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Nonstandard Methods in Geometric Functional Analysis

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Nonstandard methods in the modern sense consist of the explicit or implicit appeal to two different models of set theory—"standard" and "nonstandard"—to investigate concrete mathematical objects and problems. The main development of such methods dates to the last thirty years, and they have now crystallized in several directions (see [29], [42] and the bibliography cited there). The main directions are now known as infinitesimal and Boolean analysis. In this paper we shall outline new applications of nonstandard methods to problems arising in the area of our personal interests, grouped together under the general heading of geometric functional analysis [48]; we shall also point out some promising directions of further research.

§1. Infinitesimal analysis

1.1. Infinitesimal analysis, following its creator A. Robinson, is frequently referred to by the expressive but rather unfortunate phrase "nonstandard analysis"; nowadays one most frequently speaks of classical or Robinsonian nonstandard analysis. Infinitesimal analysis is characterized by the use of certain conceptions, long familiar in the practice of natural sciences but frowned upon in twentieth-century mathematics, involving the notions of actual infinitely large and infinitely small quantities.

1.2. Modern expositions of nonstandard analysis rely on formulas of E. Nelson's internal set theory IST [58] and its later developments, the external set theories of K. Hrbacek (EXT) [49] and T. Kawai (NST) [53]. From the standpoint of the "working mathematician-Philistine," the essence of these theories is as follows.

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Ordinary mathematical objects and properties are called internal (and considered, if a rigorous formalization is desired, within the framework of Zermelo-Fraenkel set theory ZFC). One introduces a new predicate $\mathrm{St}(x)$, expressing the property of an object x to be standard (qualitatively speaking—obtained through existence and uniqueness theorems, i.e., the set of natural numbers is standard, but the infinitely large natural numbers are nonstandard). Mathematical formulas and concepts in whose construction the new predicate is used will be called external. "Cantorian" sets endowed with external properties are referred to as external. In Nelson's theory, such sets are considered only as terms of a metalanguage, which is used only for convenience. In EXT and NST one can treat them as objects of Zermelo theory, which requires elaboration of a formalism and introduction of a new primary predicate $\mathrm{Int}(x)$, stating that the object x is internal. The available formalisms ensure that the extension of ZFC is conservative, i.e., when proving mathematical statements whose formulations do not involve external concepts, we may legitimately invoke the theories IST, EXT, and NST, as no less reliable than ZFC.

1.3. A point of crucial importance is that the new theories contain additional rules, easily motivated at the intuitive level, which are known as the principles of nonstandard analysis. We present their rigorous formulations in IST.

(1) Transfer principle:

$(\forall^{\mathrm{st}} x_1) \cdots (\forall^{\mathrm{st}} x_n)((\forall^{\mathrm{st}} x)\varphi(x, x_1, \dots, x_n) \to (\forall x)\varphi(x, x_1, \dots, x_n))$,

where φ is an internal formula and $\varphi = \varphi(x, x_1, \dots, x_n)$ (i.e., φ does not contain any free variables other than those listed).

(2) Idealization principle:

$$(\forall x_1) \cdots (\forall x_n)(\forall^{\mathrm{st\,fin}} z)(\exists x)(\forall y \in z)\varphi(x, y, x_1, \dots, x_n)$$
$$\leftrightarrow (\exists x)(\forall^{\mathrm{st}} y)\varphi(x, y, x_1, \dots, x_n),$$

where φ is an internal formula and $\varphi = \varphi(x, y, x_1, \dots, x_n)$.

(3) Standardization principle:

$$(\forall x_1) \cdots (\forall x_n)(\forall^{\mathrm{st}} x)(\exists^{\mathrm{st}} y)(\forall^{\mathrm{st}} z) z \in y \leftrightarrow z \in x \wedge \varphi(z, x_1, \dots, x_n),$$

where $\varphi = \varphi(z, x_1, \dots, x_n)$ is an arbitrary formula. The index st indicates that the quantifier in question is relativized to standard sets; the index st fin has the analogous meaning with regard to standard finite sets.

§2. Boolean-valued analysis

2.1. Boolean-valued analysis is characterized by the extensive use of the terms lowering and lifting, cyclic hulls and mixing. The development of this trend, which emerged under the impetus of P. J. Cohen's remarkable work on the continuum hypothesis, leads to essentially new ideas and results, first and foremost, in the theory of Kantorovich spaces and von Neumann algebras. The modeling device offered by Boolean-valued analysis makes it possible, in particular, to consider the elements of functional classes as numbers, which substantially facilitates the analysis and creates a unique possibility of automatically extending the scope of classical theorems.

2.2. The construction of a Boolean-valued model begins with a complete Boolean algebra B. For every ordinal $\alpha \in O_n$ one defines

$$V_\alpha^{(B)} := \{x\colon (\exists \beta \in \alpha) x\colon \mathrm{dom}(x) \to B \wedge \mathrm{dom}(x) \in V_\beta^{(B)}\}.$$

After this recursive definition, one introduces the Boolean-valued universe $V^{(B)}$ or class of B-sets:

$$V^{(B)} := \bigcup_{\alpha \in O_n} V_\alpha^{(B)}.$$

2.3. Taking an arbitrary formula of ZFC and interpreting the connectives and quantifiers in the natural way in the Boolean algebra B, one defines its truth value $[\![\varphi]\!]$, which depends on the way in which φ is built up from atomic formulas $x = y$ and $x \in y$. The truth values of the latter are defined for $x, y \in V^{(B)}$ by a recursion schema:

$$[\![x \in y]\!] := \bigvee_{z \in \mathrm{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \bigvee_{z \in \mathrm{dom}(x)} x(z) \Rightarrow [\![z \in y]\!] \wedge \bigwedge_{z \in \mathrm{dom}(y)} y(z) \Rightarrow [\![z \in x]\!]$$

(the sign $\Rightarrow$ symbolizes implication in B).

The universe $V^{(B)}$ with the above valuation rule is a ("nonstandard") model of set theory in the following sense.

2.4. Transfer principle. For any theorem φ of ZFC, the formula $[\![\varphi]\!] = 1$ is valid, i.e., φ is true inside $V^{(B)}$.

2.5. In the class $V^{(B)}$ there is a natural equivalence $x \sim y := [\![x = y]\!] = 1$, which preserves truth values. In this connection, one can use a special device to go over to a separated universe $\overline{V}^{(B)}$, in which $x = y \leftrightarrow [\![x = y]\!] = 1$. In fact, the identification $V^{(B)} := \overline{V}^{(B)}$ is usually assumed without special mention. The basic properties of $V^{(B)}$ are expressed by the following assertions.

2.6. Mixing principle. Let $(b_\xi)_{\xi \in \Xi}$ be a partition of unity in B, i.e., $\xi \neq \eta \to b_\xi \wedge b_n = 0$, $V_{\xi \in \Xi} b_\xi = 1$. For any family $(x_\xi)_{\xi \in \Xi}$ of the universe $V^{(B)}$ there exists a (unique) mixture of $(x_\xi)_{\xi \in \Xi}$ with probabilities $(b_\xi)_{\xi \in \Xi}$, i.e., an element x of the separated universe, denoted by $\sum_{\xi \in \Xi} b_\xi x_\xi$ or $\mathrm{mix}_{\xi \in Z} b_\xi x_\xi$, such that $[\![x = x_\xi]\!] \geq b_\xi$ for $\xi \in \Xi$.

2.7. Maximum principle. For every formula φ of ZFC there is an element $x_0 \in V^{(B)}$ for which

$$[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

In particular, $V^{(B)}$ contains an object $\mathscr{R}$ which plays the role of the field R inside $V^{(B)}$.

2.8. Besides the above principles, there is an important procedure for passing to $V^{(B)}$ from the ordinary von Neumann universe V, where the latter is defined by the recursion schema

$$V_\alpha := \{x : (\exists \beta \in \alpha) x \in P(V_\beta)\}, \qquad V := \bigcup_{\alpha \in O_n} V_\alpha.$$

This procedure is defined by the rule

$$\varnothing^\wedge := \varnothing, \quad \mathrm{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \mathrm{im}(x^\wedge) := \{1\}.$$

The element $x^\wedge \in V^{(B)}$ is known as the standard name of x. We thus have a canonical embedding of V into $V^{(B)}$. Apart from this we have a technique of lowerings and liftings of sets and correspondences.

2.9. Given an element $x \in V^{(B)}$, its lowering $x\downarrow$ is defined by the rule $x\downarrow := \{t \in V^{(B)} : [\![t \in x]\!] = 1\}$. The set $x\downarrow$ is cyclic, i.e., closed with respect to mixing of its elements.

2.10. Let F be a correspondence from X to Y inside $V^{(B)}$. There exists a correspondence $F\downarrow$ —and it is unique— from $X\downarrow$ to $Y\downarrow$ such that for any subset A of X inside $V^{(B)}$, we have $F(A)\downarrow = F\downarrow(A\downarrow)$.

In particular, a map $f : X^\wedge \to Y$ inside $V^{(B)}$ defines a function-lowering $f\downarrow : X \to Y\downarrow$ such that $f\downarrow(x) = f(x^\wedge)$ $(x \in X)$.

2.11. Let $x \in P(V^{(B)})$. Define $\varnothing\uparrow := \varnothing$ and $\mathrm{dom}(x\uparrow) = x$, $\mathrm{im}(x\uparrow) = \{1\}$. The element $x\uparrow$ is called the lifting of x. It is easy to see that $x\uparrow\downarrow$ is the least cyclic set containing x, i.e., its cyclic hull: $x\uparrow\downarrow = \mathrm{mix}(x)$.

2.12. Let $X, Y \in P(V^{(B)})$ and let F be a correspondence form X to Y. There exists a correspondence $F\uparrow$ —and it is unique— from $X\uparrow$ to $Y\uparrow$ inside $V^{(B)}$ such that $\mathrm{dom}(F\uparrow) = \mathrm{dom}(F)\uparrow$ and for every subset A of $\mathrm{dom}(F)$ we have $F\uparrow(A\uparrow) = F(A)\uparrow$ if and only if F is extensional, i.e.,

$$y_1 \in F(x_1) \to [\![x_1 = x_2]\!] \le \bigvee_{y_2 \in F(x_2)} [\![y_1 = y_2]\!].$$

In particular, a map $f : X \to Y\downarrow$ generates a function $f\uparrow : X^\wedge \to Y$ such that $f\uparrow(x^\wedge) = f(x)$ for $x \in X$. If necessary in specific cases, the lowering and lifting procedures can be iterated.

§3. Vector lattices

3.1. There are several excellent monographs on the theory of vector lattices [4], [18], [19], [55], [70]. Vector lattices are also commonly known as Riesz spaces, and order-complete vector lattices as Kantorovich spaces or K-spaces. A K-space is said to be extended if any set of pairwise disjoint positive elements in it has a supremum. The most important examples of extended K-spaces are the following:

(1) the space $M(\Omega, \Sigma, \mu)$ of equivalence classes of measurable functions, where (Ω, Σ, μ) is a measure space with μ a σ-finite measure (or, more generally, a space with the direct sum property, see [18]);

(2) the space $C_\infty(Q)$ of continuous functions defined on an extremally disconnected compact space Q with values in the extended real line, taking the values $\pm\infty$ only on a nowhere dense set [4], [19], [55];

(3) the space $\overline{A}$ of selfadjoint (not necessarily bounded) operators associated with a von Neumann algebra (see [66]).

To save space, we shall restrict attention to the real case, since the analysis of complex K-spaces is entirely analogous. The symbol $\mathfrak{P}(E)$ will denote the Boolean algebra of order projections in a K-space E. If E contains an order unit, $\mathfrak{E}(E)$ is the Boolean algebra of unit elements (fragments of the identity) in E. The algebras $\mathfrak{P}(E)$ and $\mathfrak{E}(E)$ are isomorphic and known as the base of E. Throughout the sequel, B will be a fixed complete Boolean algebra. The basis for Boolean-valued analycis of vector lattices is the following result.

3.2. Theorem (Gordon [6]). *Let $\mathscr{R}$ be the field of real numbers in the model $V^{(B)}$. The algebraic system $\mathscr{R}\downarrow$ (i.e., $\mathscr{R}$ with lowered operations and order) is an extended K-space. Moreover, there exists an isomorphism χ of the Boolean algebra B onto the base $\mathfrak{P}(E)$ such that*

$$b \leq [\![x = y]\!] \leftrightarrow \chi(b)x = \chi(b)y,$$
$$b \leq [\![x \leq y]\!] \leftrightarrow \chi(b)x \leq \chi(b)y$$

for all $x, y \in \mathscr{R}\downarrow$ and $b \in B$.

Throughout the sequel, R will denote the field of real numbers inside $V^{(B)}$. If the base of a K-space E is isomorphic to B, then E itself is isomorphic to the foundation $E_0 \subset \mathscr{R}\downarrow$, and in this situation E is extended if and only if $E_0 = \mathscr{R}\downarrow$. Under these circumstances one says that $\mathscr{R}\downarrow$ is a maximal extension and $\mathscr{R}$ a Boolean-valued realization of the K-space E. It is noteworthy that Boolean-valued realizations of certain structures lead to subsystems of the field $\mathscr{R}$.

3.3. Theorem [25].

(1) *A subgroup of the additive group of $\mathscr{R}$ is a Boolean-valued realization of an archimedean lattice-ordered group.*

(2) *A vector sublattice of $\mathscr{R}$, considered as a vector lattice over the field $R^\wedge$ is a Boolean-valued realization of an archimedean vector lattice.*

(3) *An archimedean f-ring contains two mutually complementary components, one of which is a group with zero multiplication realized as in* (1), *and the other has a subring of the ring $\mathscr{R}$ as a Boolean-valued realization.*

(4) *An archimedean f-algebra contains two mutually complementary components, one of which is a vector lattice with zero multiplication realized as in*

(2), *and the other is realized as a subring and sublattice of* $\mathscr{R}$, *considered as an* f*-algebra over* $R^{\wedge}$.

3.4. Gordon's theorem implies the main structural properties of K-spaces. We shall dwell on the realization of K-spaces and functional calculus. Let Q be a Stonean compact subspace of the Boolean algebra B and define $C_{\infty}(Q)$ as in 3.1 (2). We call a map $e: R \to B$ a resolution of unity in B if (1) $e(s) \leq e(t)$ $(s \leq t)$; (2) $\bigvee_{t \in R} e(t) = 1$, $\bigwedge_{t \in R} e(t) = 0$; (3) $\bigvee_{s<t} e(s) = e(t)$ $(t \in R)$. Let $B(R)$ be the set of all resolutions of unity in B. The sets $C_{\infty}(Q)$ and $B(R)$ can be endowed canonically with the structure of an extended K-space (see [4] and [19]).

3.5. Theorem ([29], [50]). *The extended* K*-space* $\mathscr{R}\downarrow$ *is (algebraically and order) isomorphic to each of the* K*-spaces* $B(R)$ *and* $C_{\infty}(Q)$. *Under this isomorphism an element* $x \in \mathscr{R}\downarrow$ *is mapped onto a resolution of unity* $t \to e_t^x$ $(t \in R)$ *and onto a function* $\overline{x}: Q \to \overline{R}$ *by the formulas*

$$e_t^x := [\![x < t^{\wedge}]\!] \qquad (t \in R),$$
$$\overline{x}(q) := \inf\{t \in R: [\![x < t^{\wedge}]\!] \in q\} \qquad (q \in Q).$$

3.6. Let $\mathscr{B}_R$ and $\mathscr{B}(R)$ be the σ-algebra of Borel sets and the vector lattice of Borel functions, respectively, on the real line. We identify B with the algebra of fragments of the identity in $\mathscr{R}\downarrow$ (see 3.2). For every $x \in \mathscr{R}\downarrow$ there exists a unique spectra measure (= sequentially o-continuous Boolean homomorphism $\mu: \mathscr{B}_R \to B$) such that $\mu(-\infty, t) = e_t^x$ $(t \in R)$. The measure μ defines an integral

$$I_x(f) := \int_R f(t)\, d\mu(t) \qquad (f \in \mathscr{B}(R)).$$

In this situation $I_x(f)$ is the unique element of $\mathfrak{R}\downarrow$ for which

$$[\![I_x(f) < t^{\wedge}]\!] = \mu(\{f < t\}).$$

3.7. Theorem ([29], [50]). *The map* $I_x: \mathscr{B}(R) \to \mathscr{R}\downarrow$ *is the unique sequentially* o*-continuous lattice and algebraic homomorphism satisfying the condition*

$$I_x(\mathrm{id}_R) = x.$$

3.8. For other aspects of Boolean-valued analysis of vector lattices, see [7], [8], [24], [29], [50], [51], [65].

§4. Positive operators

4.1. General information about positive and order-bounded operators may be found in [24], [29]. Take arbitrary K-spaces Z and E. A positive operator $\Phi: Z \to E$ will be called a Maharam operator if it is order continuous and $\Phi([0, z]) = [0, \Phi(z)]$ for every $z \in Z^+$, where $[a, b] := \{c: a \leq c \leq b\}$ is an order interval. Let mZ be a maximal extension of Z and $D(\Phi)^+$ the set of all $0 \leq z \in mZ$ such that $\{\Phi z' : z' \in Z,\ 0 \leq z' \leq z\}$ is bounded. Then $D(\Phi) := D(\Phi)^+ - D(\Phi)^+$ is a foundation in mZ and Φ extends to a Maharam operator on the whole of $D(\Phi)$. We say that Φ is essentially positive if $\Phi \geq 0$ and $\Phi(|z|) = 0$ implies $z = 0$.

4.2. Theorem [22]. *Let Φ be an essentially positive Maharam operator. There exist a K-space $\mathscr{Z}$ and an essentially positive o-continuous functional $\varphi : \mathscr{Z} \to \mathscr{R}$ in the model $V^{(B)}$, and there exists an isomorphism ι from $D(\Phi)$ onto the K-space $\mathscr{Z} \downarrow$ such that $\Phi = \varphi \downarrow \circ \iota$.*

4.3. The above result reduces the investigation of Maharam operators to analysis of the class of o-continuous positive functionals. What is the situation with regard to arbitrary positive operators? Various approaches based on Boolean-valued analysis may be adopted here. Let us consider an order-bounded operator from a vector lattice Z into $E : \mathscr{R} \downarrow$. There exists an order-bounded $R^{\wedge}$-linear functional $\varphi : Z^{\wedge} \to R$ inside $V^{(B)}$ for which $\Phi = \varphi \downarrow \circ j$, where $j : z \to z^{\wedge}$ $(z \in Z)$. The map $\Phi \to \varphi$ is an isomorphism of the space of all order-bounded operators $L_r(Z, E)$ onto $\widetilde{\mathscr{Z}} \downarrow$, where $\widetilde{\mathscr{Z}}$ is the space of order-bounded functionals on $\mathscr{Z}$. In particular, $\Phi \geq 0$ if and only if $[\![\varphi \geq 0]\!] = 1$. The disadvantage of this device is that the map $\Phi \to \varphi$ does not preserve order-continuity.

On the other hand, for a positive operator $\Phi : Z \to E$ one can construct an essentially positive Maharam operator $\overline{\Phi}$ and a lattice homomorphism $h : Z \to D(\overline{\Phi})$ such that $\Phi = \overline{\Phi} \circ h$, where the pair $(h, \overline{\Phi})$ is minimal in a certain sense (see [1]). Appealing to Theorem 4.2, we obtain a representation $\Phi = \varphi \downarrow \circ \iota'$, where $\iota' := h \circ \iota$ and φ is an essentially positive o-continuous functional in the model $V^{(B)}$. The disadvantage of this approach is that the space $D(\overline{\Phi})$ may prove to be invisible. However, in a fairly general situation, $D(\overline{\Phi})$ is realized as the space of functions (in two variables) on $P \times Q$, where P and Q are Stonean compact subspaces of Z and E, respectively (see [55]).

4.4. The above arguments are easily applied to the algebra of fragments of an arbitrary positive operator Φ acting from a vector lattice Z to a K-space E with filter of units ξ and base $\mathfrak{P}(E)$ (see [1] and [39]). We dwell on the representation of the projection S of an operator T onto the component $\{\Phi\}^{dd}$ generated by the operator Φ. Let us call a set of operators $\mathscr{P}$ in $L_r(Z, E)$ a generating set if $\Phi x^+ = \sup\{p\Phi x : p \in \mathscr{P}\}$ for all $x \in Z$. To study interesting fragments by lifting into a Boolean-valued universe, one can reduce everything to the case of functionals. For the latter, using infinitesimal representations, one readily proves that

$$Sx \underline{\rightharpoonup} \inf{}^{*}\{{}^{\circ}pTx : p^{d}\Phi x \approx 0,\ p \in \mathscr{P}\},$$
$$Sx \underline{\rightharpoonup} \inf{}^{*}\{{}^{\circ}Ty : \Phi(x - y) \approx 0,\ 0 \leq y \leq x\},$$

where * is the standardization symbol, ${}^{\circ}$ the "standard part" operation, $\approx$

denotes infinite smallness and $\rightharpoonup$ denotes the exactness of the formula, i.e., the attainability of equality.

Interpreting the above nonstandard representations and performing the lowering, one arrives at the following formulas [29]:

$$Sx = \sup_{\varepsilon \in \xi} \inf \{ \pi T_y + \pi^d T_x : 0 \le y \le x,\ \pi \in \mathfrak{P}(E),\ \pi\Phi(x-y) \le \varepsilon \},$$

$$Sx = \sup_{\varepsilon \in \xi} \inf \{ (\pi p)^d T_x : p\Phi_x \le \varepsilon,\ p \in \mathscr{P},\ \pi \in \mathfrak{P}(E) \}.$$

§5. Banach-Kantorovich spaces

5.1. A Banach-Kantorovich space consists of a (real or complex) vector space X, a K-space E, and a vector norm $|\cdot| : X \to E$ such that the following conditions hold: (1) the norm is decomposable, i.e., if $|x| = e_1 + e_2$, where $x \in X$ and $e_1, e_2 \in E^+$, then $x = x_1 + x_2$ and $|x_k| = e_k$ $(k := 1, 2)$ for suitable $x_1, x_2 \in X$; (2) X is o-complete, i.e., for any net $(x_\alpha) \subset X$, if $o\text{-}\lim |x_\alpha - x_\beta| = 0$, then $o\text{-}\lim |x_\alpha - x| = 0$ for some $x \in X$. We shall assume that $\{|x| : x \in X\}^{dd} = E \subset \mathscr{R}\downarrow$. If E is extended, i.e., $E = \mathscr{R}\downarrow$, then X is also said to be extended. An example of an extended Banach-Kantorovich space is the space $M(\Omega, \Sigma, \mu, Y)$ of (equivalence classes of) strongly measurable vector-valued functions with values in a Banach space Y.

5.2. Theorem [23]. *Let x be a Banach space in the model $V^{(B)}$. Then the lowering $x\downarrow$ is an extended Banach-Kantorovich space. Conversely, if X is an extended Banach-Kantorovich space, there exists a unique* (*up to linear isometry*) *Banach space x in $V^{(B)}$ whose lowering is linearly isometric to X.*

5.3. Let us call the bounded part of the space $x\downarrow$ the restricted descent of x. The restricted descents of Banach spaces in $V^{(B)}$ constitute the class of B-cyclic Banach spaces. Let B be the complete Boolean algebra of norm one projections in a Banach space X. We shall say that X is cyclic with respect to B, or B-cyclic, if, for an arbitrary partition of unity $(\pi_\xi)_{\xi \in \Xi} \subset B$ and any bounded family $(x_\xi)_{\xi \in \Xi} \subset X$ there exists a unique element $x \in X$ such that $\pi_\xi x_\xi = \pi_\xi x$ $(\xi \in \Xi)$ and $\|x\| \le \sup_{\xi \in \Xi} \|x_\xi\|$. Let $A(B)$ denote an arbitrary commutative AW^*-algebra whose complete Boolean algebra of idempotents is isomorphic to B. If X is an AW^*-model over $A(B)$ (see [52]), then X is a B-cyclic Banach space. All the aforesaid leads to the following realization theorem.

5.4. Theorem [59]. *The restricted descent of a complex Hilbert space in the model $V^{(B)}$ is an AW^*-module over the algebra $A(B)$. Conversely, for any AW^*-module X over $A(B)$ there exists a unique* (*up to unitary equivalence*) *Hilbert space inside $V^{(B)}$ whose restricted descent is unitarily equivalent to X.*

5.5. Let X and Y be Banach-Kantorovich spaces with norming lattices E and F, respectively. A linear operator $T: X \to Y$ is said to be majorizable if there exists a positive operator $S: E \to F$ such that $|Tx| \leq S(|x|)$ for all $x \in X$. If $E = F$ and S is an orthomorphism, a majorizable operator is also called bounded, since in that situation T coincides with the lowering from $V^{(B)}$ of a bounded linear operator acting in Banach spaces. By interpreting Riesz-Schauder theory in a Boolean-valued model one arrives at a new concept of cyclic compactness and obtains corresponding results on the solvability of operator equations in Banach-Kantorovich spaces [24]. General majorizable operators have a far more complicated structure and their analysis requires appeal to a considerable variety of methods (see [24], [26], [31]).

5.6. Banach-Kantorovich spaces and majorizable operators were first introduced by L. V. Kantorovich in [16]. It was he who proposed the first applications to the solution of operator equations by the method of successive approximations (see [17], [19]). These objects possess a rich structure and have several important applications in the area of spaces of measurable vector-valued functions and linear operators in such spaces [26]. In particular, the study of Banach-Kantorovich spaces leads to the notion of Banach spaces with mixed norms, which is enormously useful in connection with the isometric classification of Banach function spaces (see [26]).

§6. Banach algebras

6.1. Certain classes of Banach algebras yield some beautiful variations on the theme outlined in the previous section. Call a C^*-algebra A a B-C^*-algebra if A is cyclic with respect to a Boolean algebra of projections B, where any projection in B is multiplicative, involutive and of unit norm. If A is an AW^*-algebra and B a regular subalgebra of the Boolean algebra of central projections $\mathfrak{P}(A)$, then A is a B-C^*-algebra. We shall therefore say that A is a B-AW^*-algebra if B is a regular subalgebra of $\mathfrak{P}_C(A)$. Now let A be a JB-algebra and B and $\mathfrak{P}_C(A)$ the same as before. If A is a cyclic Banach space with respect to B, we shall say that A is a B-JB-algebra. An isomorphism that commutes with the projections in B will be called a B-isomorphism. The following theorem, though in a slightly different form, was proved in [67].

6.2. Theorem [67]. *The restricted descent of a C^*-algebra in the model $V^{(B)}$ is a B-C^*-algebra. Conversely, for every B-C^*-algebra A, there exists inside $V^{(B)}$ a unique (up to $*$-isomorphism) C^*-algebra $\mathscr{A}$ such that the restricted descent of $\mathscr{A}$ is $*$-B-isomorphic to A.*

6.3. Theorem. *The restricted descent of an AW^*-algebra (JB-algebra) from the model $V^{(B)}$ is a B-AW^*-algebra (B-JB-algebra). Conversely, for any B-AW^*-algebra (B-JB-algebra) A there exists a unique (to within an* *isomorphism)* AW^**-algebra* (JB*-algebra*) $\mathscr{A}$ *whose restricted descent is* B*-isomorphic to* A. *In addition,* $\mathscr{A}$ *will be a factor in* $V^{(B)}$ *if and only if* $B = \mathscr{P}_c(A)$. *The formulated statement concerning* AW^**-algebras is obtained in* [59] *and* [60].

6.4. The Boolean-valued realization of von Neumann algebras [66] is also worthy of mention. The above realization theorems form the foundation for Boolean-valued analysis of all these classes of Banach algebras (see [59]–[62], [66], [67]). In particular, it was shown in [59] that for all infinite cardinals α and β there exists an AW^*-algebra that is simultaneously α- and β-homogeneous (a conjecture of Kaplansky in [52]). This fact is related to the location of cardinal numbers under embeddings in $V^{(B)}$ (see [44], [68]).

§7. Convex analysis

7.1. The subdifferential is one of the most important concepts in convex analysis (see [24], [28]). In this section, referring to a few examples, we shall show how to use Boolean-valued analysis to study the internal structure of subdifferentials. Take a vector space X, a K-space E, and a sublinear operator $P: X \to E$. The subdifferential ∂P of P at zero is also called the supporting set of P [28]. By Gordon's theorem, we may assume that $E \subset \mathscr{R}\downarrow$, so that we can "convert" P inside a suitable model into an $\mathscr{R}$-valued sublinear operator, i.e., a sublinear functional. To be precise:

7.2. Theorem [54]. *There exist a Banach space* x *and a continuous sublinear functional* $p: x \to \mathscr{R}$ *in the model* $V^{(B)}$ *such that there is an isomorphic embedding of* X *into the Banach-Kantorovich space* $x\downarrow$ *with* $[\![(\imath X)\uparrow$ *is dense in* $x]\!] = 1$ *and* $P = p \circ \imath$. *In this situation, for any* $U \in \partial P$ *there is a unique element* $u \in V^{(B)}$ *for which* $[\![u \in \partial p]\!] = 1$ *and* $U = u\downarrow \circ \imath$. *The map* $U \to u$ *is an affine isomorphism of the convex sets* ∂P *and* $(\partial p)\downarrow$.

7.3. Thus, the study of ∂P largely reduces to that of ∂p. For example, let us look at the extremal structure of the subdifferential ∂P. Let $\mathrm{Ch}(P)$ denote the set of extreme points of ∂P. It should be noted that by Theorem 2 the relations $U \in \mathrm{Ch}(P)$ and $[\![u \in \mathrm{Ch}(p)]\!] = 1$ are equivalent, and one can then use the classical Kreĭn-Mil′man Theorem and Mil′man's inversion of it for ∂p. For a rigorous formulation of the result, we need some more definitions. The weak closure σ-$\mathrm{cl}(\Omega)$ (cyclic hull $\mathrm{mix}(\Omega)$) is the set of all operators $T \in L(X, E)$ of the form $Tx = o\text{-}\lim T_\alpha x$ $(x \in X)$, where (T_α) is a net in Ω (resp., $Tx = o\text{-}\sum \pi_\xi T_\xi x$ $(x \in X)$, where $(T_\xi) \subset \Omega$ and (π_ξ) is a partition of unity in $\mathfrak{P}(E)$). The weak cyclic closure of Ω is the set σ-$\mathrm{cl}(\mathrm{mix}(\Omega))$. If σ-$\mathrm{cl}\,\Omega = \Omega$ or $\mathrm{mix}(\Omega) = \Omega$, one says that Ω is weakly o-closed or cyclic, respectively. The definition of weak r-closedness is analogous.

7.4. Theorem [27], [28]. (1) *For any sublinear operator* $P: X \to E$ *the subdifferential coincides with the weakly cyclic closure of the convex hull of its extreme points* $\mathrm{Ch}(P)$.

(2) *If* $P: X \to E$ *is a sublinear operator and* $T \in L(Y, X)$, *then* $\mathrm{Ch}(P \circ T) \subset \mathrm{Ch}(P) \circ T$.

A set $\Omega \subset L(X, E)$ is operator convex (weakly bounded) if $\alpha\Omega + \beta\Omega \subset \Omega$ for any $\alpha, \beta \in \mathscr{R}\downarrow^{+}$, $\alpha + \beta = 1$ (the set $\{T_x : T \in \Omega\}$ is order bounded for all $x \in X$).

7.5. Theorem [24], [27]. *For a weakly bounded set* $\Omega \subset L(X, E)$, *the following assertions are equivalent*:

(1) $\Omega = \partial P$ *for some sublinear* $P: X \to E$;

(2) Ω *is convex, cyclic, and weakly r-closed*;

(3) Ω *is convex, cyclic, and weakly o-closed*;

(4) Ω *is operator convex and weakly o-closed.*

7.6. Let $\Phi: Z \to E$ be a positive operator, P a sublinear operator from a vector space X to a K-space Z. The term disintegration in K-spaces refers to those parts of the calculus of subdifferentials based on the formula $\partial(\Phi \circ P) = \Phi \circ \partial P$. This formula is not always true, but it is known to be valid if Φ is an order-continuous functional $(E = R)$. The general case is analyzed with the help of Theorem 4.2. Let $\Phi, \varphi, \imath$ be the same as in 4.2. There exists an $R^{\wedge}$-sublinear operator $\rho: X^{\wedge} \to \mathscr{Z}$ inside $V^{(B)}$ for which $\rho\downarrow \circ j = \imath \circ P$ (cf. 4.3). From this and 7.2 we conclude that

$$\Phi \circ P = \Phi \circ \imath^{-1} \circ (\imath \circ P) = \varphi\downarrow \circ \rho\downarrow \circ j = (\varphi \circ \rho)\downarrow \circ j,$$
$$\partial(\Phi \circ P) = \{u\downarrow \circ j : [\![u \in \partial(\varphi \circ \rho) = \varphi \circ \partial \rho]\!] = 1\}.$$

These arguments yield the following result.

7.7. Theorem [22]. *Let* Φ *be a positive order-continuous operator. The formula* $\sigma(\Phi \circ P) = \Phi \circ \partial P$ *is valid for any sublinear operator* P *if and only if* Φ *is a Maharam operator.*

7.8. Further developments of disintegration in K-spaces may be found in [24] and [28]. On nonstandard methods in convex analysis see also [33], [34], [36], and [54].

§8. Monadology

8.1. A central concept of infinitesimal analysis is the monad. According to Euclid's definition, "a monad is that through which the many become one." In the formal theory, a monad $\mu(\mathscr{F})$ is defined as an external list of the standard elements of a standard filter $\mathscr{F}$, i.e., $x \in \mu(\mathscr{F}) \leftrightarrow (\forall^{\mathrm{st}} F \in \mathscr{F}) x \in F$. A syntactic characterization of external sets that are monads was proposed not long ago by Benninghofen and Richter [45]. It is useful to emphasize that every monad is a union of ultramonads—monads of ultrafilters. For such a monad U the assertions $(\exists x \in U)\varphi(x)$ and $(\forall x)\varphi(x)$, where $\varphi = \varphi(x)$ is an external formula, are equivalent. Hence it is clear that ultramonads are the genuine "elementary" objects of infinitesimal analysis.

8.2. For applications to the theory of operators, it is of essential importance to construct a synthetic theory in the framework of which both the nonstandard methods offered by Boolean-valued models and external set theories can be used. Only preliminary results have so far been achieved in this direction; they pertain to the study of topological-type notions related with mixing—cyclic filters, topologies and so on, which play major roles in K-spaces. We shall point out one of the possible approaches to cyclic monadology.

8.3. Fix a standard complete Boolean algebra B and an external set A consisting of elements of a separated Boolean-valued universe $V^{(B)}$. An element $x \in V^{(B)}$ is a member of the cyclic hull $\operatorname{mix}(A)$ if and only if, for some internal family $(a_\xi)_{\xi\in\Xi}$ of elements of A and an internal partition of unity $(b_\xi)_{\xi\in\Xi}$ in B, we have
$x = \operatorname{mix}_{\xi\in\Xi} b_\xi x_\xi$. A monad $\mu(\mathscr{F})$ is said to be cyclic if $\mu(\mathscr{F}) = \operatorname{mix}\mu(\mathscr{F})$. A point is said to be essential if it lies in the monad of some pro-ultrafilter—a maximal cyclic filter or, more rigorously, an ultrafilter in $V^{(B)}$.

8.4. Theorem. (1) *A standard filter is cyclic if and only if its monad is cyclic.*
(2) *A filter is extensional if and only if its monad is the cyclic monadic hull of the set of its essential points.*

As corollaries we cite the following Boolean-valued analogs of some classical criteria of A. Robinson.

8.5. Theorem. (1) *A standard set is the lowering of a compact space if and only if each of its essential points is near-standard.*
(2) *A standard set is the lowering of a totally bounded space if and only if each of its essential points is pre-near-standard.*

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