$ on the sphere such that for any $0 \leqslant \nu_1 \leqslant \nu$ there is a measure $0 \leqslant \mu_1 \leqslant \mu$ for which $\mu_1 - \nu_1 \in \widetilde{\mathfrak{V}}^*_{n-1}$ and $(\mu - \mu_1)(e_n) \geqslant (\nu - \nu_1)(e_n)$.*

Note that the polar $\widetilde{\mathfrak{V}}^*_{n-1}$ can also be described by means of decompositions.

EXAMPLE 4.5. We define an *m-tope* as a polyhedron in $R^n$ which is the convex hull of not more than $m$ points. Let $M_m$ be the set of all $m$-topes. Denote by $\widetilde{M}_m$ the closed conical hull of $M_m$ in the space of convex sets. We call the elements of $\widetilde{M}_m$ *m-hedra.* Thus, an $m$-hedron is "a continuous positive combination of $m$-topes". For the case $m = 2$ we obtain the usual polyhedra (see [37]). Using the Minkowski duality and the decomposition theorem it is not difficult to give a characterization of the support functions of $m$-hedra. We will only outline the characteristic idea for obtaining similar representations. It is necessary to point out that $\mathfrak{x} \in \widetilde{M}_m \iff \mu(\mathfrak{x}) \geqslant 0$ for all $\mu \in M^*_m$. Since $M_m$ consists of functions of the form $z_1 \vee \cdots \vee z_p$ $(p \leqslant m)$, where $z_i \in R^n$ (we recall that we identify convex compacta with their support functions), it follows that $M^*_m$ can be described in terms of the decomposition property. So we may confine our attention to the case of discrete measures $\mu$, because we have the following useful result:

PROPOSITION 4.1. *Let $H$ be a convex cone in* $[\mathfrak{V}_n]$, $H \subset \mathfrak{V}_n$. *Then the discrete measures are dense in the polar $H^*$.*

The proof uses the theorem on simplicial approximation and the well-known theorem of Aleksandrov on the reconstruction of a convex surface from its surface function ([1]) and can be found in [62].

The following example is based on the same technique. It deals with the tie-up between the Minkowski operations and those of taking the convex hull and the intersection on Minkowski balls. These questions are related, for instance, to interpolation theorems (see [82]).

EXAMPLE 4.6. Let $S_1, \ldots, S_m$ be convex balanced bodies in $R^n$ (that is, symmetrical with respect to the origin). Then their so-called Pinsker hull $\pi(S_1, \ldots, S_m)$ is the smallest closed cone in the space of convex sets $[\mathfrak{V}_n]$, that contains $S_1, \ldots, S_m$, and is closed with respect to the operation of taking the convex hull of their union.

The arguments outlined in this last example come down to the following result.

THEOREM 4.3. *A convex figure $S$ ($S \neq \{0\}$) is in $\pi(S_1, \ldots, S_m)$ if and only if for any vectors $x_1, \ldots, x_p$ (not all zero) we have the inequality (where $S^0$ is the polar of $S$)*

$$\frac{S}{\sum_{k=1}^{p} \|x_k\|_{S^0}} \leqslant \frac{S_1}{\sum_{k=1}^{p} \|x_k\|_{S_1^0}} \vee \cdots \vee \frac{S_m}{\sum_{k=1}^{p} \|x_k\|_{S_m^0}}.$$

In other words,

$$S = \bigwedge_{(x_1, \ldots, x_p) \neq 0} \sum_{k=1}^{p} S(x_k) \left[ \frac{S_1}{\sum_{k=1}^{p} S_1(x_k)} \vee \cdots \vee \frac{S_m}{\sum_{k=1}^{p} S_m(x_k)} \right].$$

Here is an interesting consequence of this fact.

PROPOSITION 4.2. *Let $Q(S_1, S_2, \ldots, S_m)$ be the smallest closed cone that contains $S_1, \ldots, S_m$ and is closed with respect to intersections. Then the smallest closed sublattice $M(S_1, \ldots, S_m)$ in $\mathfrak{V}_n$, containing the balls $S_1, \ldots, S_m$ is $\pi(Q(S_1, \ldots, S_m))$. Furthermore, a non-zero $S$ is contained in $M(S_1, \ldots, S_m)$ if and only if*

$$S = \bigwedge_{x \neq 0} S(x) \bigvee_{S_0 \in Q(S_1, \ldots, S_m)} \frac{S_0}{S_0(x)}.$$

We outline the main idea of a proof of Proposition 4.2.

Clearly we merely have to verify that $\pi(Q(S_1, \ldots, S_m))$ is closed under intersections. By Theorem 4.3 it suffices to show that if

$\sum_{k=1}^{p} S_i'(x_k) \geqslant S_i'(y)$ $(i = 1, 2)$ for those $x_1, \ldots, x_p$ and $y$ for which

$\sum_{k=1}^{p} (S_0(x_k) \geqslant S_0(y)$ for all $S_0 \in Q(S_1, \ldots, S_m)$, then

$\sum_{k=1}^{p} (S_1' \wedge S_2')(x_k) \geqslant S_1' \wedge S_2'(y)$. Since the cone $Q(S_1, \ldots, S_m)$ is closed

with respect to intersections, according to the appropriate modification of the decomposition property the matter reduces to measures of the form $\bar{\mu} \overset{\Delta}{=} \alpha\varepsilon_x - \beta\varepsilon_y$, where $\bar{\mu} \in (Q(S_1, \ldots, S_m))^*$, and the required result follows from this.

EXAMPLE 4.7. Let $H$ be a minorant closed cone in $C(Q)$ and let $\mathrm{lat}(H)$ be the smallest closed cone in $C(Q)$ that is a lattice containing $H$. Since in $C(Q)$ the distributive law holds, $\mathrm{lat}(H)$ can be obtained as a cone of $P(H)$-concave functions. Combining Theorems 1.1 and 3.1 we obtain the following proposition.

PROPOSITION 4.3. *The difference of positive measures $\mu$, $\nu$ is contained in* $[\mathrm{lat}(H)]^*$ *if and only if for each partition* $\{\mu_1, \ldots, \mu_n\}$ *of $\mu$ there is a partition* $\{\nu_1, \ldots, \nu_n\}$ *of $\nu$ such that $\mu_k$ is an H-distribution of $\nu_k$* $(k = 1, \ldots, n)$.

Further examples can be found in [67], [105].

**2.5. $H$-maximal measures.** In this subsection we establish a connection between the theory of $H$-convex functions and Choquet's theory. Historically Choquet theory arose from the problem of balayage in potential theory, but for us its main interest is that it has evolved so as to provide an approach to an important dual description of $H$-convex functions.

The fundamental problem in Choquet theory is that of obtaining an integral representation of points of a convex (in our case $H$-convex) compact set. The basic apparatus for this is that of maximal measures (that is, roughly speaking, measures concentrated on the boundary points of the compact set). In the present situation the concept of an $H$-maximal measure is introduced as follows.

Let $H$ be a cone in $C(Q)$. A positive measure which is maximal with respect to the ordering $\underset{H}{\succ\succ}(\mu \underset{H}{\succ\succ} \nu \Leftrightarrow \mu - \nu \in H^*)$, is called *H-maximal.* It can be shown that there exist $H$-maximal measures on $H$ if and only if $H$ is minorant. Furthermore, given a positive measure there is an $H$-maximal measure majorizing it (in the sense of $\underset{H}{\succ\succ}$).

An immediate check (see, for example, [12]) establishes the following proposition.

PROPOSITION 5.1. *A measure $\mu$ is H-maximal if and only if for each function $f \in C(Q)$*

$$\mu(f) = \sup_{h \leqslant f,\, h \in H} \mu(h).$$

In particular, we have the corollary:

THE PRINCIPLE OF PRESERVATION OF INEQUALITIES. *A function $f$ is $H$-convex relative to the minorant cone $H$ if and only if $\mu(f) \geqslant f(z)$ for each $z \in Q$ and positive measure $\mu$ such that $\mu(h) \geqslant h(z)$ $(h \in H)$.*

In 2.1 we saw how to define $H$-convex functions by means of the preservation of operator inequalities. We shall return to this question presently; for the moment we establish the following fact:

MAKOBODSKII'S THEOREM. *A measure $\mu$ is maximal with respect to the cone of $H$-convex functions $P(H)$ if and only if $\mu(f) = \mu(\mathrm{co}_H f)$ for each continuous function $f$.*

PROOF. From Proposition 5.1. we have

$$\mu(f) = \sup_{\varphi \leqslant f, \varphi \in P(H)} \mu(\varphi) = \mu(x \longmapsto \sup \{\varphi(x) : \varphi \leqslant f, \ \varphi \in P(H)\}) = \mu(\mathrm{co}_H f).$$

On the other hand, if $\mu(\mathrm{co}_H f) = \mu(f)$, then

$$\mu(f) = \mu(x \longmapsto \sup \{\varphi(x) : \varphi \leqslant f, \ \varphi \in P(H)\}) = \sup_{\varphi \leqslant f, \varphi \in P(H)} \mu(\varphi),$$ that is,

$\mu$ is $P(H)$-maximal.

We now introduce the following definition. A point $z \in Q$ is said to belong to the *Choquet boundary* $b(H)$ of the cone $H$ if $(\mathrm{co}_H f)(z) = f(z)$ for any $f \in C(Q)$ (it would be more correct to call it the "Milman–Choquet boundary" (see, for instance, [78]), but the terminology "Choquet boundary" is more customary). In other words, according to Makobodskii's theorem a point $z$ is in $b(H)$ if and only if the measure $\varepsilon_z$ is $P(H)$-maximal (or, what is the same thing, $H$-maximal).

Note that a point of the Choquet boundary is an analogue to a boundary point of a convex compact set (more precisely, the Choquet boundary of the subspace of affine functions on a convex compact subset of a locally convex space coincides with the set of boundary points of this compact set).

Choquet's theorem on the construction of maximal measures becomes particularly simple for the case of a metrizable space (in the general case it does not make sense to talk about "a measure concentrated on the set of boundary points"). For this reason we confine ourselves below to "almost metric" spaces. First of all we give the necessary definitions.

A continuous function $f$ is said to *isolate the cone $H$* if $S_f \stackrel{\Delta}{=} \{z \in Q : (\mathrm{co}_H f)(z) = f(z)\}$ coincides with the Choquet boundary of $H$, that is, if $f$ is not an $H$-convex function at the points of the complement of the Choquet boundary $b(H)$ (recall that in the present case $H$-convexity is a local property). A cone $H$ is called a Choquet cone if (a) its Choquet boundary is a non-empty Borel set, (b) there exists a function isolating $H$.

CHOQUET'S THEOREM. *Let $H$ be a Choquet cone. Then $P(H)$-maximal measures are concentrated on the Choquet boundary $b(H)$;*

*furthermore, there exists for each $z \in Q$ a positive measure $\mu$ concentrated on the Choquet boundary such that $\mu(h) \geqslant h(z)$ for all $h \in H$.*

PROOF. Let $\mu$ be a $P(H)$-maximal measure. By Makobodskii's theorem $\mu(f) = \mu(\mathrm{co}_H f)$, where $f$ is a function isolating $H$. Clearly $b(H) = S_f$. On the other hand, if $A$ is open and $A \subset Q \setminus S_f$, then since $f(z) > \mathrm{co}_H\, f(z)$ for $z \notin A$, we have $\mu(A) = 0$. Thus, $\mu$ is concentrated on $b(H)$. The second part of the theorem is obvious.

REMARK. Choquet's theorem gives an exhaustive characterization of maximal measures. The fact is that a measure whose support is concentratec on the Choquet boundary of the cone $H$ is evidently $P(H)$-maximal (by Makobodskii's theorem).

The classical Choquet theorem [105] is obtained from the above result in the following way: we choose as a function isolating the cone of affine functions an arbitrary strictly concave function (whose existence, as is well-known, is equivalent to the metrizability of the original space). We now apply this result to the study of convex sets in the sense of Fan. Let $Y$ be a topological space, and let $H$ be a cone of continuous functions on $Y$ that contains a straight line passing through 1 and is such that the traces of elements of $H$ on $Q$ form a Choquet cone, where $Q$ is a Fan-convex compact set. In other words,

$$Q \triangleq \{z \in Y\colon h(z) \leqslant \sup_{x \in Q} h(x) \ (h \in H)\}.$$

FAN'S THEOREM. *The set $Q$ is the $H$-convex hull of the Choquet boundary $b(H)$, that is, $Q = \{z \in Y\colon h(z) \leqslant \sup_{x \in b(H)} h(x) \ (h \in H)\}$.*

PROOF. The inclusion $\{z \in Y\colon h(z) \leqslant \sup_{x \in b(H)} h(x) \ (h \in H)\} \subset Q$ is obvious. Let $z \in Q$. Then by Choquet's theorem there exists a measure $\mu \geqslant 0$ concentrated on $b(H)$ such that $\mu(h) \geqslant h(z)$ for $h \in H$, that is, $h(z) \leqslant \sup_{x \in b(H)} h(x) \cdot \mu(1) = \sup_{x \in b(H)} h(x)$. This proves the result.

Finally we look at the notion of a *boundary* of a cone $H$. This, by definition, is a set $B \subset Q$ such that $\sup_{x \in Q} h(x) = \sup_{x \in B} h(x)$ for all $h \in H$.

It follows immediately from the above results that the Choquet boundary $b(H)$ is contained in any closed boundary of $H$.

The closure of the Choquet boundary is called the Shilov boundary. In particular, if the Choquet boundary of the cone $H$ is a boundary of $H$ (this is by no means the case in general, but is true, for instance, for Choquet cones), then the Shilov boundary is a minimal closed boundary of this cone. For further properties of the Choquet and Shilov boundaries see [105].

**2.6. Weak $H$-convexity.** We saw in the previous subsection that $H$-convex functions can be represented by means of integral inequalities. It is well known, however, that there are two methods for defining "good" classes of convex functions: by means of the discrete Jensen inequalities, and as upper envelopes of affine functions or subclasses of them. In general, the classes of functions determined by these two methods are not necessarily the same. In this subsection we establish a connection between these two ways of inducing convexity.

Thus, let $Q$ be a set, $X$ a $K$-lineal in $R^Q$ (where the supremum of two elements taken in $X$ or in $R^Q$ is the same), and let $H$ be a cone in $X$. A function $f \in X$ is *weakly $H$-convex* if for any points $z, x_1, \ldots, x_n \in Q$ and non-negative numbers $\alpha_1, \ldots, \alpha_n$ such that $\sum_{k=1}^{n} \alpha_k h(x_k) \geqslant h(z)$ $(h \in H)$ we have $\sum_{k=1}^{n} \alpha_k f(x_k) \geqslant f(z)$. We denote the set of all weakly $H$-convex functions by $P_w(H)$. It is clear that $P(H) \triangleq P(H, X, \widetilde{R}^Q)$ is contained in $P_w(H)$. The converse is false, in general. The difference between $H$-convexity and weak $H$-convexity is obvious. The latter class is closed in the topology of simple convergence; the second is, in general, not. The decomposition theorem shows that the above circumstance characterizes the difference.

THEOREM 6.1. *$P_w(H) = \overline{P(\overline{H})}$, where the bar denotes the closure with respect to the topology of simple convergence in $X$.*

The following is a useful corollary of this theorem.

PROPOSITION 6.1. *Let $H$ be a closed cone in the $K$-semi-lineal $\mathfrak{V}_n$. Then every continuous weakly $H$-convex function is the uniform limit of a sequence of $H$-convex functions.*

This proposition lies behind the phenomenon described in Example 4.6.

## § 3. Supremal generators

**3.0. Preliminary remarks.** We introduce in this subsection dual characteristics of $H$-convex elements related to the convergence of sequences of operators and functionals that are based on the operator principle of preservation of inequalities. It will be convenient to set out the results in the form of theorems on supremal generators, that is, on cones $H$ such that every element of the space is (in some sense or another) $H$-convex. We adopt this style of exposition because most research concerned with this topic is devoted to the study of the definability of operators by their values on subspaces.

The existence of restricted (finite) cones which completely determine the convergence properties of operators enables us to simplify considerably the analysis of convergence phenomena. At the same time, generators find applications in a number of other problems (quasilinearization, classification of compacta, etc.).

Research prior to the theory of generators (see the Commentary) has, in the main, been concerned with the problem of convergence to the identity of sequences of (usually positive) operators.

The approach based on Minkowski duality allows us to grasp the nature of similar phenomena, and also to obtain by unified natural methods necessary and sufficient conditions for convergence to an arbitrary positive operator, to an operator with an abstract norm, and so on.

**3.1. Supremal generators with respect to a $K$-space.** Let $Y$ be a $K$-space and $X$ a vector subspace of $Y$. A minorant cone $H$ in $X$ is said to be a *supremal generator of* $X$ with respect to $Y$ if any element $x \in X$ is $H$-convex, that is, $x = \sup \{h \in H\colon h \leqslant x\}$. It turns out that a supremal generator can be characterized as follows: convergence to the identity of a sequence of positive operators on the generator $H$ implies convergence of this sequence on the whole of $X$. This is stated more precisely in the next theorem (using the notation of 2.2).

THEOREM 1.1. *Let $H$ be a minorant cone in the vector subspace $X$ of the $K$-space $Y$. Let $E\colon X \to Y$ be the inclusion operator. Then the following statements are equivalent:*

(1) *$H$ is a supremal generator of $X$ with respect to $Y$.*

(2) *For any sequence of operators $(T_n)$ such that $T_n \in \mathscr{L}^+(X, Y)$ and $\underline{\lim}_n\, T_n h \geqslant h$ for all $h \in H$, we have $T_n x \overset{(0)}{\to} x$ $(x \in X)$.*

(3) $\operatorname{Spr}(E, H) = \{E\}$.

PROOF. (1) $\Rightarrow$ (2). Let $x \in X$ and $h \in U_x \overset{\Delta}{=} \{ h \in H\colon h \leqslant x\}$. Then $T_n x \geqslant T_n h$, and therefore $\underline{\lim}_n\, T_n x \geqslant \underline{\lim}_n\, T_n h \geqslant h$. Since $\sup U_x = x$, we have $\underline{\lim}_n\, T_n x \geqslant x$. Repeating the argument for $-x$, we see that $\underline{\lim}_n\, T_n(-x) \geqslant -x$ or, what is the same, $\overline{\lim}_n\, T_n x \leqslant x$. This completes the proof.

(2) $\Rightarrow$ (3). This is obvious.

(3) $\Rightarrow$ (1). This follows from the operator principle of preservation of inequalities (Theorem 2.2.1).

REMARK 1. In the proof of (1) $\Rightarrow$ (2) we have nowhere used the fact that $H$ is a cone. Instead of the inclusion operator $E$ we can, more generally, consider a non-linear operator $T$ sending $X$ into a $K$-space $Z$, where $T$ is monotone, odd (that is, $T(-x) = -Tx$ $(x \in X)$) and commuting on $H$ with the sup operation (that is, $Tx = \sup_{h \in U_x} Th$ for any $x \in X$). Also, the sequence $(T_n)$ must consist of monotone odd operators sending $X$ into $Z$. If $H$ is a supremal generator, then an operator

$T \in \mathscr{L}^+(X, Z)$ has the property $\mathrm{Spr}(T, H) = \{T\}$ if and only if $T$ commutes on $H$ with the sup operation.

REMARK 2. In certain cases the "qualified" convergence of $(T_n)$ on a supremal generator $H$ implies its "qualified" convergence on the whole of $X$. Thus, for example, if the cone $H$ is the conical hull of at most a countable number of elements and $(T_n)$ is such that for all $h \in H$ there exists $(*) - \lim\limits_n T_n h \stackrel{\Delta}{=} b_h$, where $b_h \geqslant h$, then $T_n x \stackrel{(*)}{\to} x$ for all $x \in X$.

3.2. **Supremal generators of $C(Q)$ relative to $B(Q)$.** Let $Q$ be a compact topological space, $C(Q)$ the $K$-lineal of continuous functions on $Q$, and $B(Q)$ the $K$-space of bounded functions on $Q$. The study of supremal generators of $C(Q)$ relative to $B(Q)$ uses the results of § 2. From Proposition 5.1 in § 2 and the principle of preservation of inequalities it follows that a cone $H$ is a supremal generator of $C(Q)$ relative to $B(Q)$ if and only if the Dirac measure $\varepsilon_x$ is $H$-maximal for any $x \in Q$. This can be restated as follows: the Choquet boundary of a cone $H$ is $Q$. (As was mentioned in § 2, the existence of an $H$-maximal measure is equivalent to the cone $H$ being minorant.)

Theorem 1.1 describes the supremal generator $H$ in terms of the positive germ $\mathrm{Spr}(E, H)$ of the inclusion operator $E\colon C(Q) \to B(Q)$. It is of interest to clarify whether a generator can be characterized by means of generators in $\mathscr{L}^+(C(Q), C(Q))$. For an operator $T \in \mathscr{L}^+(C(Q), C(Q))$ we set

$$\mathrm{Spr}_{C(Q)}(T, H) \stackrel{\Delta}{=} \{T' \in \mathscr{L}^+(C(Q), C(Q))\colon T'h \geqslant Th \quad (h \in H)\}.$$

Let $I\colon C(Q) \to C(Q)$ be the identity operator. It is easy to see that the claim "if $\mathrm{Spr}_{C(Q)}(I, H) = \{I\}$, then $H$ is a supremal generator of $C(Q)$ relative to $B(Q)$" is, in general, false. However, one can find a set of operators $T_c(Q)$ such that from $\mathrm{Spr}_{C(Q)}(T, H) = \{T\}$ for all $T \in T_c(Q)$ it follows that $H$ is a generator. Such a set is that of all operators of the form $T_\varphi$, where $\varphi$ is a continuous mapping of the compactum $Q$ into itself, $T_\varphi\colon f \mapsto f \circ \varphi$ $(f \in C(Q))$.

The above remarks form a part of the following theorem.

THEOREM 2.1. *Let $H$ be a cone in $C(Q)$. Then the following statements are equivalent:*

(1) *$H$ is a supremal generator of $C(Q)$ relative to $B(Q)$.*

(2) *For any $x \in Q$ the measure $\varepsilon_x$ is $H$-maximal.*

(3) *If $x \in Q$ and the sequence of measures $(\mu_n)$ is such that $\varliminf\limits_n \mu_n(h) \geqslant h(x)$ for all $h \in H$, then $(\mu_n)$ converges to $\varepsilon_x$ (in the $\sigma(C'(Q), C(Q))$-topology).*

(4) *For any operator $T \in T_c(Q)$ the positive germ* $\mathrm{Spr}_{C(Q)}(T, H)$ *is the same as* $\{T\}$.

(5) *If $T \in T_c(Q)$ and the sequence $(T_n)$ in $\mathscr{L}^+$ $(C(Q), C(Q))$ is such that* $\lim_n T_n h \geqslant Th$ *for all $h \in H$ (where the convergence is uniform), then* $\| T_n f - Tf \| \to 0$ *for any $f \in C(Q)$.*

(6) *For any operator $T \in \mathscr{L}^+(C(Q), B(Q)$ the decompositional germ* $\mathrm{Dpr}(T, H)$ *coincides with* $\{T\}$.

Note that (5) shows that in the given situation one can, apart from (o)-convergence, talk about uniform convergence (see Remark 2 to Theorem 1.1). Theorem 2.1 and modifications of it have various applications. Note, in particular, that Ray's theorem on the resolvent (see, for instance, [105]) is an immediate consequence.

We now formulate criteria for a supremal generator in terms of "almost peaked" functions.

THEOREM 2.2. *A minorant cone $H$ is a supremal generator of $C(Q)$ in the sense of $B(Q)$ if and only if it has the following property* (A): *for any $\varepsilon > 0$, $z \in Q$, and neighbourhood $U$ of $z$ there exists an $h \in H$ which is a "support to the Uryson peak", that is, such that*

$$(2.1) \quad h(z) > 1 - \varepsilon, \quad h(x) \leqslant 1 \quad (x \in U), \quad h(x) \leqslant 0 \quad (x \in Q \setminus U).$$

REMARK. Theorem 2.2 is a Bishop–de Leeuw type statement (see, for instance, [9]), giving a description of the Choquet boundary of functional algebras in terms of supports to Uryson peaks.

The above theorem often enables us to decide easily whether a given cone is a generator. The following example illustrates this.

EXAMPLE 2.1. Let $Q$ be a compact subset of the interior of the positive orthant $R^n_+$ of $R^n$. Let us show that the cone $H$ spanned by the generators $-1, x \mapsto x_1, \ldots, x \mapsto x_n, x \mapsto -|x|^2$ is a supremal generator of $C(Q)$ relative to $B(Q)$. (Here $x_1, \ldots, x_n$ are the coordinates of $x$.) With this aim we take $\varepsilon > 0$, $z \in Q$, and $U$ a neighbourhood of $z$, and we consider the function $h'$: $x \mapsto -|x - z|^2$. Then it is clear that $h' \in H$. Now we choose a small $\delta > 0$ such that $\delta < |z|^2$ and $-\delta > \max_{x \in Q \setminus U} h'(x)$, and we set $h'' \stackrel{\Delta}{=} (h' + \delta 1)/\delta$. Then it is not difficult to verify that $h'' \in H$; in addition, $h''$ satisfies condition (A) for the given $\varepsilon$, $z$ and $U$. The required result now follows from Theorem 2.2.

This example demonstrates the existence of finite cones in $C(Q)$ that are supremal generators relative to $B(Q)$.

EXAMPLE 2.2. The following result (originally due to Korovkin [54]) is an easy corollary to Theorem 2.2. A three-dimensional subspace of $C([a, b])$ is a supremal generator of $C([a, b])$ relative to $B([a, b])$ if and only if it contains three functions forming a Chebyshev system on $[a, b]$.

We now touch upon the question of generators of $C(Q)$ relative to a $K$-space $S(Q)$. Let $\mu$ be a Baire measure on a compact set $Q$ whose support is the whole of $Q$. We denote by $S(Q)$ the $K$-space of all $\mu$-measurable functions defined on $Q$ (more precisely, classes of $\mu$-equivalent functions). Note that $C(Q)$ can be regarded as a vector subspace of $S(Q)$. Then we have the following simple result:

PROPOSITION 2.1. *Let $H$ be a cone in $C(Q)$ whose Choquet boundary contains a measurable subset of full measure; then $H$ is a supremal generator of $C(Q)$ relative to $S(Q)$.*

Combining this with Remark 2 after Theorem 1, we can prove the following result.

THEOREM 2.3. *Let $H$ be a separable cone in $C(Q)$ whose Choquet boundary contains a measurable subset of full measure; let $Z$ be a $K$-space normally embedded in $S(Q)$ and containing $C(Q)$; finally, let $(T_n)$ be a sequence of operators in $\mathcal{L}^+(C(Q), Z)$ such that for all $h \in H$ the limit $(*)\text{-}\lim\limits_n T_n h \triangleq b_h$ exists, where $b_h \geqslant h$. Then $T_n f \overset{(*)}{\to} f$ for all $f \in C(Q)$.*

REMARK 1. We recall that in the $K$-space $S(Q)$ $(*)$-convergence is the same as convergence in measure. If $Z$ is a $KB$-space, then $(*)$-convergence is the same as convergence in norm.

REMARK 2. Using the Banach–Steinhaus theorem one can modify Theorem 2.3 as follows. Let $Z_1$ be a $B$-space and $Z_2$ a $KB$-space, where $C(Q) \subset Z_1 \subset Z_2 \subset S(Q)$, $C(Q)$ is dense in $Z_1$ and $Z_2$ is normally embedded in $S(Q)$. Let $H$ be the same cone as in Theorem 2.3, $T_n \in \mathcal{L}^+(Z_1, Z_2)$, $\sup\limits_n \| T_n \| < +\infty$, and suppose that $\lim\limits_n \| T_n h - b_h \|_{Z_2} = 0$, where $h \in H$, $b_h \geqslant h$. Then $\| T_n f - f \|_{Z_2} \to 0$ for all $f \in Z_1$.

REMARK 3. We can consider the convergence of a sequence $(T_n)$ not only to the inclusion operator but, more generally, to any operator commuting with the sup operation. Results of this type for subspaces have been obtained by Krasnosel'skii and Lifshits [56], [57] using the "Theorem on complete shadows".

3.3. **Finite generators.** Of considerable interest are the finite supremal generators of a vector space $X$ with respect to a $K$-space $Y$ (that is, those spanned by finitely many generators). It turns out that if $X$ is a $K$-lineal (in the sense of the order induced by $Y$), then the existence of a finite generator in $X$ implies that $X$ is a $K$-lineal of bounded elements. Furthermore, we have

PROPOSITION 3.1. *A $K$-lineal $X$ containing a minorant cone $H$ is a $K$-lineal of bounded elements.*

For the proof it suffices to note that the element $-u$, where $u$ is the infimum of the generators of the cone, is a unit in $X$ bounding each element.

By the Krein–Kakutani theorem, for each Archimedean $K$-lineal of bounded elements $X$ there exists a compact set $Q$ such that $X$ is (algebraically and order) isomorphic to a dense sublattice of $C(Q)$. We can therefore confine ourselves to the study of finite generators of $C(Q)$ relative to $B(Q)$.

It is appropriate to point out that any dense subspace of $C(Q)$ is a supremal generator of $C(Q)$ with respect to $B(Q)$ (see Theorem 2.1). This fact enables us to give, within the setting of the $K$-lineal of bounded elements $X$, a criterion for a cone $H$ in $X$ (where the latter is realized as a dense subspace of $C(Q)$) to be a supremal generator of $X$ (and therefore also of $C(Q)$) with respect to $B(Q)$. Namely, $H$ has this property if and only if a minimal upper semilattice generating $H$ (that is, a set of elements of the form $h_1 \vee \cdots \vee h_m$, where $h_i \in H$, $(i = 1, \ldots, m)$) is dense in $X$.

We now turn our attention to the problem of what sort of a compactum $Q$ has to be so that $C(Q)$ has a finite generator, and also what is the minimal "dimension" of a generator. In this connection we introduce the following definitions. A number $m$ is called the *supremal rank* of a compact set $Q$ (or of the space $C(Q)$) if: (1) there exists a family $f_1, \ldots, f_m$ of continuous functions on $Q$ such that the cone spanned by the generators $-1, f_1, \ldots, f_m$, is a supremal generator of $C(Q)$ with respect to $B(Q)$; (2) no family $g_1, \ldots, g_r$ $(r < m)$ has this property. A number $m$ is called the *complete supremal rank* of $Q$ if (1) there exists a cone in $C(Q)$ spanned by $m$ generators that is a supremal generator of $C(Q)$ with respect to $B(Q)$; (2) no cone spanned by fewer than $m$ generators has this property. These definitions are justified, in particular, by the fact that homeomorphic compacta have the same (complete) supremal rank. We denote the supremal and complete supremal ranks of $Q$ by $\operatorname{sim}(Q)$ and $\operatorname{Sim}(Q)$, respectively. If $\operatorname{Sim}(Q)$ is defined, then it follows at once from the definitions that either $\operatorname{Sim}(Q) = \operatorname{sim}(Q) + 1$ or $\operatorname{Sim}(Q) = \operatorname{sim}(Q)$.

Example 2.1 shows that if $Q$ is a compact subset of $R$, then $\operatorname{sim}(Q) \leqslant n+1$. The following theorem shows that this estimate cannot be improved.

THEOREM 3.1. *The supremal rank* $\operatorname{sim}(Q)$ *of a compactum* $Q$ *is* $n + 1$ $(n \geqslant 1)$ *if and only if the smallest dimension of a Euclidean space into which* $Q$ *can be topologically embedded is equal to* $n$.

COROLLARY. The space $C(Q)$ has a finite supremal generator relative to $B(Q)$ if and only if $Q$ is finite-dimensional.

The following result is due to Rutkovskii.

THEOREM 3.2. *A compactum* $Q$ *is homeomorphic to a finite-dimensional sphere if and only if* $\operatorname{sim}(Q) = \operatorname{Sim}(Q)$.

Next we recall the definition of a Korovkin system ($K$-system) [115]. This is a system $\varphi \overset{\Delta}{=} (f_1, \ldots, f_m)$ of continuous functions on a compactum $Q$ such that the subspace of $C(Q)$ generated by it is a supremal generator of $C(Q)$. It is not hard to see that if $\varphi = (f_1, \ldots, f_m)$ is a $K$-system on $Q$, then there exist functions $g_2, \ldots, g_m$ such that $\psi \overset{\Delta}{=} (-1, g_2, \ldots, g_m)$ is also a $K$-system. It follows from Theorem 3.2 and from what has just been said that the supremal rank is a somewhat more refined characterization of a compactum than the minimal rank of Korovkin systems.

We conclude this subsection by considering supremal generators of a $K$-space $Y$ in itself (in the notation of §3.1 the subspace $X$ is the whole of $Y$).

THEOREM 3.3 [98]. *Let $Y$ be a $K$-space. The following statements are equivalent:*

(1) *$Y$ has a finite supremal generator in itself.*

(2) *$Y$ has a supremal generator in itself that is spanned by three generators.*

(3) *$Y$ is isomorphic to the $K$-completion $\hat{C}(Q)$ of $C(Q)$, where $Q$ is a finite-dimensional compactum.*

(4) *$Y$ is a $K$-space of bounded elements whose basis is a separable Boolean algebra.*

REMARK. If $Y$ is not isomorphic to $R^1$ or $R^2$, then it does not contain supremal generators in itself spanned by two generators.

It follows from Theorem 3.3, for instance, that the space $l^\infty$ of all bounded sequences has generators in itself spanned by three generators; the space $L^\infty([0, 1])$ of all almost-everywhere bounded functions on $[0, 1]$ (in the Lebesgue measure) has no finite generator in itself.

**3.4. The supremal generator with respect to an operator.** Here we study an important case of the construction of a supremal generator. As a preliminary we extend the definition of a minorant cone to ordered vector spaces (which are not necessarily lattices). A cone $H$ in $X$ is said to be minorant if for any $x \in X$ the set $U_x \triangleq \{h \in H: h \leqslant x\}$ is non-empty.

A cone $H$ in an ordered vector space $X$ is called a *supremal generator of $X$ relative to a positive operator* $T: X \to Y$ (where $Y$ is a $K$-space) if $H$ is minorant and $Tx = \sup_{h \in U_x} Th$ $(x \in X)$. If $X \subset Y$ and $T \triangleq E$ is the inclusion operator, then this coincides with the standard definition. In our case the following analogue to Theorem 1.1 holds.

THEOREM 4.1. *Let $X$ be an ordered vector space, $Y$ a $K$-space, $H$ a minorant cone in $X$, and $T \in \mathscr{L}^+(X, Y)$. Then the following statements are equivalent:*

(1) *$H$ is a supremal generator relative to $T$.*

(2) *If the sequence $(T_n)$, where $T_n \in \mathscr{L}^+(X, Y)$ is such that $\underline{\lim}_n T_n h \geqslant Th$ for all $h \in H$, then $(o)\text{-}\lim_n T_n x = Tx$ for all $x \in X$.*

(3) $\mathrm{Spr}(T, H) = \{T\}$.

We illustrate this with the following example.

EXAMPLE 4.1. Let $G$ be a region (assumed to be bounded, for simplicity) in $R^n$ with compact boundary $\partial G$. By $H_G$ we denote the space of harmonic bounded functions in $G$. Clearly $H_G$ is a normal subspace of the $K$-space of differences of positive harmonic functions on $G$. Hence $H_G$ is a $K$-space. We denote by $H_{\partial G}$ the subspace of $C(\partial G)$ consisting of the traces on $\partial G$ of functions in $HC_{\bar{G}}$ (where $HC_{\bar{G}}$ is the space of continuous functions on $\overline{G}$ that are harmonic in $G$). Note that $H_{\partial G}$ and $HC_{\overline{G}}$ are isometric.

Consider now the operator $T$: $C(\partial G) \to H_G$ that associates with a function $f \in C(\partial G)$ the corresponding solution to the generalized Dirichlet problem. It is well known that $T \in \mathscr{L}^+(C(\partial G), H_G)$; furthermore, $T$ takes $f$ in $H_{\partial G}$ to a member of $HC_{\overline{G}}$ whose trace on $\partial G$ is $f$. Then we have:

KELDYSH'S THEOREM [49]. *The positive germ* $\mathrm{Spr}(T, H_{\partial G})$ *of the operator $T$ on $H_{\partial G}$ is* $\{T\}$.

It follows from this and Theorem 4.1 that the solution $Tf$ (corresponding to $f$) of the generalized Dirichlet problem is represented as follows:

$$Tf = \sup \{h: h \in HC_{\overline{G}},\ h(x) \leqslant f(x)\ (x \in \partial G)\},$$

where the sup is, naturally, computed in the $K$-space $H_G$. It is clear that Keldysh's theorem is in turn a simple consequence of the above representation.

3.5. **The supremal generator relative to a functional.** Let $X$ be a locally convex space ordered by the cone[1] $K$. Here we study supremal generators relative to functionals $\mu$ in $X'$. It is clear that in the present situation Theorem 4.1 is valid (with $Y \overset{\Delta}{=} R$). If $X$ is such that every additive homogeneous positive functional is continuous, then we can confine ourselves in Theorem 4.1 to the continuous functionals (more precisely, in (2) we may suppose that the functionals $\mu_n$ (which are the $T_n$ in the notation of Theorem 4.1) belong to $X'$, and in (3) we can set $\mathrm{Spr}_X(\mu, H) \overset{\Delta}{=} \{\mu' \in K^*: \mu'(h) \geqslant \mu(h)(h \in H)\} = K^* \cap (\mu + H^*))$. This theorem is particularly useful when the cone $K$ is solid. The fact is that in this case the equality $\mathrm{Spr}(\mu, H) = \{\mu\}$ for some $\mu \in K^*$ implies that $H$ is minorant, so that the latter requirement can be dropped from the condition of the theorem.

As an example of a generator relative to a functional we consider a *supremal generator* of the space $C(Q)$ *of order $n$*. This, by definition, is a cone in $C(Q)$ that is a supremal generator of $C(Q)$ with respect to a probability measure concentrated at not more than $n$ points.

PROPOSITION 5.1. *A cone $H$ is a supremal generator of $C(Q)$ of order $n$ if and only if for any $f \in C(Q)$, any $\varepsilon > 0$, and any points $x_1, \dots, x_n \in Q$ there exists an $h \in H$ such that $h(x) \leqslant f(x)$ $(x \in Q)$, $h(x_i) > f(x_i) - \varepsilon$ $(i = 1, \dots, n)$.*

Using this result it is not difficult to give examples of generators of order $n$. We are only interested in finite generators of this type. These generators are closely related to *generalized Korovkin systems of order $n$* ($K$-systems) [115]. Such a system is by definition a family of continuous functions $f_1, \dots, f_n$ on $Q$ for which the subspace spanned by

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[1] The cone $K$ may contain straight lines, in which case $X$ is merely endowed with a pre-ordering. We still say, however, that $X$ is *ordered* by this cone.

it is a generator of order $n$. Suppose that $f_1, \ldots, f_m$ are generators of a finite generator of order $n$. For $x \in Q$ we set $\psi(x) \triangleq (f_1(x), \ldots, f_m(x))$. The compactum $\psi(Q)$ is homeomorphic to $Q$ and is such that the traces on $\psi(Q)$ of the coordinate functions $x \mapsto x_i$ $(i = 1, \ldots, m)$ are generators of a generator of order $n$ in $C(\psi(Q))$. This fact enables us to reduce the study of the generators we are interested in to the description of compacta in a real space on which the traces of the coordinate functions are "supremal generators". The requisite description of these compacta is obtained by examining the structure of their convex hulls. Here is a relevant result.

SHASHKIN'S THEOREM [115]. *For the system* $(-1, f_1, \ldots, f_m)$ *to be a* $K_n$*-system it is necessary and sufficient that the mapping* $\tilde{\psi}: x \mapsto (f_1(x), \ldots, f_m(x))$ *is injective and that the convex hull of any* $r$ *points* $(1 \leqslant r \leqslant n)$ *of* $\tilde{\psi}(Q)$ *is a face of the convex hull* $\operatorname{co}(\tilde{\psi}(Q))$ *of this set.*

A closer look at the mapping $\psi$ shows that the following is true:

THEOREM 5.1. *For* $f_1, \ldots, f_m$ *to be generators of a generator of order* $n$ *in* $C(Q)$ *it is necessary and sufficient that no positive linear combination (of length* $r \leqslant n$*) of rows of the matrix*

$$\begin{pmatrix} f_1(x_1) & \ldots & f_m(x_1) \\ \vdots & & \vdots \\ f_1(x_{m+n+1}) & \ldots & f_m(x_{m+n+1}) \end{pmatrix}$$

*majorizes any positive linear combination of its other rows (where* $x_1, \ldots, x_{m+n+1}$ *are any distinct points of* $Q$*).*

A corresponding result for $K_n$-systems was proved by Shashkin [115].

The following result is due to Rutkovskii:

THEOREM 5.2. *The minimal dimension of a* $K_n$*-system in* $C(Q)$ *is the minimal number of generators that a generator of order* $n$ *in* $C(Q)$ *can have.*

We now consider supremal generators of $H$ relative to a functional $\mu$ in a locally convex space $X$ ordered by a cone $K$. If $K$ is non-solid, then the condition on $H$ to be minorant in the definition of a generator becomes very restrictive. This can be avoided as follows. A cone $H$ is said to be a *generalized supremal generator* of $X$ relative to $\mu$ if for any $x \in X$ the closure of the superlinear functional $q: x \to [-\infty, +\infty)$ defined by $q(x) \triangleq \sup\limits_{h \leqslant x,\, h \in H} \mu(h)$ is the same as $\mu$; in other words,

$\mu(x) = \sup\limits_{(x_\alpha)} \overline{\lim\limits_{\alpha}} \sup\limits_{h \leqslant x_\alpha,\, h \in H} \mu(h)$, where the outer sup is taken over all nets $(x_\alpha)_{\alpha \in A}$ such that $x_\alpha \in H + K$ $(\alpha \in A)$ and $x_\alpha \xrightarrow[A]{} x$. If $K$ is solid, then generalized and ordinary generators are the same.

THEOREM 5.3. *Let $H$ be a cone in $X$, $\mu \in K^*$. Then the following statements are equivalent:*

(1) *$H$ is a generalized supremal generator of $X$ relative to $\mu$.*

(2) *If $(\mu_n)$ is an equicontinuous sequence, $\mu_n \in K^*$, and $\underline{\lim}_n \mu_n(h) \geqslant \mu(h)$ for all $h \in H$, then $\mu_n \to \mu$ (in $\sigma(X', X)$).*

(3) $\mathrm{Spr}_X(\mu, H) \overset{\Delta}{=} K^* \cap (\mu + H^*) = \{\mu\}$.

EXAMPLE 5.1. A point $u$ is called a *point of smoothness* of a cone $K$ if $u \in K$ and there exists a unique (to within a factor) non-zero functional $\mu_u$ in $K^*$ such that $\mu_u(u) = 0$.

PROPOSITION 5.2. *Suppose that $u$ is a point of smoothness of the cone $K$ and that $u_1$ in $X$ is such that $\mu_u(u_1) \neq 0$. Then for any equicontinuous sequence of positive functionals $(\mu_n)$ such that $\mu_n(u) \to 0$, $\mu_n(u_1) \to \mu_u(u_1)$, we have $\mu_n \to \mu_u$ (in $\sigma(X', X)$).*

To prove this it suffices to remark that $\mathrm{Spr}_X(\mu_u, H) = \{\mu_u\}$. (Here $H$ is the plane in $X$ spanned by $u$ and $u_1$).

Points of smoothness were introduced by Klimov, Krasnosel'skii and Lifshits in [51]. A proof can be found there of Proposition 5.2 for Banach spaces. The paper also introduces the notion of a saturated and of a completely saturated subspace; these are used to obtain a number of results on the uniqueness of an extension of the identity operator and of convergence to the identity operator. These results easily follow from Theorems 5.3 and 4.1. Note that in our present situation uniqueness of an extension of the identity operator does not imply convergence of a sequence of operators. The notion of a point of smoothness has been generalized by Labsker [71]. His results also follow immediately from Theorem 5.3.

In the next two subsections we shall apply the above results to the study of operators and functionals that are, in general, non-positive.

**3.6. The ordering superstructure and weak convergence of functionals.** Let $X$ be a locally convex space and let $U$ be a convex subset of $X'$ that contains the origin and is closed with respect to $\sigma(X', X)$. We denote by $p$ the support function of $U$ and by $\hat{p}$ the Minkowski gauge functional of this set: $p(x) \overset{\Delta}{=} \sup_{\mu \in U} \mu(x)$ $(x \in X)$: $\hat{p}(\mu) \overset{\Delta}{=} \inf\{\lambda > 0\colon \mu \in \lambda U\}$. Note that $\hat{p}(\mu) = \sup\{\mu(x)\colon p(x) \leqslant 1\}$ $(\mu \in X')$. We endow $X \times R$ with a pre-ordering via the cone $K$ of the supergraph of the functional $p$: $K \overset{\Delta}{=} \{(x, t) \in X \times R\colon t \geqslant p(x)\}$. It is not difficult to verify that $K^* = \{(\mu, s) \in X' \times R\colon s \geqslant \hat{p}(-\mu)\}$. This fact, together with Theorem 5.3, establishes the following result:

THEOREM 6.1. *Let $H$ be a cone in $X$. The following statements are equivalent:*

(1) *The cone* $H \times (-R_+)$ *is a generalized supremal generator of* $X \times R$ *relative to the functional* $(\mu_0, \hat{p}(-\mu_0))$, *where* $-\mu_0$ belongs to the *conical hull of* $U$.

(2) *If* $(\mu_n)$ *is an equicontinuous sequence in* $X'$ *such that* $\overline{\lim_n}\, \hat{p}(-\mu_n) \leqslant \hat{p}(-\mu_0)$ *and* $\underline{\lim_n}\, \mu_n(h) \geqslant \mu_0(h)$ $(h \in H)$, *then* $\mu_n \to \mu$ *(in* $\sigma(X', X)$).

(3) *If* $\mu \subset X'$, $\hat{p}(-\mu) \leqslant \hat{p}(-\mu_0)$ *and* $\mu(h) \geqslant \mu_0(h)$ $(h \in H)$, *then* $\mu = \mu_0$.

REMARK 1. Condition (2) can be replaced by the equivalent condition

(2′): (2) holds and $\lim_n \hat{p}(-\mu_n) = \hat{p}(-\mu_0)$.

REMARK 2. If $K$ is solid, then in (1) we can replace generalized by ordinary generators, and in (2) we can omit the equicontinuity requirement.

REMARK 3. If $U$ is a cone, then the conditions of Theorems 6.1 and 5.3 are the same. If $U$ is not a cone, then $\partial^+ U \triangleq \{\mu \in U\colon \hat{p}(\mu) = 1\} \neq \emptyset$. Condition (3) in the given situation means that the functional $\nu \triangleq -\mu$ from $U$, which is majorized on $H$ by a functional $\nu_0$ in $\partial^+ U$, coincides with $\nu_0$.

REMARK 4. If $h \in X$ is such that the functional $p$ has a unique support $\nu_0$ for which $p(h) = \nu_0(h)$, then the ray $H \triangleq \{\alpha h\}_{\alpha \leqslant 0}$ satisfies each of the equivalent conditions of Theorem 6.1 (relative to the functional $\mu_0 \triangleq -\nu_0$).

We now turn our attention to the case when $X$ is a normed space and $U$ is the unit ball in $X'$. Then $p$ is the norm on $X$ and $\hat{p}$ is the norm on $X'$. The space $X \times R$ ordered by the cone $K$, the supergraph of the norm, is called the *ordering superstructure* of $X$. (Observe that $K$ is solid.) By means of this superstructure and by applying Theorem 6.1, we can obtain a number of results on weak convergence. We mention two typical examples.

SHMUL'YAN'S THEOREM. *Let* $(x_n)$ *be a sequence of elements of a uniformly convex space* $X$. *Then it converges in norm to* $x_0 \neq 0$ *if and only if* $\overline{\lim_n} \|x_n\| \leqslant \|x_0\|$ *and* $\underline{\lim_n}\, \mu_0(x_n) \geqslant \mu_0(x_0)$, *where* $\mu_0$ *is the unique element of* $X'$ *such that* $\|\mu_0\| = \mu_0(x_0)/\|x_0\| = 1$.

THEOREM 6.2. *Let* $X$ *be a reflexive Banach space. The following statements are equivalent:*

(1) *The unit ball in* $X$ *is smooth.*

(2) *For each subspace* $H$, *any functional* $\mu_0 \in H'$, *and any sequencee* $(\mu_n) \subset X'$ *such that* $\overline{\lim_n} \|\mu_n\| \leqslant \sup_{\|x\| \leqslant 1, x \in H} |\mu_0(h)|$ *and* $\lim_n \mu_n(h) = \mu_0(h)$ $(h \in H)$, *the sequence* $(\mu_n)$ *is weakly convergent.*

(3) *Property* (2) *holds for one-dimensional subspaces.*

The above group of questions is closely connected with the problem of the uniqueness of extension of functionals (see, for instance, [35], [104], [114]).

If $H$ is a cone in $X$, then, as is easily seen, the cone $H \times (-R_+)$ is a supremal generator of the ordered superstructure relative to the functional $(\mu, \|\mu\|)$ if and only if $\mu(x) = \sup\limits_{h \in H} [\mu(h) - \|\mu\| \, \|x - h\|]$ for all $x \in X$. This shows that these questions are closely related to the Hahn–Banach theorem.

3.7. **Convergence of operators with an abstract norm.** A natural generalization of a bounded linear functional (or, what is the same, an operator from $X$ to the $K$-space $R$) is a linear operator $T$ from $X$ to a $K$-space $Y$ having an abstract norm $|T|$ (we recall that $|T| \triangleq \sup\limits_{\|x\| \leqslant 1} |Tx|$). It can be shown that $T$: $X \to Y$ has an abstract norm not exceeding $a \in Y$, $a \geqslant 0$, if and only if the operator $(T, a)$ acting on the ordering superstructure of $X$ into $Y$ according to the rule $(T, a)$: $(x, t) \longrightarrow Tx + ta$ is positive. Using this fact we can give an analogue to Theorem 6.1 pertaining to the present situation (it is clear that the generators in condition (1) must be relative to the operator $(T, |T|)$).

In this subsection we are concerned with a basic way of making the above discussion quite specific when $X$ is a $KN$-lineal of bounded elements. Suppose that $X$ is contained in a $K$-space $Y$ and is given the ordering induced by $Y$. If $H$ is a cone in $X$, then we denote by $\underset{\sim}{H}$ the conical hull of the element $(-1, -1)$ (where $1$ is the unit in $X$) and the cone $\{(h, -h) \in X \times X : h \in H\}$ in $X \times X$. Then we have

THEOREM 7.1. *Let $H$ be a cone in the $KN$-lineal of bounded elements $X$. Then the following statements are equivalent:*

(1) *$H \times (-R_+)$ is a supremal generator of the ordering superstructure of $X$ relative to the operator $(E, 1)$ (where $E$: $X \to Y$ is the inclusion operator).*

(2) *$\underset{\sim}{H}$ is a supremal generator of $X \times X$ relative to $\underset{\sim}{E}$: $(x_1, x_2) \mapsto x_1$.*

(3) *For any $x \in X$ we have $x = \sup\limits_{h \in H} (h - \|x - h\| 1)$.*

(4) *If $(T_n)$, where $T_n$: $X \to Y$, is such that $\overline{\lim\limits_n} |T_n| \leqslant 1$ and $\underline{\lim\limits_n}\, T_n h \geqslant h$ $(h \in H)$, then $(0)$-$\lim\limits_n T_n x = x$ for all $x \in X$.*

(5) *If $T$: $X \to Y$, $|T| \leqslant 1$ and $Th \geqslant h$ $(h \in H)$, then $T = E$.*

If $H$ is a subspace, then we can talk about ordinary supremal generators; more precisely, each of the above conditions is equivalent to the assertion that $H$ is a supremal generator of $X \times X$ with respect to the $K$-space $Y \times Y$.

This fact allows one occasionally to compute the minimal dimension of such subspaces (see [117]).

It is of interest to determine when in the above theorem the abstract norm can be replaced by the "ordinary" norm (it is understood here that $Y$ is a normed space). Simple examples show that this cannot always be done. However, when $Y$ is the $K$-space of all bounded functions, the situation is different. In more detail, we have:

PROPOSITION 7.1. *Let $T$ be an operator from the normed space $X$ to the KN-space $B(Q)$ of bounded elements. Then $|T| \leqslant 1$ if and only if $\|T\| < 1$.*

This proposition and Theorem 7.1 imply, in particular, certain results in [117].

In conclusion we mention that the method given in this section can also be applied to the case of compact (positive or non-expanding) operators in $C(Q)$. For this purpose, apart from the concept of $H$-convexity, use is made of Michael's theorem [76] on the choice of a continuous selector. We state a result of this kind for operators with a prescribed norm.

THEOREM 7.2. *Let $X$ be a normed space, $H$ a cone in $X$, and $T$: $X \to C(Q)$ a compact operator. Then for every $\varepsilon > 0$ we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1), where*

*(1) $H \times (-R_+)$ is a supremal generator of $X \times R$ relative to each of the functionals $(T_z, \|T\|)$, where $T_z$: $x \mapsto (Tz)(x)$ $(z \in Q)$.*

*(2) For any $T'$: $X \to B(Q)$ such that $\|T'\| < \|T\|$ and $T'h > Th$ $(h \in H)$ it follows that $T' = T$.*

*(3) For any sequence $(T_n)$ of operators $T_n$: $X \to C(Q)$ such that $\overline{\lim_n} \|T_n\| < \|T\|$ and the uniform limit $\lim_n T_n h \geqslant Th$ $(h \in H)$ it follows that $(T_n)$ is strongly convergent to $T$.*

*(4) If $T'$ is a compact operator $T'$: $X \to C(Q)$ such that $\|T'\| \leqslant (1 + \varepsilon) \|T\|$ and $T'h \geqslant Th$ $(h \in H)$, then $T' = T$.*

**3.8. Some examples of supremal generators.** Supremal generators find applications also in problems not connected with convergence and uniqueness of extensions. We illustrate this with examples of various kinds.

EXAMPLE 8.1. This is an application of generators to a generalization of Bellman and Kalaba's quasilinearization method [6], [43]. Following these authors we explain the idea of this method by an example of the Cauchy problem for the Riccati equation

$$(8.1) \qquad v' + v^2 + p(x)v + q(x) = 0, \quad v(x_0) = v_0.$$

Since the function $x \mapsto x^2$ is convex, $x = \max\limits_{v \in R} (2xu - u^2)$, so that (8.1) can be re-written in the form

$$(8.2) \qquad v' = \min_{u \in C(R)} [u^2 - 2uv - p(x)v - q(x)], \quad v(x_0) = v_0.$$

Let $w(u,\cdot)$ be a solution to the Cauchy problem

$$w' = u^2 - 2uw - p(x)w - q(x), \quad w(x_0) = v_0.$$

Let $[x_1, x_2]$ be an interval on which the solution $v$ to (8.1) exists. Taking into account the fact that the operator associating with $g$ the solution to the Cauchy problem $y' = f(x)y + g(x)$, $y(x_0) = v_0$ is monotone, it is not difficult to see that

$$v(x) = \min_{u \in U} w(u, x), \tag{8.3}$$

where $U$ is the space $C^1[x_1, x_2])$ of continuous differentiable functions on $[x_1, x_2]$.

Using the fact that the cone of concave quadratic trinomials supremally generates the space of continuous functions on the interval, we can use the method of Bellman and Kalaba in a much more general situation. Consider the Cauchy problem

$$y' = f(x, y), \quad y(x_0) = y_0, \tag{8.4}$$

where $f$ is continuous on the rectangle $[a, b] \times [c, d]$ and has continuous partial derivatives $f'_y$, $f''_{yy}$ there; we suppose that the problem (8.4) has a solution on $[a, b]$. Under the given conditions for each $x \in [a, b]$ and $y \in [c, d]$ the following formula holds:

$$f(x, y) = \max_{c \leqslant t \leqslant d} [-K(y-t)^2 + f'_y(x, t)(y-t) + f(x, t)], \tag{8.5}$$

where $K$ is a positive number such that $K \geqslant \max\limits_{a \leqslant x \leqslant b, c \leqslant y \leqslant d} \left(-\frac{1}{2} f''_{yy}(x, y)\right)$.

Using the idea of quasilinearization it is not difficult to verify that the solution $y$ to (8.4) can be represented as an upper envelope (pointwise maximum) of solutions to Riccati equations whose right-hand side is determined by the quadratic trinomials under the max sign in (8.5). Using (8.3) we can prove the following:

THEOREM 8.1. *There exists an interval $[a', b'] \subset [a, b]$ such that the solution to* (8.4) *defined on this interval takes the form*

$$y(x) = \max_{v \in V} \min_{w \in W} u(v, w, x), \tag{8.6}$$

*where $V \triangleq \{v \in C^1([a', b']) : v(x) \in [c, d]\ (x \in [a', b'])\}$, $W \triangleq C^1([a', b'])$, and $u(v, w, \cdot)$ is a solution to the linear equation*

$$u' = [-2K(v-w) + f'_y(x, v)]u - K(w^2 - v^2) - f'_y(x, v)v + f(x, v)$$

*under the initial condition $u(x_0) = y_0$. Furthermore, the maxmin in* (8.6) *is attained at $v = w = y$.*

This theorem allows us to evaluate the solution to (8.4) as a solution to the Riccati equation. It also enables us to construct a monotonely convergent interative process for the solution to (8.4); at each stage one has to solve the Riccati equation. This construction can be applied to a number of partial differential equations.

EXAMPLE 8.2. Let $Q$ be a convex compact subset of a locally convex space $V$, let $A$ be the subspace of affine functions (in $C(Q)$), and $\widetilde{P}$ the cone of concave functions (in $C(Q)$). The following theorem shows that every continuous function on $Q$ is obtained from the affine functions via the "maxmin" operation.

THEOREM 8.2. *If $f \in C(Q)$, then* $f(x) = \sup\limits_{p \in \widetilde{P}, p \leqslant f} \inf\limits_{h \in A, h \geqslant p} h(x)$.

For the proof it suffices to check, using Theorem 2.1, for example, that $\widetilde{P}$ is a supremal generator of $C(Q)$ relative to $B(Q)$.

EXAMPLE 8.3 (THE DIRICHLET PROBLEM FOR CONVEX FUNCTIONS). Let $S$ be a solid strictly convex compact subset of $R^n$ with boundary $\partial S$. It is easy to verify that every continuous function on $\partial S$ is convex, that is, the subspace $A$ of traces on $\partial S$ of affine functions generates $C(\partial S)$ with respect to $B(\partial S)$. From this it follows that every lower semicontinuous function $f$ on $\partial S$ is convex. For the given function $f$ and $x \in S$, set $\widetilde{f}(x) = \sup\{h(x)\colon h \in A,\ h(y) \leqslant f(y) \quad (y \in \partial S\}$. Then $\widetilde{f}$ is convex and coincides on the boundary with $f$.

Hence the following simple proposition, which is mentioned, for example, by Maiergoiz [75], holds: every lower semicontinuous function on $\partial S$ can be extended to a convex function defined on $S$.

EXAMPLE 8.4. Let $n \geqslant 4$. Denote by $A$ the set of $(n \times n)$-matrices with fixed principal diagonal, and give $A$ the natural ordering.

THEOREM 8.3. *For each matrix $a \in A$ there exist singular matrices $\underline{a}, \overline{a} \in A$ such that $\underline{a} \leqslant a \leqslant \overline{a}$.*

To prove this it suffices to take a three-dimensional generator in $R^n$ and to minorize each row of $a$ by a (row) vector of the generator. The matrix $\underline{a}$ determined by these rows has rank not exceeding three and minorizes $a$. It is easy to construct it in such a way that it is a member of $A$. The matrix $\overline{a}$ is constructed similarly.

It is interesting to note that for $n < 4$ Theorem 8.3 is no longer valid.

## §4. An application to extremal problems of geometry

**4.0. Preliminary remarks.** In this section we are concerned with some applications of the above results to extremal problems in the geometry of convex sets; this is the topic which historically was at the basis of Minkowski duality theory.

The history of the development of the study of isoperimetric problems within the framework of convex geometry is fairly fully set out, for instance, in [11], [17], [110]. For us it is important to note that at the beginning of the century these investigations led to two fundamental results, Blaschke's theorem of choice (which establishes the local compactness of the cone $\mathfrak{B}_n$ in $[\mathfrak{B}_n]$) and the Brunn–Minkowski theorem (on the quasi-concavity of the mixed volume). Blaschke's theorem enables us to prove theorems on the existence of solutions. The Brunn–Minkowski theorem in its turn enables us to perfect the technique of various kinds of symmetrization. Let us recall the idea behind this last method by an example on the Bieberbach problem: to find a convex figure of maximum volume with a given diameter. According to the Brunn–Minkowski theorem, under a Minkowski symmetrization the volume does not decrease and the diameter is not altered. Thus, the solution lies in the class of centrally symmetric figures. Now a centrally symmetric figure is contained in a ball of the same diameter. Hence the Bieberbach problem reduces to the following simple problem: among the figures lying in a ball of given diameter to find the figure of maximum volume. In this example of the application of the very simple technique of symmetrization we see the elegance and the weakness of similar methods. The basis of the method of symmetrization is the extremely specific structure of the constraints. No wonder then that interest in extremal problems in the geometry of convex sets has latterly been on the wane. The centre of gravity of research has shifted to extremal problems of another sort and in a class of different objects. However, one of the most complicated examples of solved extremal problems of complex geometry is still the problem of maximizing the area of a plane figure with given perimeter and radii of inscribed and circumscribed circles; this is an extremal problem with three constraints [14].

A. D. Aleksandrov [1] was the first to draw attention to the fact that a formal application of the method of Lagrange multipliers to the isoperimetric problem immediately leads to the answer. Since the method of duality of mathematical programming (which is an exact analogue of the method of Lagrange multipliers) enables us to make a uniform study of problems with many constraints, it is appropriate to apply the general methods of convex analysis to the study of problems of isoperimetric type.

To put this approach into effect we have to choose, first of all, a suitable vector space in which the investigation takes place. The most natural (although, in principle, not the only) space of this kind is undoubtedly the space of convex sets constructed in § 1. The reasons for this are the simple behaviour of geometric functionals with respect to the topological and algebraic structure of this space, the explicit treatment of linear functionals (measures closely connected with the convex figures in Aleksandrov's theorem) and finally, the standard explicit form of the polar of a cone of convex sets.

**4.1. The general problem of isoperimetric type.** We consider the following situation. We are given convex bodies $\mathfrak{A}_1^i, \ldots, \mathfrak{A}_{n-m_i}^i, \mathfrak{B}^i$ $(i=0, 1, \ldots, s)$ and numbers $b_1, \ldots, b_s \in R_+$.

PROBLEM 1.1. Among the convex figures satisfying the conditions

$V_{m_i, k_i}(\mathfrak{A}^i, \mathfrak{x}, \mathfrak{B}^i) \leqslant b_i$ $(i = 1, \ldots, s)$, to find a figure attaining the maximum of the function $V_{m_0, k_0}(\mathfrak{A}^0, \mathfrak{x}, \mathfrak{B}^0)$.

(Here $V_{m,k}(\cdot, \cdot, \cdot)$ is the corresponding mixed volume. We adhere to the following standard notation[1] for mixed volumes and mixed surface functions [17]:

$$V_{m,k}(\mathfrak{A}, \mathfrak{x}, \mathfrak{B}) \triangleq V(\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-m}, \mathfrak{x}, \ldots, \mathfrak{x}, \underbrace{\mathfrak{B}, \ldots, \mathfrak{B}}_{k}),$$

$$V_m(\mathfrak{A}, \mathfrak{B}) \triangleq V(\mathfrak{A}_1, \ldots, \mathfrak{A}, \underbrace{\mathfrak{B}, \ldots, \mathfrak{B}}_{m}),$$

$$\mu_{m,k}(\mathfrak{A}, \mathfrak{x}, \mathfrak{B}) \triangleq \mu(\mathfrak{A}_1, \ldots, \mathfrak{A}_{n-m}, \mathfrak{x}, \ldots, \mathfrak{x}, \underbrace{\mathfrak{B}, \ldots, \mathfrak{B}}_{k}),$$

$$\mu_m(\mathfrak{A}, \mathfrak{B}) \triangleq \mu(\mathfrak{A}, \ldots, \mathfrak{A}, \underbrace{\mathfrak{B}, \ldots, \mathfrak{B}}_{m}).$$

Thus, in particular,

$$V_{m,k}(\mathfrak{A}, \mathfrak{x}, \mathfrak{B}) = \frac{1}{n} \int\limits_{Z_n} \mathfrak{x}\, d\mu_{m,k}(\mathfrak{A}, \mathfrak{x}, \mathfrak{B}).$$

Note that $V(\mathfrak{x}) = V(\mathfrak{x}, \ldots, \mathfrak{x})$; the area of $S(x) = nV(\mathfrak{x}, \ldots, \mathfrak{x}, \mathfrak{z}_n) = nV_{n-1}(\mathfrak{x}, \mathfrak{z}_n) = nV_1(\mathfrak{x}, \mathfrak{z}_n)$, $\mathfrak{z}_n$ is the unit (Euclidean) ball in $R^n$ $(n \geqslant 2)$; $\mu(\mathfrak{x}) \triangleq \mu(\mathfrak{x}, \ldots, \mathfrak{x})$ is the surface function of $\mathfrak{x}$.

We suppose, as in [1], that the $V_{m_i, k_i}(\mathfrak{A}^i, \cdot, \mathfrak{B}^i)$ are extended with preservation of continuity and multilinearity to $[\mathfrak{V}_n]$ (or $C(Z_n)$), and we denote these extensions by $G_i$ $(i = 0, 1, \ldots, s)$. Then Problem 1.1 can be stated as the following problem in mathematical programming. To find an element $\mathfrak{x} \in [\mathfrak{V}_n]$ such that (a) $\mathfrak{x} \in \mathfrak{V}_n$; (b) $G_i(\mathfrak{x}) \leqslant b_i$ $(i = 1, \ldots, s)$; (c) $G_0(\mathfrak{x})$ attains its maximum.

An important fact is that the $G_i$ have Gateaux derivatives. In fact, it follows from the multilinearity of the $G_i$ that for an arbitrary derivative $(G_i)'_{\mathfrak{x}}$ of $G_i$ at $\mathfrak{x}$ the following formula holds (see [66]):

$$(G_i)'_{\mathfrak{x}}(g) = \frac{m_i - k_i}{n} \int\limits_{Z_n} g\, d\mu_{m_i, k_i}(\mathfrak{A}^i, \mathfrak{x}, \mathfrak{B}^i).$$

We now give the main theorem concerning Problem 1.1. This is the well-known approach developed by Dubovitskii and Milyutin [33]. First of all, we denote by $\mathfrak{V}_{n,\mathfrak{x}}$ the cone of possible directions at $\mathfrak{x} \in \mathfrak{V}_n$, that is,

$\mathfrak{V}_{n,\mathfrak{x}} \triangleq \{g \in [\mathfrak{V}_n]\colon \exists \alpha_0 > 0\colon \mathfrak{x} + \alpha \cdot g \in \mathfrak{V}_n)\ (0 \leqslant \alpha \leqslant \alpha_0)\}$.

---

[1] We denote by $V$ the mixed volume of $n$ bodies. We recall that, in particular, $V$ is a multilinear symmetric form in $n$ variables. The mixed surface function of $(n-1)$ variables is denoted by $\mu$.

THEOREM 1.1. *If the admissible body $\bar{\mathfrak{x}}$ is a solution to Problem* 1.1, *then there exist numbers $\bar{\alpha}_1, \ldots, \bar{\alpha}_s \in R_+$ complementing the non-rigidity, such that the following Euler–Lagrange equation holds:*

$$\sum_{i=1}^{s} \bar{\alpha}_i \mu_{m_i, k_i}(\mathfrak{A}^i, \bar{\mathfrak{x}}, \mathfrak{X}^i) - \mu_{m_0, k_0}(\mathfrak{A}^0, \bar{\mathfrak{x}}, \mathfrak{X}^0) \in \mathfrak{B}^*_{n, \bar{\mathfrak{x}}}.$$

Thus, the application of the general principle of duality (in the form of the method of possible directions) reduces the question of finding a solution to Problem 1.1 to the problem of representing elements of the cone $\mathfrak{B}^*_{n,\bar{\mathfrak{x}}}$ and to uniqueness theorems for surface functions. A very simple fact concerning the structure of $\mathfrak{B}^*_{n,\bar{\mathfrak{x}}}$ is the following:

PROPOSITION 1.1. *If $\bar{\mathfrak{x}}$ is a regular body, then $\mathfrak{B}^*_{n,\bar{\mathfrak{x}}} = \{0\}$.*

We give an example of application of this.

EXAMPLE 1.1. To find a regular body $\bar{\mathfrak{x}}$ from the conditions: (a) $V_1(\mathfrak{x}, \mathfrak{x}_i) \leqslant b_i$ $(i = 1, \ldots, s)$ ($\mathfrak{x}_1, \ldots, \mathfrak{x}_s$ being regular bodies), (b) $V(\mathfrak{x})$ attains its maximum.

By Theorem 1.1 and Proposition 1.1 there exist numbers $\bar{\alpha}_1, \ldots, \bar{\alpha}_s \in R_+$ such that $\mu(\bar{\mathfrak{x}}) = \sum_{i=1}^{s} \bar{\alpha}_i \mu_1(\bar{\mathfrak{x}}, \mathfrak{x}_i) = \mu_1(\bar{\mathfrak{x}}, \sum_{i=1}^{s} \bar{\alpha}_i \mathfrak{x}_i)$.

It then follows from the Aleksandrov–Volkov theorem [2] that the solution $\bar{\mathfrak{x}}$, to within a parallel shift, has the form $\sum_{i=1}^{s} \bar{\alpha}_i \mathfrak{x}_i$. By the same token, the original problem reduces to that of finding parameters $\bar{\alpha}_1, \ldots, \bar{\alpha}_s$, in other words, to a finite-dimensional problem in mathematical programming.

**4.2. The polar of the cone of possible directions.** We consider in more detail the representation of the cone $\mathfrak{B}^*_{n,\bar{\mathfrak{x}}}$. The cone $R^n$ (in $C(Z_n)$), as we have seen, has the Reshetnyak–Loomis property. In connection with this and with Aleksandrov's theorem it is convenient to introduce the following definition. A convex figure $\mathfrak{y}$ is said to be "$T$-antecedent" to a figure $\mathfrak{x}$ if $\mu(\mathfrak{x}) \underset{R^n}{\gg} \mu(\mathfrak{y})$; in this situation we use the notation $\mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y}$. The notion of $T$-antecedence has a simple intrinsic characterization. From Theorem 2.4.1 we derive:

PROPOSITION 2.1. *A figure $\mathfrak{y}$ is T-antecedent to $\mathfrak{x}$ if and only if $V_1(\mathfrak{x}, \mathfrak{z}) \geqslant V_1(\mathfrak{y}, \mathfrak{z})$ for any convex surface $\mathfrak{z}$.*

The connection between $\underset{T}{\succcurlyeq}$ and the relations $\underset{T}{\geqslant}$ and $\underset{T}{=}$ plays an important role. We say that $\mathfrak{y}$ is a "$T$-constituent" of $\mathfrak{x}$ ($\mathfrak{x} \underset{T}{\geqslant} \mathfrak{y}$) if $\mathfrak{y}$

can be put into $\mathfrak{x}$ by a parallel shift. The notation $\mathfrak{x} \underset{T}{=} \mathfrak{y}$ means that $\mathfrak{x}$ and $\mathfrak{y}$ coincide to within a shift. Note that if one of the compacta $\mathfrak{x}$ and $\mathfrak{y}$ is solid, then $(\mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y}$ and $\mathfrak{y} \underset{T}{\succcurlyeq} \mathfrak{x}) \Rightarrow \mathfrak{x} \underset{T}{=} \mathfrak{y}$. In addition to this, for any $\mathfrak{x}$ and $\mathfrak{y}$ we have $(\mathfrak{x} \underset{T}{\geqslant} \mathfrak{y} \Rightarrow \mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y})$.

PROPOSITION 2.2. *Let $n = 2$. Then* $(\mathfrak{x} \underset{T}{\geqslant} \mathfrak{y} \Leftrightarrow \mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y})$.

It is interesting to note that for $n > 2$ this no longer holds. Thus, for $n > 2$, $\alpha > 1$, the figures $\mathfrak{z}_n$ and $\alpha\mathfrak{z}_{n-2}$ are not comparable under $\underset{T}{\geqslant}$, although $\mathfrak{z}_n \underset{T}{\succcurlyeq} \alpha\mathfrak{z}_{n-2}$. For $\mu(\alpha\mathfrak{z}_{n-2}) = \alpha^{1/n-1}\mu(\mathfrak{z}_{n-2}) = 0$. Furthermore, $\mu(\mathfrak{z}_n)(x) = 0$ $(x \in R^n)$, so that $\mu(\mathfrak{z}_n) \underset{R^n}{\geqslant} 0$. One must not suppose that this fact is connected only with the degeneracy of $\mathfrak{z}_{n-2}$. Take, for instance, $\mathfrak{x} \overset{\Delta}{=} 2\mathfrak{z}_n$, $\mathfrak{B} \overset{\Delta}{=} \mathfrak{x} + \gamma\mathfrak{z}_{n-2}$, where $\gamma \overset{\Delta}{=} \beta^{1/n-2}$ and where $\beta$ is chosen so that $1 < \beta < 2^{n-1}(2^{n-1} - 1)^{-1}$. Then since $\gamma > 0$, $\mathfrak{x}$ and $\mathfrak{y}$ are not comparable under the relation $\underset{T}{\geqslant}$; on the other hand, it is not difficult to show that $V_1(\mathfrak{x}, \mathfrak{z}) \geqslant V_1(\mathfrak{y}, \mathfrak{z})$ for any $\mathfrak{z} \in \mathfrak{B}_n$; in other words, by Proposition 2.2, $\mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y}$. From Aleksandrov's theorem [1] it follows that

$$\mathfrak{B}_n^* = \{\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \colon \mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y}\}. \tag{2.1}$$

For let $\mu \overset{\Delta}{=} \mu_1 - \mu_2$, where $\mu_1, \mu_2 \geqslant 0$, $\mu_1 \underset{R^n}{\geqslant} \mu_2$. Then it is clear that $\int\limits_{Z_n} z\, d\mu_1 = \int\limits_{Z_n} z\, d\mu_2 = (u, z)$, where $u$ is a vector in $R^n$. It is obvious that the measures $\tilde{\mu}_1 \overset{\Delta}{=} \mu_1 + \mu(\mathfrak{z}_n) + |u|\,\varepsilon_{-\frac{u}{|u|}}$; $\tilde{\mu}_2 \overset{\Delta}{=} \mu_2 + \mu(\mathfrak{z}_n) + |u|\,\varepsilon_{-\frac{u}{|u|}}$ are of Aleksandrov type [66], where $\mu = \tilde{\mu}_1 - \tilde{\mu}_2$. Taking (2.1) and the preceding proposition into account we obtain the next theorem.

THEOREM 2.1. *The following representations hold:*

1. $$\mathfrak{B}_{n,\bar{\mathfrak{x}}}^* = \{\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \colon \mathfrak{x} \underset{T}{\succcurlyeq} \mathfrak{y},\ V_1(\mathfrak{x}, \bar{\mathfrak{x}}) = V_1(\mathfrak{y}, \bar{\mathfrak{x}})\}.$$

2. $$\mathfrak{B}_{2,\bar{\mathfrak{x}}}^* = \{\mu(\mathfrak{x}) - \mu(\mathfrak{y}) \colon \mathfrak{x} \underset{T}{\geqslant} \mathfrak{y},\ V(\mathfrak{x}, \bar{\mathfrak{x}}) = V(\mathfrak{y}, \bar{\mathfrak{x}})\}.$$

**4.3. The general solution of the plane isoperimetric problem.** We consider the general problem of maximizing an area under arbitrary linear constraints, that is, restrictions on the mixed area and the support distances of the required plane figure.

PROBLEM 3.1. We are given a polygon $P \triangleq \bigcap_{i=1}^{s} \{z \in R^2 : (z, z_i) \leqslant c_i\}$ $(z_i \in Z_2)$ and figures $\mathfrak{y}_1, \ldots, \mathfrak{y}_m$. It is required to find among the figures lying in $P$ for which $V(\mathfrak{y}_j, \mathfrak{x}) \leqslant b_j$ $(j = 1, \ldots, m)$, the figure with the greatest area.

We denote by $\nu_1, \ldots, \nu_p \in R^s$ the directions of the boundary rays of the cone $S \triangleq \{\alpha \in R^s_+ : \sum_{i=1}^{s} \alpha_i z_i = 0\}$ $\nu_t = (\nu_t = (\nu_t^1, \ldots, \nu_t^s))$ and consider the Aleksandrov measures $\mu_t \triangleq \sum_{i=1}^{s} \nu_t^i \varepsilon_{z_i}$. Let $\mathfrak{x}_t \in \mathfrak{V}_2$ be such that $\mu(\mathfrak{x}_t) = \mu_t$ $(t = 1, \ldots, p)$.

An application of Theorem 2.1 to this problem shows that the solution $\bar{\mathfrak{x}}$ satisfies the Euler–Lagrange equation

$$\sum_{i=1}^{s} \bar{\gamma}_i \varepsilon_{z_i} + \sum_{j=1}^{m} \bar{\gamma}_{s+j} \mu(\mathfrak{y}_j) - \mu(\bar{\mathfrak{x}}) \in \mathfrak{V}^*_{2,\bar{\mathfrak{x}}}. \tag{3.1}$$

It follows from (3.1) that $(\bar{\gamma}_1, \ldots, \bar{\gamma}_s) \in S$. In other words, there exist $\bar{\beta}_1, \ldots, \bar{\beta}_p \in R_+$ such that $\bar{\gamma}_i = \sum_{t=1}^{p} \bar{\beta}_t \nu_t^i$. Now we let $\tilde{\mathfrak{x}}$ denote the figure $\sum_{t=1}^{p} \bar{\beta}_t \mathfrak{x}_t + \sum_{j=1}^{m} \bar{\gamma}_{s+j} \mathfrak{y}_j$; so that by applying Theorem 2.1 we see that $\tilde{\mathfrak{x}} \underset{T}{\geqslant} \bar{\mathfrak{x}}$ and $V(\tilde{\mathfrak{x}}, \bar{\mathfrak{x}}) = V(\bar{\mathfrak{x}}, \bar{\mathfrak{x}})$. If $\bar{\mathfrak{x}}$ is solid, then it follows from all this that $\tilde{\mathfrak{x}} \underset{T}{=} \bar{\mathfrak{x}}$.

THEOREM 3.1. *A solid solution* $\bar{\mathfrak{x}}$ *to Problem* 3.1 *can be represented to within a shift in the form* $\bar{\alpha}_1 \mathfrak{x}_1 + \ldots + \bar{\alpha}_p \mathfrak{x}_p + \bar{\alpha}_{p+1} \mathfrak{y}_1 + \ldots + \bar{\alpha}_{p+m} \mathfrak{y}_m$.

It is not difficult to verify that the problem of determining the coefficients $\bar{\alpha}_1, \ldots, \bar{\alpha}_{p+m} \in R_+$ is a problem in quadratic programming; in particular, it can be solved in a finite number of steps.

**4.4. Convex isoperimetric problems.** Problem 3.1, which we looked at in the preceding subsection, is a special case of the isoperimetric problems that can be reduced to problems in (quasi-)convex programming.

Here is a result concerning such a problem.

PROBLEM 4.1. To find $\mathfrak{x} \in \mathfrak{V}_n$ such that (a) $V_1(\mathfrak{y}_j, \mathfrak{x}) \leqslant b_j$ $(j = 1, \ldots, m)$; (b) $V(\mathfrak{x})$ attains its maximum.

THEOREM 4.1. *An admissible body* $\bar{\mathfrak{x}}$ *is a solution to Problem* 4.1 *if and only if there exist numbers* $\bar{\alpha}_1, \ldots, \bar{\alpha}_m \in R_+$, *complementing non-rigidity such that* $\mathfrak{x}(\bar{\alpha}) \underset{T}{\succcurlyeq} \bar{\mathfrak{x}}$, $V(\bar{\mathfrak{x}}) = V_1(\mathfrak{x}(\bar{\alpha}), \bar{\mathfrak{x}})$, *where* $\mathfrak{x}(\bar{\alpha}) \overset{\Delta}{=} \bar{\alpha}_1\mathfrak{y}_1 \# \bar{\alpha}_2\mathfrak{y}_2 \# \ldots \# \bar{\alpha}_m\mathfrak{y}_m$.

Here the symbol $\#$ stands for the Blaschke sum (we recall that the Blaschke sum of two bodies $\mathfrak{x}$ and $\mathfrak{y}$ is the figure with the surface function $\mu(\mathfrak{x}) + \mu(\mathfrak{y})$; in the plane, the Minkowski and Blaschke sums are the same thing (to within a shift).

An interesting peculiarity of convex problems is the fact that their solution $\bar{\mathfrak{x}}$ induces an isoperimetric inequality of the form $\varphi(\bar{\mathfrak{x}}, \alpha) \geqslant \varphi(\mathfrak{x}, \bar{\alpha})$, where $\varphi$ is the corresponding Lagrange function. Thus, for the (generalized) Bieberbach problem of maximizing the mixed volume $V_{m,k}(\mathfrak{A}, \mathfrak{x}, \mathfrak{B})$ under the condition $d(\mathfrak{x}) = d(\bar{\mathfrak{x}})$ (where $d(\mathfrak{x})$ is the diameter of $\mathfrak{x}$), we obtain:

PROPOSITION 4.1. *The following conditions are equivalent:*

(1) $\bar{\mathfrak{x}}$ *is a solution to the Bieberbach problem.*

(2) *For any* $\mathfrak{x} \in \mathfrak{V}_n$ *we have*

$$V_{m,k}(\mathfrak{A}, \mathfrak{x}, \mathfrak{B})\, d^{m-k}(\bar{\mathfrak{x}}) - V_{m,k}(\mathfrak{A}, \bar{\mathfrak{x}}, \mathfrak{B})\, d^{m-k}(\mathfrak{x}) \leqslant 0.$$

**4.5. Constraints of inclusion type (the plane case).** In the next two sections we show how to handle the simplest operational constraints (that is, constraints of the form "the figure is contained in such and such" or "the figure is centrally-symmetric", and so on) in problems of isoperimetric type. For convenience we start by explaining the technique for obtaining optimization criteria (Euler–Lagrange equations) for plane problems; we then discuss the (fairly routine) way of modifying this technique for higher-dimensional spaces. For the model problems we consider, as a rule, only one constraint of general type (that is, on the mixed volume). It is worth emphasizing that the following account can be carried over verbatim to the case of an arbitrary number of constraints of this kind (see §4.3), whereas geometric intuition for these situations naturally breaks down.

PROBLEM 5.1 (THE INTERIOR ISOPERIMETRIC PROBLEM). Among the figures lying in a fixed body $\mathfrak{x}_0$ and having a given perimeter $S(\bar{\mathfrak{x}})$, to find the figure of greatest area.

REMARK. Apparently it would not be amiss to point out that for problems of this sort existence and uniqueness theorems hold trivially (to within a shift).

The equivalent problem in convex programming in the space $[\mathfrak{V}_2]$ can be posed as follows: to find $\mathfrak{x} \in [\mathfrak{V}_2]$ such that: (a) $\mathfrak{x} \in \mathfrak{V}_2$; (b) $\mathfrak{x} \underset{T}{\leqslant} \mathfrak{x}_0$; (c) $S(\mathfrak{x}) \leqslant S(\bar{\mathfrak{x}})$; $2\sqrt{(V(\bar{\mathfrak{x}})\ V(\mathfrak{x}))}$ attains its maximum.

In accordance with the general theory [27], the Lagrange function for this problem is defined on $\mathfrak{V}_2 \times R_+ \times C'_+(Z_2)$, where $C'_+(Z_2) \overset{\Delta}{=} \{\mu \in C'(Z_2): \mu \geqslant 0\}$, and takes the form

$$\varphi(\mathfrak{x}, \alpha, \mu) \overset{\Delta}{=} 2\sqrt{(V(\bar{\mathfrak{x}})V(\mathfrak{x}))} + \alpha(S(\bar{\mathfrak{x}}) - S(\mathfrak{x})) + \mu(\mathfrak{x}_0 - \mathfrak{x}).$$

By the Kuhn–Tucker theorem $\bar{\mathfrak{x}}$ is a solution to this problem if and only if there exist $\bar{\alpha} \in R_+$ and a measure $\bar{\mu} \in C'_+(Z_2)$ such that $\varphi$ has a saddle-point at $(\bar{\mathfrak{x}}, \bar{\alpha}, \bar{\mu})$, that is, $\varphi(\bar{\mathfrak{x}}, \alpha, \mu) \geqslant \varphi(\bar{\mathfrak{x}}, \bar{\alpha}, \bar{\mu}) \geqslant \varphi(\mathfrak{x}, \bar{\alpha}, \bar{\mu})$ $(\mathfrak{x} \in \mathfrak{V}_2, \ \alpha \in R_+, \ \mu \in C'_+(Z_2))$. The left-hand half of the saddle inequality reduces to $\bar{\mu}(\mathfrak{x}_0 - \bar{\mathfrak{x}}) = 0$, and the right-hand half – in differential form – to $\mu(\bar{\mathfrak{x}}) = \bar{\mu} + \bar{\alpha}\mu(\mathfrak{z}_2)$. By Aleksandrov's theorem $\bar{\mu} = \mu(\mathfrak{x})$ for some $\mathfrak{x} \in \mathfrak{V}_2$. Consequently we have:

THEOREM 5.1. *An admissible body* $\bar{\mathfrak{x}}$ *is a solution to the interior isoperimetric problem if and only if there exist a (critical) figure* $\mathfrak{x}$ *and a number* $\bar{\alpha} \geqslant 0$ *such that* (a) $\bar{\mathfrak{x}} \underset{T}{=} \mathfrak{x} + \bar{\alpha}\mathfrak{z}_2$; (b) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in s(\mathfrak{x})$. (Here and from now on $s(\mathfrak{x})$ is the support of $\mathfrak{x}$, that is, the support of $\mu(\mathfrak{x})$, the surface function of $\mathfrak{x}$.)

PROBLEM 5.2. (THE EXTERIOR ISOPERIMETRIC PROBLEM). Among the figures containing $\mathfrak{x}_0$ and having a given perimeter, to find the figure of greatest area.

By analogy with Theorem 5.1 we have:

THEOREM 5.2. *An admissible body* $\bar{\mathfrak{x}}$ *is a solution to the exterior isoperimetric problem if and only if there exist a critical figure* $\mathfrak{x}$ *and a number* $\bar{\alpha} \geqslant 0$ *such that* (a) $\bar{\mathfrak{x}} + \mathfrak{x} \leqslant \bar{\alpha}\mathfrak{z}_2$; (b) $V(\bar{\mathfrak{x}}) + V(\mathfrak{x}, \bar{\mathfrak{x}}) = \bar{\alpha}V(\bar{\mathfrak{x}}, \mathfrak{z}_2)$; (c) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in s(\mathfrak{x})$.

By similar methods we can analyse a condition of the form "$\bar{\mathfrak{x}}$ is centrally symmetric". The only refinement is in the description of the polar of the cone of symmetric figures.

PROPOSITION 5.1. $V(\mathfrak{x}, \mathfrak{z}) \geqslant V(\mathfrak{y}, \mathfrak{z})$ *for any centrally-symmetric figure* $\mathfrak{z}$ *if and only if* $\mathfrak{x}^s \underset{T}{\geqslant} \mathfrak{y}^s$ *(where* $z^s$ *denotes the Minkowski symmetrization of* $z$*, that is, the figure with support function* $u \mapsto \frac{1}{2}[z(u) + z(-u)]$*).*

By way of example we consider the interior isoperimetric problem in which we look for a solution in the class of the symmetric figures.

THEOREM 5.3. *An admissible body* $\bar{\mathfrak{x}}$ *is a solution, in the class of centrally-symmetric figures, to the interior isoperimetric problem if and only if there exist a critical figure* $\mathfrak{x}$ *and a number* $\bar{\alpha} \geqslant 0$ *such that*

(a) $\bar{\mathfrak{x}} \underset{T}{=} \mathfrak{x}^s + \bar{\alpha}\mathfrak{z}_2$; (b) $\bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z)$ *for all* $z \in s(\mathfrak{x})$.

For further examples see [64].

**4.6. Constraints of inclusion type (multidimensional case).** As we have seen in §4.1, for the general isoperimetric problem we can, as a rule, only obtain necessary conditions for an extremum. The arguments of the preceding subsection go over in full measure to the case of convex isoperimetric problems. Thus, the analogue of the exterior isoperimetric problem is not the exterior "isophany" problem, but the exterior Uryson problem, that is, the problem of maximizing the volume under a given integral range (that is, under conditions on $V_1(\mathfrak{z}_n, \cdot)$). We recall that in the plane the integral range and the perimeter are proportional functionals. The second special circumstance concerning the plane is that only in this case do the Blaschke and Minkowski sums coincide. Another point is that in the above theorems there was the question of dual functional conditions; in particular, in going over from plane to multidimensional optimization criteria one has to replace Minkowski sums by Blaschke sums (for instance, in the analogue to Theorem 5.3 one would be dealing with Blaschke symmetrization). The third feature of spaces of dimension $n > 2$ is the existence of non-negative translation-invariant measures that cannot be treated as surface functions of certain figures. This means that in the optimization criteria, critical measures rather than critical figures feature, as a rule. In summary we can say that the dual study of multidimensional problems has essentially just one peculiarity – the apparatus of support functions is replaced by that of surface functions.

We now give an illustrative example.

EXAMPLE 6.1 (THE EXTERIOR URYSON PROBLEM). Among those bodies containing $\mathfrak{x}_0$ and having a given integral range, to find a body $\bar{\mathfrak{x}}$ of maximal volume.

The optimization criteria are a restatement of Theorem 5.2 with the remarks we have made taken into account.

An admissible body $\bar{\mathfrak{x}}$ is a solution to the exterior Uryson problem if and only if there exist a positive critical measure $\mu$ and a number $\bar{\alpha} \geqslant 0$ such that:

$$\text{(a) } \bar{\alpha}\mu(\mathfrak{z}_n) \underset{R^n}{\gg} \mu(\bar{\mathfrak{x}}) + \mu;$$

$$\text{(b) } V(\bar{\mathfrak{x}}) + \frac{1}{n}\int\limits_{Z_n} \bar{\mathfrak{x}}\, d\mu = \bar{\alpha} V_1(\mathfrak{z}_n, \bar{\mathfrak{x}});$$

$$\text{(c) } \bar{\mathfrak{x}}(z) = \mathfrak{x}_0(z) \qquad (z \in s(\mu)).$$

Consider, for instance, the case $\mathfrak{x}_0 \overset{\Delta}{=} \mathfrak{z}_{n-1}$. Then it is clear that the required body is a spherical lens, and the critical measure is the trace of the surface function of the ball of radius $\bar{\alpha}^{1/n-1}$ on the complementary support of this lens.

If $\mathfrak{x}_0 \overset{\Delta}{=} \mathfrak{z}_1$, $n = 3$, then it follows from the theorem that the solution is in the class of so-called spindle-shaped surfaces of revolution of constant curvature [11].

In conclusion we mention that a combination of the methods set out here enables us to find solutions to a wide class of problems. In this connection it is appropriate to use the standard methods of geometry and mathematical programming alongside the techniques developed here. Let us illustrate this by a fairly typical example.

EXAMPLE 6.2. For a convex surface of given thickness $\Delta$ and integral range to maximize its volume.

First of all, according to the formulation, the problem is not "convex right through to the other side". However, an application of Minkowski symmetrization shows that the solution lies in the class of centrally-symmetric figures. Thus (see §4.0), the problem can be restated as follows: to find $\bar{\mathfrak{x}} \in \mathfrak{B}_n$ such that (a) $\mathfrak{x} \geqslant \frac{1}{2}\Delta\,\mathfrak{z}_n$; (b) $\mathfrak{x}(z_0) + \mathfrak{x}(-z_0) \leqslant \Delta$ (where $z_0$ is a vector in $Z_n$); (c) $V_1(\mathfrak{z}_n, \mathfrak{x}) = V_1(\mathfrak{z}_n, \bar{\mathfrak{x}})$; (d) $V(\mathfrak{x})$ attains its maximum.

In this problem, which is one of convex programming, the so-called Slater condition [27] is not fulfilled. However, the methods that have been developed here enable us to obtain a sufficient criterion for optimality.

An admissible body $\bar{\mathfrak{x}}$ is a solution to the above problem if there are a critical positive measure $\mu$ and numbers $\bar{\alpha}, \bar{\beta} \in R_+$ such that:

$$\text{(a)}\quad \mu(\bar{\mathfrak{x}}) + \mu \underset{R^n}{\ll} \bar{\alpha}\mu(\mathfrak{z}_n) + \bar{\beta}(\varepsilon_{z_0} + \varepsilon_{-z_0});$$

$$\text{(b)}\quad V(\bar{\mathfrak{x}}) + \frac{1}{n}\int\limits_{Z_n} \bar{\mathfrak{x}}\,d\mu = \bar{\alpha}V_1(\mathfrak{z}_n, \bar{\mathfrak{x}}) + \frac{1}{n}\bar{\beta}[\bar{\mathfrak{x}}(z_0) + \bar{\mathfrak{x}}(-z_0)];$$

$$\text{(c)}\quad \bar{\mathfrak{x}}(z) = \frac{1}{2}\Delta \quad \text{for all} \quad z \in s(\mu).$$

Thus, a figure of the form $\bar{\alpha}\,\mathfrak{z}_n \# \bar{\beta}\mathfrak{z}_{n-1}$, having given integral range and thickness is a solution to the above problem. In particular, for $n = 3$ a solution must be sought in the class of so-called cheese-shaped surfaces of revolution of constant curvature [11].

## Commentary

To §1. The basic facts concerning convex functions can be found in Rockafellar's monograph [91], which contains an ample bibliography. Among other works we mention the monographs by Hardy, Littlewood and Pólya [112] and Rutitskii [58]; there is also the survey [5] of early works in the theory of convex functions.

For various generalizations of convexity see [28]. The construction of $H$-convexity is, of course, not new and is typical in the theory of ordered algebraic systems [109]. $H$-convexity in concrete situations occurs in a number of papers, amongst others, [30], [34], [42], [111]. Of special interest are the articles by Fan [102] and by Boboc and Cornea [12], [13], where a related approach to continuous $H$-convex functions is set forth. The scheme of § 1.3 goes back to Minkowski and Fenchel [86], [106], [107]. The definitive version of this construction is due to Hörmander [113]. The specific property of quadratic trinomials (equivalent to generating) was pointed by Bowman and Korovkin (see [24], [53]). $R^n_+$-stable sets have been systematically employed, for instance, in the theory of growth of entire functions (see [21], [22], [75], [92]). The general properties of stable sets have been studied by Rubinov [94]. $R^n_+$-normal sets have arisen mainly from the needs of mathematical economics [77]. The theory of such sets and their connection with the extension of sublinear functionals were studied by Rubinov [94]. See also [79], [85] concerning sublinear functionals and operators. For convexity on non-convex sets and geometric applications of this idea, see [18], [19]. A large area of research on convex sets has been covered by Klee (see [50]), and also more recent works cited, for instance, in [100]).

The construction of the space of convex sets goes back to Neyman and Birkhoff [109]. One of the first papers to use this concept in an essential way is certainly [1]. See also [26], [38], [86] – [88], [113].

Concerning the Fenchel–Moreau scheme and its connection with problems in other disciplines see [16], [21], [41], [72], [83], [91], [106], [107]. For the connections with extremal problems see, for instance, the large bibliography in the book [100].

To §2. The notion of decomposition (for sublinear functions) was introduced by Reshetnyak [89]. Loomis [73] gave an abstract definition of decomposition; see also [31]. The definitive result on convex functions was obtained in [47]. The theorem on polars for sublinear functions is due to Kutateladze [60]. The proof of the decomposition theorem is taken from this article. For weakly measurable distributions and topics leading to the theorem of Hardy–Littlewood–Pólya–Blackwell–Stein–Sherman–Cartier, see [10], [105], [112]. For Strassen's theorem and the related problem on the subdifferential of the integral of convex functions see [27], [121]. An account of definitive results in this direction can be found in Ioffe and Levin [40].

Concerning Choquet theory, see [4], [105], [119] and also the widespread literature devoted in the main to the so-called geometric simplexes (see, in particular, [123], [124], which contains an ample bibliography).

The principle of preservation of inequalities goes back to Kadison [48]; subsequently it has been employed in [12], [13], [122], where the related construct of convexity is considered.

Subsections 2.1, 2.2 are largely taken from [67]. Later results are due to Kutateladze [62], [65].

To §3. The first substantial results (on the question of defining convergence of sequences of operators) were obtained by Korovkin [53]. The general criteria for subspaces of $C(Q)$ such that convergence on the subspace ensures convergence on the whole space were obtained by Shashkin (in terms of the Choquet boundary) (see [115], [116]). Here there are also interesting connections of a topological character. The relation with supremal generators (in $C[a, b]$) was first observed, apparently, by Baskakov [3]. The article [29] is devoted to convergence in the $L^p$-spaces. The approach making use of points of smoothness is proposed in [51], [56], [57] (the articles [20], [71] are also relevant). In [57] the connection with convergence of non-expanding operators is also noted. The first results along these lines were obtained by Shashkin [117] and Rublev [97]. For ordered superstructures see, for instance, [32], [57].

Concerning Shmul'yan's theorem and the related phenomenon of uniqueness of the Hahn–Banach extension see, in particular, [35], [36], [103], [104], [114], [118]. A number of related questions of convergence, not discussed in §3, can be found, for example, in [15], [55], [70], [81], [99], [108], [84].

Concerning quasi-linearization see [6], [43], [44], [52]. The article [59] also touches on this. Concerning generators, see in particular [68], [69], [95]. Some results are published here for the first time.

To §4. With regard to extremal problems of geometry, there is an extensive literature, in particular, in the survey article of Bonnesen and Fenchel [14] and Busemann; see also Hadwiger's book [110] and the cycle of articles by Aleksandrov [1], where there is also an ample bibliography.

The general scheme whereby the problems of isoperimetric type are treated as problems in programming in a space of convex sets is due to Kutateladze and Rubinov [66]. Further results along these lines are due to Kutateladze [63], [64]. Concerning the Bieberbach problem, see [7], [61]; concerning the Uryson problem, see [101] and also the survey article by Lyusternik [74].

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