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Boolean-Valued Introduction to the Theory of Vector Lattices

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The theory of vector lattices appeared in early thirties of this century and is connected with the names of L. V. Kantorovich, F. Riesz, and H. Freudenthal. The study of vector spaces equipped with an order relation compatible with a given norm structure was evidently motivated by the general circumstances that brought to life functional analysis in those years. Here the general inclination to abstraction and uniform approach to studying functions, operations on functions, and equations related to them should be noted. A remarkable circumstance was that the comparison of the elements could be added to the properties of functional objects under consideration. At the same time, the general concept of a Banach space ignored a specific aspect of the functional spaces—the existence of a natural order structure in them, which makes these spaces vector lattices.

Along with the theory of ordered spaces, the theory of Banach algebras was being developed almost at the same time. Although at the beginning these two theories advanced in parallel, soon their paths parted. Banach algebras were found to be effective in function theory, in the spectral theory of operators, and in other related fields. The theory of vector lattices was developing more slowly and its achievements related to the characterization of various types of ordered spaces and to the description of operators acting in them was rather unpretentious and specialized.

In the middle of the seventies the renewed interest in the theory of vector lattices led to its fast development which was related to the general explosive developments in functional analysis; there were also some specific reasons, the main one being the use of ordered vector space in the mathematical approach to social phenomena, economics in particular. The scientific work and the unique personality of L. V. Kantorovich also played important role in the development of the theory of ordered spaces and in relating this theory to economics and optimization. Another, though less evident, reason for the interest in vector lattices was their rather unexpected role in the theory of nonstandard—Boolean-valued—models of set theory. Constructed by D. Scott, R. Solovay, and P. Vopenka in connection with the well-known results by P. G. Cohen about the continuum hypothesis, these models proved to be inseparably linked with the theory of vector lattices. Indeed, it was discovered that the elements of such lattices serve as images of real numbers in a suitably selected Boolean model. This fact not only gives a precise meaning to the initial idea that abstract ordered spaces are derived from real numbers, but also provides a new possibility to infer common

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properties of vector lattices by using the fact that they, in a precise sense, depict the sublattices of the field $\mathbb{R}$. Indeed, this possibility was taken as a basis for the present minicourse of lectures.

The main attention in these lectures is paid to the fundamental concepts. For brevity, we usually skip the proofs of the formulated theorems.

The bibliography, both in the field of vector lattices and in nonstandard analysis, is by no means complete. With few exceptions, the list of references consists of monographs and survey articles containing extensive bibliographies. Other original works are cited for specific reasons.

LECTURE I. Vector lattices

We start with a brief description of basic concepts of the theory of vector lattices.[1] Details can be found in **[7, 13, 14, 38, 41, 51]**.

1.1. Let F be a linearly ordered field. An ordered vector space over F is a pair $(E, \leq)$, where E is a vector space over the field F and $\leq$ is an order relation on E such that, in addition, the following conditions are fulfilled:

(1) if $x \leq y$ and $u \leq v$, then $x + u \leq y + v$ for any x, y, u, $v \in E$;

(2) if $x \leq y$, then $\lambda x \leq \lambda y$ for any x, $y \in E$ and $0 \leq \lambda \in F$.

Thus, in an ordered vector space inequalities can be added together and multiplied by positive elements from F. This can be expressed as follows: $\leq$ is an order relation compatible with the vector space structure or, in short, $\leq$ is a vector order.

The definition of a vector order on a vector space E over the field F is equivalent to specifying a certain set (called the positive cone) $E_+ \subset E$ with the following properties:

$$E_+ + E_+ \subset E_+, \quad \lambda E_+ \subset E_+ \quad (0 \leq \lambda \in F), \quad E_+ \cap (-E_+) = \{0\}.$$

Moreover, the order $\leq$ and the cone E_+ are connected by the relation

$$x \leq y \leftrightarrow y - x \in E_+ \quad (x, y \in E).$$

The elements of the cone E_+ are called positive.

1.2. An ordered vector space that is also a lattice is called a *vector lattice*. Hence, for any finite set $\{x_1, \dots, x_n\}$ in a vector lattice E, there exist the least upper bound $\sup\{x_1, \dots, x_n\} =: x_1 \vee \dots \vee x_n$ and the greatest lower bound $\inf\{x_1, \dots, x_n\} =: x_1 \wedge \dots \wedge x_n$ (these elements are, of course, unique). In particular, any element x of a vector lattice has a positive part $x^+ := x \vee 0$, a negative part $x^- := (-x)^+ = -x \wedge 0$, and a modulus $|x| := x \vee (-x)$.

The disjunction (disjointness relation) $\perp$ in a vector lattice E is defined by the following formula:

$$\perp := \{(x, y) \in E \times E : \quad |x| \wedge |y| = 0\}.$$

A set of the form

$$M^\perp := \{x \in E : (\forall y \in M)\, x \perp y\},$$

where M is an arbitrary nonempty set in E, is called a *component* or a *band* of the

[1] *Translator's note.* Vector lattices are also called Riesz spaces.

vector lattice E. The set of all bands of a vector lattice, ordered by inclusion, forms a complete Boolean algebra $\mathfrak{B}(E)$ under the following Boolean operations:

$$L \wedge K := L \cap K, \quad L \vee K := (L \cup K)^{\perp\perp}, \quad L^* := L^{\perp} \quad (L, K \in \mathfrak{B}(E)).$$

The algebra $\mathfrak{B}(E)$ is called the *base* of E.

An element $1 \in E$ is called an (*order*) *unit*, if $\{1\}^{\perp\perp} = E$, i.e., if E has no nonzero elements that are disjoint with 1. Let $e \wedge (1 - e) = 0$ for some $0 \leq e \in E$. Then e is said to be a unit element (with respect to 1). The set $\mathfrak{E}(1) := \mathfrak{E}(E)$ of all unit elements with the order induced from E is a Boolean algebra. The lattice operations in $\mathfrak{E}(1)$ are inherited from E, and the Boolean complement has the form $e^* := 1 - e$ $(e \in \mathfrak{E}(E))$.

Let K be a band of a vector lattice E. If there exists an element $\sup\{u \in K : 0 \leq u \leq x\}$ in E, then this element is called the *projection* of x to the band K and is denoted by $[K]x$ (or $\mathrm{Pr}_K\, x$). For an arbitrary $x \in E$ one defines $[K]x := [K]x^+ - [K]x^-$. The projection of an element $x \in E$ on K exists if and only if there is a decomposition $x = y + z$, where $y \in K$, $z \in K^{\perp}$. Moreover, in that case, $y = [K]x$ and $z = [K^{\perp}]x$. We shall assume that any element $x \in E$ has a projection on K. Then the operator $x \mapsto [K]x$ $(x \in E)$ is linear, idempotent, and $0 \leq [K]x \leq x$, for all $0 \leq x \in E$. One says that E is a vector lattice with the projection (principal projection) property if for any band (principal band) $K \in \mathfrak{B}(E)$ the projection operator $[K]$ is defined.[2]

1.3. A linear subspace I of a vector lattice is called an *order ideal* or an *o-ideal* (or just an ideal, if the rest is clear from the context), whenever the inequality $|x| \leq |y|$ implies that $x \in I$, for any $x \in E$ and $y \in I$.[3]

If an ideal I has the aditional property $I^{\perp\perp} = E$ (or $I^{\perp} = \{0\}$, which is the same), then it is called a *foundation* of E.[4]

A subspace $E_0 \subset E$ is called a *sublattice* of E if $x \wedge y, x \vee y \in E_0$ for any x, $y \in E_0$. It is then said that the sublattice E_0 is minorizing (or that it is a minorant) if for any $0 \neq x \in E_+$ there exists an element $x_0 \in E_0$ satisfying the inequalities $0 < x_0 \leq x$. We say that E_0 is a majorizing (or massive) sublattice if for any $x \in E$ there exists $x_0 \in E_0$ such that $x \leq x_0$. Thus, E_0 is a minorizing (majorizing) sublattice if and only if $E_+ \setminus \{0\} = E_+ + (E_{0+} \setminus \{0\})$ (respectively, $E = E_+ + E_0$).

Everywhere below, whenever the field F is not indicated explicitly, a vector lattice over the linear ordered field $\mathbb{R}$ of real numbers is implied. An order interval in E is a set of the form $[a, b] := \{x \in E : a \leq x \leq b\}$, where $a, b \in E$. A set in E is called (*order*) *bounded* (or *o-bounded*) if it is contained in some order interval. It is possible to introduce a seminorm on the ideal $I(u) := \bigcup_{n=1}^{\infty} [-nu, nu]$ generated by the element $0 \leq u \in E$:

$$\|x\|_u := \inf\{\lambda \in \mathbb{R}_+ : |x| \leq \lambda u\} \quad (x \in I(u)).$$

If $I(u) = E$, then u is called a strong unit and E is a vector lattice of bounded elements. The seminorm $\|\cdot\|_u$ is a norm if and only if the lattice $I(u)$ is Archimedean; i.e., for any $x \in I(u)$ the order boundedness of the set $\{n|x| : n \in \mathbb{N}\}$ implies that $x = 0$.

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[2] *Translator's note.* The principal band generated by an element f is $\{f\}^{\perp\perp}$.

[3] *Translator's note.* A set D is called solid if $f \in D$, $|h| \leq |f| \Rightarrow h \in D$.

[4] *Translator's note.* This property is often called "quasi order dense" in the literature (cf. **[34,** p. 110]); "foundation" is the literal translation of the Russian term and is more evocative.

An element $x \geq 0$ of a vector lattice is said to be discrete if $[0, x] = [0, 1]x$, i.e., if $0 \leq y \leq x$ implies $y = \lambda x$ for some $0 \leq \lambda \leq 1$. A vector lattice E is called discrete if for every $0 \neq y \in E_+$ there exists a discrete element $x \in E$ such that $0 < x \leq y$. If E has no nonzero discrete elements we say that E is continuous.

1.4. A vector lattice over the field of real numbers in which every nonempty order bounded set has an infimum and a supremum is called a *Kantorovich space*, or, in short, a *K-space*.[5] Sometimes instead of a K-space a more descriptive term is applied, namely (relatively) order complete vector lattice. If infima and suprema exist only for countable bounded sets, then the corresponding vector lattice is called a *K_σ-space*. Any K_σ-space, hence any K-space, is Archimedean. It is said that a K-space (K_σ-space) is extended[6] if any set (any countable set) in it consisting of pairwise disjoint elements is bounded.

A K-space has a projection onto every band. The set of all projections onto the bands of E is denoted by the symbol $\mathfrak{P}(E)$. For the projections π and ρ we define $\pi \leq \rho$ if and only if $\pi x \leq \rho x$ for all $0 \leq x \in E$.

THEOREM. *Let E be an arbitrary K-space. Projecting onto bands defines an isomorphism $K \mapsto [K]$ of Boolean algebras $\mathfrak{B}(E)$ and $\mathfrak{P}(E)$. If there exists a unit in E then the mappings $\pi \mapsto \pi 1$ from $\mathfrak{P}(E)$ into $\mathfrak{E}(E)$ and $e \mapsto \{e\}^{\perp\perp}$ from $\mathfrak{E}(E)$ into $\mathfrak{B}(E)$ are isomorphisms of Boolean algebras.*

1.5. The projection π_u onto the principal band $\{u\}^{\perp\perp}$, where $0 \leq u \in E$, can be computed by means of a simpler rule than is indicated in 1.2:

$$\pi_u x = \sup\{x \wedge (nu) : n \in \mathbb{N}\}.$$

In particular, a K_σ-space contains the projection of any element onto every principal band.

Let E be a K-space with unit 1. The projection of the unit onto the band $\{x\}^{\perp\perp}$ is called the *trace* of the element x and is denoted by the symbol e_x. Thus, $e_x = \sup\{1 \wedge (n|x|) : n \in \mathbb{N}\}$. The trace e_x can be used both as a unit in $\{x\}^{\perp\perp}$ and as a unit element in E. For any real number λ, e_λ^x denotes the trace of the positive part of the element $\lambda 1 - x$, i.e., $e_\lambda^x = e_{(\lambda 1 - x)^+}$. The associated function $\lambda \mapsto e_\lambda^x$ is called the *spectral function* or the *characteristic of x*.

1.6. An ordered space E over F is called an *ordered algebra* over F, if it is an algebra over F and, moreover, the following condition is satisfied: if $x, y \in E$ with $x \geq 0$ and $y \geq 0$, then $xy \geq 0$. In order to characterize the positive cone E_+ of an ordered algebra E, another property should be added to those mentioned in 1.1: $E_+ \cdot E_+ \subset E_+$. We say that E is a lattice ordered algebra if E is a vector lattice and an ordered algebra, simultaneously. A lattice ordered algebra is called an *f-algebra* if for any $a, x, y \in E_+$ from the condition $x \wedge y = 0$ it follows that $(ax) \wedge y = 0$ and $(xa) \wedge y = 0$. An f-algebra is called *faithful* if for any two elements x and y the equality $xy = 0$ implies $x \perp y$. It is not difficult to show that an f-algebra is faithful if and only if it has no nonzero nilpotent elements. The faithfulness of an f-algebra is also equivalent to the absence of strictly positive elements with zero square.

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[5] *Translator's note.* This is the Russian terminology for a Dedekind complete vector lattice.

[6] *Translator's note.* Literally translated from the Russian; in the French translation of [13] this is called "achevé", and in [30] the phrase "universally complete" is used for this notion.

1.7. The complexification $E \oplus iE$ (i is the imaginary unit) of a real vector lattice E is called a *complex vector lattice*. In addition, it is often required that

$$|z| := \sup\left\{ \operatorname{Re}(e^{i\theta}z) \colon 0 \le \theta \le \pi \right\}$$

for any element $z \in E \oplus iE$. In the case of a K-space or an arbitrary Banach lattice this requirement is automatically fulfilled. Thus, a complex K-space is the complexification of a real K-space. Speaking of order properties of a complex vector lattice $E \oplus iE$, we have in mind its real part E. The notions of a sublattice, ideal, bands of projection, etc. are naturally extended to the case of a complex vector lattice with the help of suitable complexifications.

1.8. Various types of convergence are related with the order relation in a vector lattice. Let $(A, \le)$ be an upwards-filtered set (i.e., filtered with respect to increase). A net $(x_\alpha) := (x_\alpha)_{\alpha \in A}$ in E is said to be increasing (decreasing) if $x_\alpha \le x_\beta$ ($x_\beta \le x_\alpha$) for $\alpha \le \beta$ ($\alpha, \beta \in A$).

It is said that a net (x_α) o-converges to an element $x \in E$ if there exists a decreasing net $(e_\alpha)_{\alpha \in A}$ in E with the properties $\inf\{e_\alpha : \alpha \in A\} = 0$ and $|x_\alpha - x| \le e_\alpha$ ($\alpha \in A$). If this is the case, x is called an o-limit of the net (x_α) and this is denoted by $x = o\text{-}\lim x_\alpha$ or $x_\alpha \xrightarrow{(o)} x$. In a K-space E, the upper and lower o-limits for an order bounded net are introduced by the following formulas:

$$\limsup_{\alpha \in A} x_\alpha := \overline{\lim_{\alpha \in A}}\, x_\alpha := \inf_{\alpha \in A} \sup_{\beta \ge \alpha} x_\beta;$$

$$\liminf_{\alpha \in A} x_\alpha := \underline{\lim_{\alpha \in A}}\, x_\alpha := \sup_{\alpha \in A} \inf_{\beta \ge \alpha} x_\beta.$$

There is an evident relation between these objects:

$$x = o\text{-}\lim_{\alpha \in A} x_\alpha \leftrightarrow \limsup_{\alpha \in A} x_\alpha = x = \liminf_{\alpha \in A} x_\alpha.$$

It is said that the net $(x_\alpha)_{\alpha \in A}$ regulator converges to $x \in E$ if there exist an element $0 \le u \in E$, which is called a regulator of convergence, and a net of numbers $(\lambda_\alpha)_{\alpha \in A}$ with the properties $\lim \lambda_\alpha = 0$ and $|x_\alpha - x| \le \lambda_\alpha u$ ($\alpha \in A$). In addition, x is called a r-limit of the net (x_α) and this is denoted as $x = r\text{-}\lim_{\alpha \in A} x_\alpha$ or $x_\alpha \xrightarrow{(r)} x$. Clearly, regulator convergence is convergence in the normed space $(I(u), \|\cdot\|_u)$.

The presence of o-convergence in a K-space allows us to define the sum of an infinite family $(x_\xi)_{\xi \in \Xi}$. Indeed, let $A := \mathfrak{P}_{\text{fin}}(\Xi)$ be the set of all finite subsets of Ξ. We write $y_\alpha := x_{\xi_1} + \ldots + x_{\xi_n}$ for $\alpha := \{\xi_1, \ldots, \xi_n\} \in A$, obtaining a net $(y_\alpha)_{\alpha \in A}$, where A is naturally ordered by inclusion. If $x := o\text{-}\lim_{\alpha \in A} y_\alpha$ exists, then the element x is called the *o-sum* of the family (x_ξ) and it is denoted as $x = o\text{-}\sum_{\xi \in \Xi} x_\xi$ or simply $x = \sum_{\xi \in \Xi} x_\xi$. It is clear that for $x_\xi \ge 0$ ($\xi \in \Xi$) the o-sum of the family (x_ξ) exists if and only if the family $(y_\alpha)_{\alpha \in A}$ is order bounded; moreover,

$$o\text{-}\sum_{\xi \in \Xi} x_\xi = \sup_{\alpha \in A} y_\alpha.$$

If the elements of the family (x_ξ) are pairwise disjoint, then

$$o\text{-}\sum_{\xi \in \Xi} x_\xi = \sup_{\xi \in \Xi} x_\xi^+ - \sup_{\xi \in \Xi} x_\xi^-.$$

Any K-space is o-complete in the following sense. If a net $(x_\alpha)_{\alpha\in A}$ in E satisfies the condition $\limsup|x_\alpha - x_\beta| := \inf_{\gamma\in A}\sup_{\alpha,\beta\geq\gamma}|x_\alpha - x_\beta| = 0$, then there exists an element $x \in E$ such that $x = o\text{-}\lim x_\alpha$.

1.15. Comments. (a) The history of functional analysis in ordered vector spaces is usually related to the contributions of G. Birkhoff, L. V. Kantorovich, M. G. Krein, H. Nakano, F. Riesz, H. Freudenthal, etc. At the present time, the theory and applications of ordered vector spaces forms an extensive domain in mathematics which, essentially, is one of the main branches of contemporary functional analysis. The field is well presented in monographs; see **[1, 7, 13, 14, 16, 18, 20, 25, 26, 29, 31, 32, 36–39, 41, 50, 51]**. We should also mention the surveys with extensive bibliographies **[2–4]**. The necessary information on the theory of Boolean algebras can be found in **[6, 24, 30]**.

(b) L. V. Kantorovich singled out the most important class of ordered vector spaces—the order complete vector lattices, i.e., K-spaces. They were introduced in Kantorovich's first fundamental work on the subject **[11]**, where he wrote: "In this note I define a new type of spaces which I call linear semi-ordered spaces. The introduction of these spaces allows us to study linear operations of one general class (operations with values belonging to such a space) as linear functionals."

In the same paper Kantorovich formulated an important methodological principle—a heuristic transfer principle for the K-spaces. As an example of application of this principle one can take Theorem 3 from **[11]** which is also called the Hahn-Banach theorem. It states that the Kantorovich principle can be realized in the case of the classical theorem on the majorized extension of a linear functional, i.e., the real numbers in the Hahn-Banach-Kantorovich theorem can be replaced by the elements of an arbitrary K-space, and the linear functionals by linear operators with values in this K-space.

LECTURE 2. Boolean-valued models

This lecture presents a short survey of necessary information from the theory of Boolean-valued models. The details can be found in **[10, 18, 21, 40, 45–47]**.

The main feature of the method of Boolean-valued models lies in the comparative analysis of two models—standard and nonstandard (Boolean-valued)—using a certain technique of descents and ascents. In addition, a syntactic comparison of formal strings has often to be applied. Therefore, before starting our study of the technique of descents and ascents we need to have a more precise idea about the status of mathematical objects within the framework of formalized set theory.

2.1. At present, the Zermelo-Frenkel set theory is the most widely used axiomatic basis of mathematics. We recall briefly some of the concepts of this theory, concentrating on the details that will be necessary below. It should be noted that regarding formal set theory we shall use (since it is unavoidable) the level of rigor that is accepted in mathematics; we shall introduce abbreviations with the help of the definition operator $:=$ and we will not go into the concomitant details.

(1) The alphabet of the Zermelo-Frenkel theory (shortened ZF or ZFC) consists of symbols for variables; parentheses (,); propositional connectives ($=$ operations of the propositional algebra); $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$, $\rceil$; the quantifiers $\forall$, $\exists$; the sign of equality $=$; and a symbol for a special two-place predicate $\in$. Conceptually, the range of the variables of ZF is conceived as the world (universe) of sets. In other words, the

universe of ZF has no other objects but sets. Instead of $\in (x, y)$, one writes $x \in y$ and says that x is an element of y.

(2) The formulas of ZF are defined by the usual procedure. In other words, the ZF formulas are finite strings obtained from the atomic formulas of the form $x = y$ and $x \in y$, where x, y are variables of ZF, with the help of reasonable placements of parentheses, quantifiers, and propositional connectives. Natural meaning is given to the terms of free and bound variables (or, equivalently, to the concept of the action area of a quantifier).

(3) When studying ZF theory, it is convenient to use the expressions that are absent in its formal language. In particular, it is appropriate to use the notions of a class and of a definable class, and also corresponding symbols for classes of the form $A\varphi := A_{\varphi(\cdot)} := \{x : \varphi(x)\}$ and $A\psi := A_{\psi(\cdot,y)} := \{x : \psi(x, y)\}$, where φ, ψ are formulas from ZF, and y is a selected set of variables. If one wishes to make the resulting notations more precise (or eliminate them) one can assume that the use of classes and classifiers is only related to the usual conventions about the introduction of abbreviations. This convention, sometimes called the Church scheme, is postulated as follows:

$$z \in \{x : \varphi(x)\} \leftrightarrow \varphi(z),$$
$$z \in \{x : \psi(x, y)\} \leftrightarrow \psi(z, y).$$

When working with ZF, abbreviations widely used in mathematics are involved. Some of them are:

$\cup x := \{z : (\exists y \in x)\, z \in y\}$;

$\cap x := \{z : (\forall y \in x)\, z \in y\}$;

$x \subset y := (\forall z)\,(z \in x \to z \in y)$;

$\mathscr{P}(x) :=$ the class of all subsets of $x := \{z : z \subset x\}$;

$V :=$ the class of all sets $:= \{x : x = x\}$.

We note that in further discussions more complicated descriptions, in which a lot is implicitly understood, are allowed:

$\mathrm{Funct}(f) := f$ is a function;

$\mathrm{dom}(f) :=$ the domain of definition of f;

$\mathrm{im}(f) :=$ the range of values of f;

$\varphi \vDash \psi := \langle\langle \psi$ is derivable from $\varphi \rangle\rangle$;

the class A is a set $:= A \in V := (\exists x)(\forall y)\, y \in x \leftrightarrow y \in A$.

Similar simplifications without special stipulations are used in writing down complicated concepts and formulas. For instance, the following formulas of ZF, which are rather large in the language of ZF itself, are simply written down as:

$f : x \to y := f$ is a function from x to y;

E is a K-space.

2.2. In ZF set theory we accept the usual axioms and inference rules for first-order theories with equality, which fix the standard methods of classical reasoning (syllogisms, law of the excluded middle, modus ponens, generalization, etc.). Besides, the following special and characteristic axioms are assumed.

(1) **Axiom of extensionality**:

$$(\forall x)(\forall y) \quad (x \subset y \wedge y \subset x \to x = y).$$

(2) **Axiom of union**:

$$(\forall x) \quad \cup x \in V.$$

(3) **Axiom of power set**:

$$(\forall x) \quad \mathscr{P}(x) \in V.$$

(4) **Axiom scheme of replacement**:

$$(\forall x)(((\forall y)(\forall z)(\forall u)\varphi(y,z) \wedge \varphi(y,u) \to z = u) \\ \to \{z : (\exists y \in x)\varphi(y,z)\} \in V).$$

(5) **Axiom of foundation**:

$$(\forall x)(x \neq \varnothing \to (\exists y \in x)(y \cap x = \varnothing)).$$

(6) **Axiom of infinity**:

$$(\exists \omega)((\varnothing \in \omega) \wedge (\forall x \in \omega)(x \cup \{x\} \in \omega)).$$

(7) **Axiom of choice**:

$$(\forall F)(\forall x)(\forall y)((x \neq \varnothing \wedge F : x \to \mathscr{P}(y)) \\ \to ((\exists f) f : x \to y \wedge (\forall z \in x) \quad f(z) \in F(z))).$$

The precise concept of the class of all sets as the von Neumann universe V is based on the presented axiomatics. The initial object in this construction is the empty set $\varnothing$. A simple step for introducing new sets consists in forming the union of sets or in taking subsets of the sets already constructed. The transfinite repetition of such steps exhausts the class of all sets. More precisely, it is assumed that $V := \bigcup_{\alpha \in \mathrm{Or}} V_\alpha$, where Or is the class of all ordinals and

$$V_0 := \varnothing; \qquad V_{\alpha+1} := \mathscr{P}(V_\alpha); \qquad V_\beta := \bigcup_{\alpha<\beta} V_\alpha \quad (\beta \text{ is a limit ordinal}).$$

The class V is the standard model of ZF theory.

2.3. Now we shall describe the construction of a Boolean-valued universe. Let B be a complete Boolean algebra. For each ordinal α we set

$$V_\alpha^{(B)} := \{x : \mathrm{Funct}(x) \wedge (\exists \beta)(\beta < \alpha \wedge \mathrm{dom}(x) \subset V_\beta^{(B)} \wedge \mathrm{im}(x) \subset B)\}.$$

Thus, in more detailed notation:
$V_0^{(B)} := \varnothing;$
$V_{\alpha+1}^{(B)} := \{x : x \text{ is a function with domain in } V_\alpha^{(B)} \text{ and with range in } B\}$;
$V_\alpha^{(B)} := \bigcup_{\beta<\alpha} V_\beta^{(B)}.$

The following class is considered as the Boolean-valued universe $V^{(B)}$:
$V^{(B)} := \bigcup_{\alpha \in Or} V_\alpha^{(B)}.$

Elements of the class $V^{(B)}$ are called *B-valued sets*. It is worth noting that $V^{(B)}$ consists only of functions. In particular, $\varnothing$ is the function with the domain $\varnothing$ and the range $\varnothing$.

2.4. Suppose we have an arbitrary formula $\varphi = \varphi(u_1, \dots, u_n)$ from ZF theory. By replacing the variables $u_1, \dots, u_n$ with elements $x_1, \dots, x_n \in V^{(B)}$ we obtain a certain statement about the objects $x_1, \dots, x_n$, the validity of which we try to estimate. The desired truth-value $[\![\varphi]\!]$ should be an element from the algebra B. In the process we wish the ZF theorems to be judged valid, i.e., the largest truth value in B, namely one, to be ascribed to them.

The assignment of truth-values is carried out by a double induction which takes into account the character of constructing formulas from the atomic ones, and assigns the truth-values for $[\![x \in y]\!]$ and $[\![x = y]\!]$, where $x, y \in V^{(B)}$, based on the method of construction of $V^{(B)}$.

It is clear that if φ and ψ are already estimated ZF formulas, and $[\![\varphi]\!] \in B$ and $[\![\psi]\!] \in B$ are their truth-values, then one should put

$$[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \wedge [\![\psi]\!];$$

$$[\![\varphi \vee \psi]\!] := [\![\varphi]\!] \vee [\![\psi]\!];$$

$$[\![\varphi \to \psi]\!] := [\![\varphi]\!] \Rightarrow [\![\psi]\!];$$

$$[\![\rceil\varphi]\!] := [\![\varphi]\!]^*;$$

$$[\![(\forall x)\varphi(x)]\!] := \bigwedge_{x \in V^{(B)}} [\![\varphi(x)]\!];$$

$$[\![(\exists x)\varphi(x)]\!] := \bigvee_{x \in V^{(B)}} [\![\varphi(x)]\!],$$

where on the right-hand sides we have the Boolean operations corresponding to the logical connectives and quantifiers from the left sides: $\wedge$ is the infimum, $\vee$ the supremum, $*$ the complement, $\bigwedge$ and $\bigvee$ denote the infimum and supremum of arbitrary sets, and the operation $\Rightarrow$ is introduced in the following way: $a \Rightarrow b := a^* \vee b$ $(a, b \in B)$. Only these definitions allow us to obtain the unit truth-value for the classical tautologies.

Now we pass to the estimation of atomic formulas $x \in y$ and $x = y$ for $x, y \in V^{(B)}$. The intuitive idea is that a B-valued set is a "fuzzy (lattice) set", i.e., a "set that contains an element z from $\operatorname{dom}(y)$ with the probability $y(z)$". Taking into account this idea as well as the goal to save both the logical truth $x \in y \leftrightarrow (\exists z \in y)x = z$ and the extensionality axiom, we are compelled to use the following recursive definition:

$$[\![x \in y]\!] := \bigvee_{z \in \operatorname{dom}(y)} y(z) \wedge [\![z = x]\!],$$

$$[\![x = y]\!] := \bigwedge_{z \in \operatorname{dom}(x)} x(z) \Rightarrow [\![z \in y]\!] \wedge \bigwedge_{z \in \operatorname{dom}(y)} y(z) \Rightarrow [\![z \in x]\!].$$

2.5. Now, we are already in a position to give meaning to formal expressions of the form $\varphi(x_1, \dots, x_n)$, where $x_1, \dots, x_n \in V^{(B)}$ and φ is a ZF formula, i.e., to give precise meaning to the expression: "a set-theoretic expression $\varphi(u_1, \dots, u_n)$ holds for the elements $x_1, \dots, x_n \in V^{(B)}$". Namely, we say that the formula $\varphi(x_1, \dots, x_n)$ is true inside $V^{(B)}$ or that the elements $x_1, \dots, x_n$ possess the property φ in $V^{(B)}$ if $[\![\varphi(x_1, \dots, x_n)]\!] = 1$. This is denoted by $V^{(B)} \models \varphi(x_1, \dots, x_n)$.

It is not difficult to be convinced that the axioms and theorems of the first-order predicate calculus with equality are valid in $V^{(B)}$. In particular,

(1) $[\![x = x]\!] = 1$,

(2) $[\![x = y]\!] = [\![y = x]\!]$,
(3) $[\![x = y]\!] \wedge [\![y = z]\!] \leq [\![x = z]\!]$,
(4) $[\![x = y]\!] \wedge [\![z \in x]\!] \leq [\![z \in y]\!]$,
(5) $[\![x = y]\!] \wedge [\![x \in z]\!] \leq [\![y \in z]\!]$.

It is useful to note that, in general, for every formula φ the following holds:

$$V^{(B)} \vDash x = y \wedge \varphi(x) \to \varphi(y),$$

i.e., written out explicitly
(6) $[\![x = y]\!] \wedge [\![\varphi(x)]\!] \leq [\![\varphi(y)]\!]$.

2.6. In a Boolean-valued universe the relation $[\![x = y]\!] = 1$ does not mean at all that the functions x and y (considered as elements of V) coincide. For instance, the zero function on any layer $V_\alpha^{(B)}$, where $\alpha \geq 1$, plays the role of the empty set in $V^{(B)}$. This makes some constructions more difficult. In view of this, we can introduce a separated Boolean-valued universe $\bar{V}^{(B)}$, which is often denoted by the same symbol $V^{(B)}$; i.e., one sets $V^{(B)} := \bar{V}^{(B)}$. In order to define $\bar{V}^{(B)}$ in the class $V^{(B)}$, the relation $\{(x, y) : [\![x = y]\!] = 1\}$ is considered, which evidently is an equivalence. Selecting an element (a representative of the smallest rank) from every equivalence class one arrives at the separated universe $\bar{V}^{(B)}$. It should be noted that for every formula φ of ZF theory and any elements $x, y \in V^{(B)}$ the following holds:

$$[\![x = y]\!] = 1 \to [\![\varphi(x)]\!] = [\![\varphi(y)]\!].$$

Therefore, in a separated universe the truth-value of formulas can be computed independently of the method of selecting representatives. In general, when dealing with a separated universe, the equivalence class is often replaced (with the necessary precaution) by a certain representative, as it is done, for instance, in the case of function spaces.[7]

2.7. The most important properties of a Boolean-valued universe are contained in the following three principles.

Transfer principle. All the theorems of the ZF theory are true in $V^{(B)}$; i.e., the transfer principle, symbolically written down as

$$V^{(B)} \vDash \text{ZF theorem},$$

is valid.

The transfer principle is established by a rather laborious test showing that all ZF axioms have the truth-value 1, and that the derivation rules preserve the validity of formulas. The transfer principle is sometimes expressed by saying that $V^{(B)}$ is a Boolean-valued model of ZF set theory.

Maximum principle. For every formula φ of ZF theory there is $x_0 \in V^{(B)}$ such that

$$[\![(\exists x)\varphi(x)]\!] = [\![\varphi(x_0)]\!].$$

In particular, if it is true that in $V^{(B)}$ there exists x for which $\varphi(x)$ holds, then, in

[7] *Translator's note.* Such as $L^2(\mathbb{R})$, where one usually thinks in terms of true functions instead of equivalence classes of functions that differ on sets of measure zero.

fact, in $V^{(B)}$ (in the sense of V!) there can be found an element x_0 such that $\varphi(x_0)$. Symbolically,

$$V^{(B)} \models (\exists x)\varphi(x) \to (\exists x_0) V^{(B)} \models \varphi(x_0).$$

In other words, for any formula φ of the ZF theory the maximum principle holds:

$$(\exists x_0 \in V^{(B)})[\![\varphi(x_0)]\!] = \bigvee_{x \in V^{(B)}} [\![\varphi(x)]\!].$$

The latter equality also explains the origin of the term "maximum principle". The proof of this principle is a simple application of the following mixing principle.

Mixing principle. Let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity in B, i.e., a family of elements from the Boolean algebra B such that $\xi \neq \eta \to b_\xi \wedge b_\eta = 0$ and $\bigvee_{\xi\in\Xi} b_\xi = 1$.

For any family of elements $(x_\xi)_{\xi\in\Xi}$ of the universe $V^{(B)}$ and any partition of unity $(b_\xi)_{\xi\in\Xi}$ there exists a (unique) mixing (x_ξ) with the probabilities (b_ξ); i.e., there exists an element x of the separated universe $V^{(B)}$ such that for all $\xi \in \Xi$ the following holds: $[\![x = x_\xi]\!] \geq b_\xi$.

The mixing x of the set (x_ξ) with respect to (b_ξ) is denoted as

$$x := \operatorname*{mix}_{\xi\in\Xi}(b_\xi x_\xi) = \operatorname{mix}\{b_\xi x_\xi : \xi \in \Xi\}.$$

2.8. The comparative analysis which was discussed at the beginning of this lecture is possible due to a close interrelation of the worlds V and $V^{(B)}$. In other words, it is necessary to have a rigorous mathematical technique that would permit us to determine interrelations between interpretations of the same fact in the models V and $V^{(B)}$. The basis of this technique consists in introducing the operations of canonical imbedding, descent, and ascent, which will be defined below. Let us begin with the canonical imbedding of the von Neuman universe. For $x \in V$ the standard name of x in $V^{(B)}$ is $x^\wedge$, so that we have the following recursion scheme:

$$\varnothing^\wedge := \varnothing, \quad \operatorname{dom}(x^\wedge) := \{y^\wedge : y \in x\}, \quad \operatorname{im}(x^\wedge) := \{1\}.$$

We note the necessary properties of the mapping $x \mapsto x^\wedge$.

(1) For every $x \in V$ and formula φ the following holds

$$[\![(\exists y \in x^\wedge)\varphi(y)]\!] = \bigvee\{[\![\varphi(z^\wedge)]\!] : z \in x\};$$

$$[\![(\forall y \in x^\wedge)\varphi(y)]\!] = \bigwedge\{[\![\varphi(z^\wedge)]\!] : z \in x\}.$$

(2) If x, y are elements from V, then using the transfinite induction, one establishes the following:

$$x \in y \leftrightarrow V^{(B)} \models x^\wedge \in y^\wedge,$$

$$x = y \leftrightarrow V^{(B)} \models x^\wedge \in y^\wedge.$$

In other words, the standard name can be considered as an imbedding of V into $V^{(B)}$. Moreover, the standard name maps V onto $V^{(2)}$, as can be noticed from the following fact.

(3) The following statement holds:

$$(\forall u \in V^{(2)})(\exists!\, x \in V)\, V^{(B)} \models u = x^\wedge.$$

(4) A formula is called *bounded* if all occurring variables are included into it under the signs of bounded quantifiers, i.e., quantifiers which range over some set.

The latter phrase means that any occurring variable x is found in the domain of action of a quantifier of the form $(\forall x \in y)$ or $(\exists x \in y)$, for some y.

The principle of bounded transfer. For any bounded formula φ of ZF theory and for any family $x_1, \dots, x_n \in V$ the following equivalence holds:

$$\varphi(x_1, \dots, x_n) \leftrightarrow V^{(B)} \models \varphi(x_1^\wedge, \dots, x_n^\wedge).$$

Let us agree, in the study of a separated universe $\bar{V}^{(B)}$ below, to keep the symbol $x^\wedge$ for a selected element of the class corresponding to x.

(5) Let us note, as an example, that the principle of bounded transfer implies

$$(\Phi \text{ is a correspondence from } x \text{ to } y)$$
$$\leftrightarrow (V^{(B)} \models \Phi \text{ is a correspondence from } x^\wedge \text{ to } y^\wedge);$$
$$(f \text{ is a function from } x \text{ to } y)$$
$$\leftrightarrow (V^{(B)} \models f^\wedge \text{ is a function from } x^\wedge \text{ to } y^\wedge).$$

Moreover, $f(a)^\wedge = f^\wedge(a^\wedge)$ for every $a \in x$.

Thus, the standard name can be regarded as a covariant functor from the category of sets (or correspondences) of V into a suitable subcategory $V^{(2)}$ of the separated universe $V^{(B)}$.

2.9. For an arbitrary element x from the (separated) Boolean-valued universe $V^{(B)}$ the descent $x\downarrow$ of this element is defined by the formula

$$x\downarrow := \{y \in V^{(B)} : [\![y \in x]\!] = 1\}.$$

Let us mention the main properties of the descent procedure.

(1) The class $x\downarrow$ is a set; i.e., $x\downarrow \in V$ for every $x \in V^{(B)}$. If $[\![x \neq \varnothing]\!] = 1$, then $x\downarrow$ is a nonempty set.

(2) Let $z \in V^{(B)}$ and $[\![z \neq \varnothing]\!] = 1$. Then for any formula φ of ZF theory the following holds:

$$[\![(\forall x \in z)\, \varphi(x)]\!] = \bigwedge\{[\![\varphi(x)]\!] : x \in z\downarrow\};$$
$$[\![(\exists x \in z)\, \varphi(x)]\!] = \bigvee\{[\![\varphi(x)]\!] : x \in z\downarrow\}.$$

In addition, there exists $x_0 \in z\downarrow$ such that $[\![\varphi(x_0)]\!] = [\![(\exists x \in z)\varphi(x)]\!]$.

(3) Let Φ be a correspondence from X into Y in $V^{(B)}$. Thus Φ, X, Y are elements of $V^{(B)}$ and, moreover, $[\![\Phi \subset X \times Y]\!] = 1$. Then there exists a unique correspondence $\Phi\downarrow$ from $X\downarrow$ into $Y\downarrow$ such that for any nonempty subset A of the set X in $V^{(B)}$ the following holds:

$$\Phi\downarrow(A\downarrow) = \Phi(A)\downarrow.$$

The correspondence $\Phi\downarrow$ from $X\downarrow$ into $Y\downarrow$ that appears in this statement is called the *descent of the correspondence* Φ from X into Y in $V^{(B)}$.

(4) The descent of the composition of correspondences in $V^{(B)}$ is the composition of the descents of these correspondences:

$$(\Psi \circ \Phi)\downarrow = \Psi\downarrow \circ \Phi\downarrow.$$

(5) If Φ is a correspondence in $V^{(B)}$, then $(\Phi^{(-1)})\downarrow = \Phi\downarrow^{-1}$.

(6) Let I_X be the identity mapping of a set $X \in V^{(B)}$. Then $(I_X)\downarrow = I_{X\downarrow}$.

(7) Let f, X, $Y \in V^{(B)}$ be such that $[\![f : X \to Y]\!] = 1$; i.e., f is a mapping from X into Y in $V^{(B)}$. Then $f\downarrow$ is the unique mapping from $X\downarrow$ into $Y\downarrow$ for which

$$[\![f\downarrow(x) = f(x)\downarrow]\!] = 1 \quad (x \in X\downarrow).$$

As illustrated by statements (1)–(7), the operation of descent can be regarded as a functor from B-valued sets and mappings (correspondences) into the category of usual (i.e., in the sense of V) sets and mappings (correspondences).

(8) For $x_1, \dots, x_n \in V^{(B)}$ we denote the corresponding ordered n-tuple in $V^{(B)}$ by $(x_1, \dots, x_n)^B$. Let us assume that P is an n-ary relation on X in $V^{(B)}$; i.e., X, $P \in V^{(B)}$ and $[\![P \subset X^n]\!] = 1$ $(n \in \omega)$. Then there exists an n-ary relation P' on $X\downarrow$ such that

$$(x_1, \dots, x_n) \in P' \leftrightarrow [\![(x_1, \dots, x_n)^B \in P]\!] = 1.$$

The relation P' is also denoted by the symbol $P\downarrow$ and is called the descent of P.

2.10. Let $x \in V$ and $x \subset V^{(B)}$; i.e., x is a set consisting of B-valued sets or, in other words, $x \in \mathscr{P}(V^{(B)})$. We put $\varnothing\uparrow := \varnothing$ and

$$\operatorname{dom}(x\uparrow) := x, \quad \operatorname{im}(x\uparrow) := \{1\}$$

if $x \neq \varnothing$. The element $x\uparrow$ (of the separated universe $V^{(B)}$, i.e., the selected representative of the class $\{y \in V^{(B)} \cdot [\![y = x\uparrow]\!] = 1\}$) is called the *ascent* of x.

(1) For any $x \in \mathscr{P}(V^{(B)})$ and any formula φ the following equalities hold:

$$[\![(\forall z \in x\uparrow)\,\varphi(z)]\!] = \bigwedge_{y\in x} [\![\varphi(y)]\!];$$

$$[\![(\exists z \in x\uparrow)\,\varphi(z)]\!] = \bigvee_{y\in x} [\![\varphi(y)]\!].$$

To introduce the ascent of a correspondence $\Phi \subset X \times Y$ it is necessary to exercise some caution related to the distinction between the source domain X and the domain of definition $\operatorname{dom}(\Phi) := \{x \in X : \Phi(x) \neq \varnothing\}$. For our further investigations this distinction is not essential; therefore, we can assume that in talking about the ascent we always consider everywhere defined correspondences, i.e., such that $\operatorname{dom}(\Phi) = X$.

(2) Let X, Y, $\Phi \in \mathscr{P}(V^{(B)})$. Further, let Φ be a correspondence from X into Y. There exists a (unique) correspondence $\Phi\uparrow$ from $X\uparrow$ into $Y\uparrow$ such that for every subset A of the set X the following is fulfilled:

$$\Phi\uparrow(A\uparrow) = \Phi(A)\uparrow$$

if and only if Φ is extensional, i.e., satisfies the condition

$$y_1 \in \Phi(x_1) \to [\![x_1 = x_2]\!] \leq \bigvee_{y_2\in\Phi(x_2)} [\![y_1 = y_2]\!]$$

for x_1, $x_2 \in \operatorname{dom}(\Phi) = X$. Further, $\Phi\uparrow = \Phi'\uparrow$, where $\Phi' := \{(x, y)^B : (x, y) \in \Phi\}$. The element $\Phi\uparrow$ is called the *ascent of the initial correspondence* Φ.

(3) The composition of extensional correspondences is extensional. Moreover, the ascent of a composition is equal to the composition of ascents (in $V^{(B)} : V^{(B)} \models (\Psi \circ \Phi)\uparrow = \Psi\uparrow \circ \Phi\uparrow$).

It should be noted that if Φ and Φ^{-1} are extensional, then $(\Phi\uparrow)^{-1} = (\Phi^{-1})\uparrow$. However, the extensionality of Φ does not guarantee the extensionality of Φ^{-1}.

(4) It is worth noting that if an extensional correspondence f is a function from X into Y, then the ascent $f\uparrow$ is a function from $X\uparrow$ into $Y\uparrow$. Here, the extensionality f can be formulated in the following way:

$$[\![x_1 = x_2]\!] \leq [\![f(x_1) = f(x_2)]\!] \quad (x_1, x_2 \in X).$$

For a set $X \subset V^{(B)}$ the symbol mix (X) denotes the set of all the mixings of the form mix $(b_\xi x_\xi)$, where $(x_\xi) \subset X$ and (b_ξ) is an arbitrary partition of unity. The following statements are called the *rules of reducing arrows* or the *rules of "descent-ascent"* and "*ascent-descent*".

(5) Let X and X' be subsets from $V^{(B)}$ and let $f : X \to X'$ be an extensional mapping. Let Y, Y', $g \in V^{(B)}$ be such that $[\![Y \neq \varnothing]\!] = [\![g : Y \to Y']\!] = 1$. Then the following relations hold:

$$X\uparrow\downarrow = \operatorname{mix}(X), \quad f\uparrow\downarrow = f, \quad Y\downarrow\uparrow = Y, \quad g\downarrow\uparrow = g.$$

2.11. Let $X \in V$, $X \neq \varnothing$; i.e., X is a nonempty set. We denote by the letter $\imath$ the imbedding $x \mapsto x^\wedge$ $(x \in X)$. Then $\imath(X)\uparrow = X^\wedge$ and $X = \imath^{-1}(X^\wedge\downarrow)$. Using these relations we can extend descent and ascent to the case when Φ is a correspondence from X into $Y\downarrow$ and $[\![\Psi$ is a correspondence from $X^\wedge$ into $Y]\!] = 1$, where $Y \in V^{(B)}$. Namely, we set $\Phi\upharpoonright := (\Phi \circ \imath)\uparrow$ and $\Psi\downharpoonleft := \Psi\downarrow \circ \imath$. We call $\Phi\upharpoonright$ the *modified ascent* of the correspondence Φ, and $\Psi\downharpoonleft$ the *modified descent* of the correspondence Ψ. (If the context precludes misunderstandings, then one can consider just ascent and descent, and use ordinary arrows.) It is not difficult to see that $\Phi\upharpoonright$ is the only correspondence in $V^{(B)}$ satisfying the following relation:

$$[\![\Phi\upharpoonright(x^\wedge) = \Phi(x)\uparrow]\!] = 1 \qquad (x \in X).$$

Analogously, $\Psi\downharpoonleft$ is the only correspondence from X into $Y\downarrow$ satisfying the following equality:

$$\Psi\downharpoonleft(x) = \Psi(x^\wedge)\downarrow \qquad (x \in X).$$

If $\Phi := f$ and $\Psi := g$ are functions, then these defining relations take the form:

$$[\![f\upharpoonright(x^\wedge) = f(x)]\!] = 1, \quad g\downharpoonleft(x) = g(x^\wedge) \quad (x \in X).$$

2.12. (1) A pair (X, d), where $X \in V$, $X \neq \varnothing$, and d is a mapping from $X \times X$ into a Boolean algebra B is called a *Boolean set* or a *B-set*, or just a *set with B-structure*, if for any x, y, $z \in X$ it satisfies the following conditions:

(a) $d(x, y) = 0 \leftrightarrow x = y$;

(b) $d(x, y) = d(y, x)$;

(c) $d(x, y) \leq d(x, z) \vee d(z, y)$.

As an example of a B-set we can take any $\varnothing \neq X \subset V^{(B)}$ by setting $d(x, y) := [\![x \neq y]\!] = [\![x = y]\!]^*$ $(x, y \in X)$. As another example, take a nonempty sets X with the "discrete B-metric" d; i.e., $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$.

(2) Let (X, d) be some B-set. There exist an element $\mathscr{X} \in V^{(B)}$ and an injection $\imath : X \to X' := \mathscr{X}\downarrow$ such that $d(x, y) = [\![\imath x \neq \imath y]\!]$ $(x, y \in X)$ and any element $x' \in X'$ admits a representation $x' = \operatorname{mix}_{\xi \in \Xi}(b_\xi \imath x_\xi)$, where $(x_\xi)_{\xi \in \Xi} \subset X$ and $(b_\xi)_{\xi \in \Xi}$ is a partition of unity in B.

The element $\mathscr{X} \in V^{(B)}$ is called the *Boolean realization* of X. If X is a discrete B-set, then $\mathscr{X} = X^\wedge$ and $\imath x = x^\wedge$ $(x \in X)$. If $X \subset V^{(B)}$, then $\imath\uparrow$ is an injection from $X\uparrow$ into $\mathscr{X}$ (in $V^{(B)}$).

A mapping f from a B-set (X, d) into a B-set (X', d') is called *nonexpanding* if $d(x, y) \geq d'(f(x), f(y))$, for all $x, y \in X$.

(3) Let X and Y be some B-sets, $\mathscr{X}$ and $\mathscr{Y}$ their Boolean realizations, and $\imath$ and κ the corresponding injections $X \to \mathscr{X}\downarrow$ and $Y \to \mathscr{Y}\downarrow$. If $f : X \to Y$ is a nonexpanding correspondence, then there exists a unique element $g \in V^{(B)}$ such that $[\![g : \mathscr{X} \to \mathscr{Y}]\!] = 1$ and $f = \kappa^{-1} \circ g\downarrow \circ \imath$.

A similar statement holds for correspondences.

(4) We shall present an example of a B-set which is important below. Let E be a vector lattice and $B := \mathfrak{B}(E)$. We put

$$d(x, y) := \{|x - y|\}^{\perp\perp} \qquad (x, y \in E).$$

It is not difficult to verify that d satisfies the conditions (b), (c) in (1), while condition (a) in (1) is fulfilled only for an Archimedean E (see 1.3).

Thus, (E, d) is a B-set if and only if the vector lattice E is Archimedean.

2.13. Starting from the results of 2.9 we can define the descent of an algebraic system. For simplicity we shall restrict ourselves to the case of a finite signature. Let $\mathfrak{A}$ be an algebraic system of finite signature in $V^{(B)}$. This means that there exist elements $A, f_1, \ldots, f_n, P_1, \ldots, P_m \in V^{(B)}$ and natural numbers $a(f_1), \ldots, a(f_n)$, $a(P_1), \ldots, a(P_m)$ such that the following conditions (all in $V^{(B)}$) are fulfilled:

$$A \neq \varnothing, \quad P_k \subset A^{a(P_k)^\wedge} \quad (k := 1, \ldots, m);$$
$$f_l : A^{a(f_l)^\wedge} \to A \quad (l := 1, \ldots, n);$$
$$\mathfrak{A} := (A, f_1, \ldots, f_n, P_1, \ldots, P_m).$$

Having obtained the descent of the set A, of the functions $f_1, \ldots, f_n$, and of the relations $P_1, \ldots, P_m$ according to the rules 1.9, we obtain an algebraic system $\mathfrak{A}\downarrow = (A\downarrow, f_1\downarrow, \ldots\ldots, f_n\downarrow, P_1\downarrow, \ldots, P_m\downarrow)$ which is called the *descent of* $\mathfrak{A}$. Thus the descent of the algebraic system $\mathfrak{A}$ is the descent of the base set A together with the descended operations and relations.

2.14. Comments. (a) As was noted above in 1.15(b), the heuristic transfer principle introduced by Kantorovich in connection with the concept of a K-space subsequently found many confirmations in the investigations of Kantorovich himself and his followers. Essentially, this principle is one of those ideas, which as the organizing and direction-giving idea in a new field, finally brought about a profound and complete theory of K-spaces, rich with various applications. Already at the beginning of the development of this theory attempts were made to formalize these heuristic arguments. There were also so-called theorems on preservation of relations, which state that if a certain proposition containing a finite number of functional relations is proved for the real numbers, then a similar fact is automatically valid for elements of a K-space (see **[7, 14]**).

However, the intrinsic mechanism controlling the phenomenon of preservation of relations, the bounds of applicability of such statements, and also the general reasons for a number of analogies and parallels with classical function theory were still obscure. The depth and the universal character of the Kantorovich principle became apparent in the framework of Boolean-valued analysis.

(b) The part of functional analysis which uses a special model-theoretic technique, the Boolean-valued models of the set theory, is called *Boolean-valued analysis*. It is of interest to note that the construction of Boolean-valued models was not related to the theory of ordered vector spaces. The necessary language and technical tools were already forged within mathematical logic by the 1960s. However, there was at that time no general idea to give life to this mathematical apparatus and to lead to progress in model theory. This idea only came with the discovery of P. Cohen, who established the absolute unsolvability (in a precise mathematical sense) of the classical continuum problem. Indeed, it was in connection with Cohen's method of forcing that there emerged Boolean-valued models of set theory, whose creation is associated with the names of P. Vopenka, D. Scott, and R. Solovay (see [**43, 45, 48, 49**]).

(c) The method of forcing is naturally divided into two parts: general and special. The general part is the technique of Boolean-valued models of set theory, i.e., the construction of a Boolean-valued universe $V^{(B)}$ and the interpretation of set-theoretic statements in it. Here the complete Boolean algebra B is totally arbitrary. The special part consists in the construction of a specific Boolean algebra B that provides necessary (rather frequently pathological, exotic) properties of objects (for instance, of a K-space) obtained from B. Both parts are of independent interest, but the most effective results are obtained by combining them. In this section, as in most investigations in Boolean-valued analysis, only the general part of the method of forcing is applied. The special part is most actively used in proofs of independence or consistency (see [**10, 27, 47**]). Further progress in Boolean-valued analysis probably will be connected with full application of the forcing method.

(d) The material in 2.1–2.8 is standard and a detailed description of it can be found in [**18, 21, 27, 47**]; see also [**10, 23**]. Various versions of the methods presented in 2.9–2.11 are widely applied in investigations of Boolean-valued models. In [**17, 22**] the descent and ascent technique is given in a form that is better adapted to problems of analysis. Indeed, in this form they are studied in [**21**]. The imbedding (2.10) of sets with Boolean structure into a Boolean-valued universe was introduced in [17]. The basis of such an imbedding is the Solovay-Tennenbaum method, proposed earlier for imbeddings of complete Boolean algebras [**44**].

LECTURE 3. Vector lattices and numerical systems

Boolean-valued analysis can be traced back to the statement by Scott and Solovay that the image of the field of real numbers in a Boolean model is an extended K-space. Depending on what Boolean algebra B (algebra of measurable sets, or of regular open sets, or of projections in a Hilbert space) is used as a base in constructing a Boolean-valued model $V^{(B)}$, different K-spaces are obtained (spaces of measurable functions, or of semicontinuous functions, or of selfadjoint operators). Thereby there arises a remarkable possibility of transferring what is known about numbers to many classical objects of analysis. This will be discussed in the present lecture.

3.1. By the field of real numbers we understand an algebraic system in which the axioms of an Archimedean ordered field (with different zero and unit element) and the axiom of completeness are fulfilled. We recall two well-known statements.

(1) The field $\mathbb{R}$ of real numbers exists and is unique up to isomorphism.

(2) If P is an Archimedean ordered field, then there exists an isomorphic imbedding h of the field P into $\mathbb{R}$ such that the image $h(P)$ is a subfield of $\mathbb{R}$ containing the subfield of rational numbers. In particular, $h(P)$ is dense in $\mathbb{R}$.

Applying to (1) consecutively the transfer and maximum principles, we can find an element $\mathscr{R} \in V^{(B)}$, for which $[\![\mathscr{R}$ is a field of real numbers $]\!] = 1$. Moreover, if any $\mathscr{R}' \in V^{(B)}$ satisfies the condition $[\![\mathscr{R}'$ is a field of real numbers $]\!] = 1$, then the condition $[\![$ the ordered fields $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic $]\!] = 1$ is also satisfied. In other words, in the model $V^{(B)}$ there exists a field of real numbers $\mathscr{R}$ which is unique up to isomorphism.

Let us also note that the formula $\varphi(\mathbb{R})$ representing the formal description of the axioms of an Archimedean ordered field is bounded and, therefore, $[\![\varphi(\mathbb{R}^\wedge)]\!] = 1$; i.e., $[\![\mathbb{R}^\wedge$ is an Archimedean ordered field $]\!] = 1$. "Having passed" the statement (2) through the transfer principle we can the conclude that $[\![\mathbb{R}^\wedge$ is isomorphic to a dense subfield of the field $\mathscr{R}]\!] = 1$. Based on this fact, we will assume below that $\mathscr{R}$ is a field of real numbers in the model $V^{(B)}$ and that $\mathbb{R}^\wedge$ is a dense subfield in it. As is easy to see, the elements $0 := 0^\wedge$ and $1 := 1^\wedge$ are the zero and the unit element of the field $\mathscr{R}$.

It should be emphasized that in the general case the equality $\mathscr{R} = \mathbb{R}^\wedge$ does not hold. Indeed, the completeness axiom for $\mathbb{R}$ is not a bounded formula, and it might fail for $\mathbb{R}^\wedge$ in $V^{(B)}$.

Now we shall consider the descent $\mathscr{R}\downarrow$ of the algebraic system $\mathscr{R}$. In other words, the descent of the carrier set of the system $\mathscr{R}$ is regarded together with the descended operations and order. For simplicity we shall denote the operations and order relation in $\mathscr{R}$ and $\mathscr{R}\downarrow$ by the same symbols $+, \cdot, \leq$. Thus, to be more precise, the addition and multiplication, and the relation of order in $\mathscr{R}\downarrow$ are introduced by the following formulas:

$$z = x + y \leftrightarrow [\![z = x + y]\!] = 1;$$
$$z = x \cdot y \leftrightarrow [\![z = x \cdot y]\!] = 1;$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = 1;$$
$$(x, y, z \in \mathscr{R}\downarrow).$$

Multiplication by real numbers can also be introduced in $\mathscr{R}\downarrow$ by the rule:

$$y = \lambda x \leftrightarrow [\![\lambda^\wedge x = y]\!] = 1 \qquad (\lambda \in \mathbb{R};\ x, y \in \mathscr{R}\downarrow).$$

3.2. Theorem (Gordon). *Let $\mathbb{R}$ be an ordered field of real numbers in the model $V^{(B)}$. Then $\mathscr{R}\downarrow$ (with operations and order descended) is an extended K-space with unit* 1. *Moreover, there exists an isomorphism χ of the Boolean algebra B onto the base $\mathscr{P}(\mathscr{R}\downarrow)$ such that the following equivalences are valid:*

$$\chi(b)x = \chi(b)y \leftrightarrow b \leq [\![x = y]\!],$$
$$\chi(b)x \leq \chi(b)y \leftrightarrow b \leq [\![x \leq y]\!]$$

for all x, $y \in \mathscr{R}\downarrow$ and $b \in B$.

3.3. The extended K-space $\mathscr{R}\downarrow$ is at the same time a faithful f-algebra with ring unit 1, where for any $b \in B$ the projection $\chi(b)$ is the operator of multiplication by the unit element $\chi(b)1$.

From what was said above it is clear that the mapping $b \mapsto \chi(b)1$ $(b \in B)$ is a Boolean isomorphism between B and the algebra of unit elements in $\mathfrak{E}(\mathscr{R}\downarrow)$. This isomorphism is denoted by the same letter χ. Thus, depending on the context, $x \mapsto \chi(b)x$ is either a band projection or the operator of multiplication by the unit element $\chi(b)$.

3.4. Everywhere below $\mathscr{R}$ is the field of real numbers in the model $V^{(B)}$. Let us explain the meaning of exact bounds and order limits in the K-space $\mathscr{R}\downarrow$.

(1) Let $(b_\xi)_{\xi\in\Xi}$ be a partition of unity in B, and $(x_\xi)_{\xi\in\Xi}$ a set in $\mathscr{R}\downarrow$. Then

$$\operatorname{mix}_{\xi\in\Xi}(b_\xi x_\xi) = o\text{-}\sum_{\xi\in\Xi}\chi(b_\xi)x_\xi.$$

(2) For a nonempty set $A \subset \mathscr{R}\downarrow$ and arbitrary $a \in \mathscr{R}$, $b \in B$ the following equivalences are valid:

$$b \leq [\![a = \sup(A\uparrow)]\!] \leftrightarrow \chi(b)a = \sup(b)A;$$

$$b \leq [\![a = \inf(A\uparrow)]\!] \leftrightarrow \chi(b)a = \inf \chi(b)A.$$

(3) Let A be an upwards filtered set and let $s : A \to \mathscr{R}\downarrow$ be a net in $\mathscr{R}\downarrow$. Then $A^\wedge$ is upwards filtered and $\sigma := s\uparrow : A^\wedge \to \mathscr{R}$ is a net in $\mathscr{R}$ (in $V^{(B)}$); moreover, for any $x \in \mathscr{R}\downarrow$ and $b \in B$ we have

$$b \leq [\![x = \lim \sigma]\!] \leftrightarrow \chi(b)x = o\text{-}\lim \chi(b) \circ s.$$

(4) Let the elements A and $\sigma \in V^{(B)}$ be such that $[\![A$ is upwards filtered and $\sigma : A \to \mathscr{R}]\!] = 1$. Then $A\downarrow$ is an upwards filtered set and thus the mapping $s := \sigma\downarrow : A\downarrow \to \mathscr{R}\downarrow$ is a net in $\mathscr{R}\downarrow$. Besides, for any $x \in \mathscr{R}\downarrow$ and $b \in B$, the following is satisfied:

$$b \leq [\![x = \lim \sigma]\!] \leftrightarrow \chi(b)x = o\text{-}\lim \chi(b) \circ s.$$

(5) Let f be a mapping from a nonempty set Ξ into $\mathscr{R}\downarrow$ and $g := f\uparrow$. Then for any $x \in \mathscr{R}\downarrow$ and $b \in B$ the following holds:

$$b \leq [\![z = \sum_{\xi\in\Xi^\wedge} g(\xi)]\!] \leftrightarrow \chi(b)x = \sum_{\xi\in\Xi}\chi(b)f(\xi).$$

3.5. For every element $x \in \mathscr{R}\downarrow$ the following relations hold:

$$e_x = \chi([\![x = 0]\!]), \quad e_\lambda^x = \chi([\![x < \lambda^\wedge]\!]) \quad (\lambda \in \mathbb{R}).$$

A real number t is not equal to zero if and only if the supremum of the set $\{1 \wedge (n|t|) : n \in \omega\}$ is equal to 1. Consequently, according to the transfer principle, for $x \in \mathscr{R}\downarrow$ we have $b := [\![x \neq 0]\!] = [\![\sup A = 1]\!]$, where $A \in V^{(B)}$ is defined by the formula $A := \{1 \wedge (n|x|) : n \in \omega^\wedge\}$. If $C := \{1 \wedge (n|x|) : n \in \omega\}$, then, by using the second formula from 2.10(1) and the representation $\omega^\wedge = (\iota\omega)\uparrow$ from 2.11, we shall prove that $[\![C\uparrow = A]\!] = 1$. So $[\![\sup(A) = \sup(C\uparrow)]\!] = 1$. Invoking 3.4(2), we derive

$$b = [\![\sup(C\uparrow) = 1]\!] = [\![\sup(C) = 1]\!] = [\![e_x = 1]\!].$$

On the other hand, $[\![e_x = 0]\!] = [\![e_x = 1]\!]^* = b^*$. According to 3.2, we can write

$$\chi(b)e_x = \chi(b)1 = \chi(b); \qquad \chi(b^*)e_x = 0 \to \chi(b)e_x = e_x.$$

Finally, $\chi(b) = e_x$.

Let us take $\lambda \in \mathbb{R}$ and define $y := (\lambda 1 - x)^+$. Since $[\![\lambda^\wedge = \lambda 1]\!] = 1$, we have $[\![y = (\lambda^\wedge - x)^+]\!] = 1$. Consequently, $e_\lambda^x = e_y = \chi([\![y = 0]\!])$. It remains to note that in $V^{(B)}$ the number $y = (\lambda^{\wedge} - x) \vee 0$ is not equal to zero only if $\lambda^{\wedge} - x > 0$, i.e., $[\![y \neq 0]\!] = [\![x < \lambda^{\wedge}]\!]$.

3.6. THEOREM. *Let E be an Archimedean vector lattice, $\mathscr{R}$ a field of real numbers in the model $V^{(B)}$, and j an isomorphism of B onto the base $\mathfrak{B}(E)$. There exists an element $\mathscr{E} \in V^{(B)}$ satisfying the following conditions*:

(1) $V^{(B)} \models \langle\langle\mathscr{E}$ *is a vector sublattice of the field $\mathscr{R}$ regarded as a vector lattice over* $\mathbb{R}^{\wedge}\rangle\rangle$;

(2) $E' := \mathscr{E}\downarrow$ *is a vector sublattice of $\mathscr{R}\downarrow$ invariant under every projection $\chi(b)$ $(b \in B)$ and such that any set of the positive pairwise disjoint sets has a supremum*;

(3) *there exists a* 0*-continuous lattice isomorphism $\imath\colon E \to E'$ such that $\imath(E)$ is a minorant sublattice in* $\mathscr{R}\downarrow$;

(4) *for every $b \in B$ the operator of projection onto a band generated in $\mathscr{R}\downarrow$ by the set $\imath(j(b))$ coincides with $\chi(b)$.*

Let us set $d(x,y) := j^{-1}(\{|x-y|\}^{\perp\perp})$. Let $\mathscr{E}$ be a Boolean realization of a B-set (E,d) and $E' := \mathscr{E}\downarrow$ (see 2.12(4)). By 2.12(2), we can say without loss of generality that $E \subset E'$, $d(x,y) = [\![x \neq y]\!]$ $(x, y \in E)$, and $E' = \operatorname{mix}(E)$. Further, in the set E' we can introduce the structure of vector lattice. For that we take a number $\lambda \in \mathbb{R}$ and elements $x, y \in E'$ of the form $x := \operatorname{mix}(b_\xi x_\xi)$, $y := \operatorname{mix}(b_\xi y_\xi)$, where $(x_\xi) \subset E$, $(y_\xi) \subset E$, and (b_ξ) is a partition of unity in B, and define

$$
\begin{aligned}
x + y &:= \operatorname{mix}(b_\xi(x_\xi + y_\xi));\\
\lambda x &:= \operatorname{mix}(b_\xi(\lambda x_\xi));\\
x \le y \leftrightarrow x &= \operatorname{mix}(b_\xi(x_\xi \wedge y_\xi)).
\end{aligned}
$$

Inside $V^{(B)}$ we define addition $\oplus$, multiplication $\odot$, and order relation $\circledleq$ on the set $\mathscr{E}$ as ascents of the corresponding operations from E'. More precisely, the operations $\oplus\colon \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ and $\odot\colon \mathbb{R}^{\wedge} \times \mathscr{E} \to \mathscr{E}$ and the relation $\circledleq \subset \mathscr{E} \times \mathscr{E}$ are defined by

$$
\begin{gathered}
[\![x \oplus y = x + y]\!] = 1;\\
[\![\lambda^{\wedge} \odot y = \lambda x]\!] = 1 \quad (x, y \in E',\ \lambda \in \mathbb{R}),\\
[\![x \circledleq y]\!] = \bigvee\{[\![x = x']\!] \wedge [\![y = y']\!]\colon x', y' \in E',\ x' \le y'\}.
\end{gathered}
$$

Then we can claim that $\mathscr{E}$ is a vector lattice over the field $\mathbb{R}^{\wedge}$ and, in particular, it is a lattice ordered group in $V^{(B)}$. It is also clear that the Archimedean axiom is valid for $\mathscr{E}$ because the lattice E' is Archimedean.

Note that if $x \in E_+$, then $\{x\}^{\perp\perp} = d(x,0) = [\![x \neq 0]\!]$; i.e., $\{x\}^{\perp} = [\![x = 0]\!]$. Consequently, for disjoint $x, y \in E$ we get $[\![x = 0]\!] \vee [\![y = 0]\!] = \{x\}^{\perp} \vee \{y\}^{\perp} = 1_B$. From this it is easy to derive that $[\![\mathscr{E}$ is linearly ordered $]\!] = 1$, since

$$
[\![(\forall x \in \mathscr{E})(\forall y \in \mathscr{E})(|x| \wedge |y| = 0 \to x = 0 \vee y = 0)]\!] = 1.
$$

It is well known that the Archimedean linearly ordered groups are isomorphic to additive subgroups of the field of real numbers. Applying this statement to $\mathscr{E}$ in $V^{(B)}$, it can be assumed without any loss of generality that $\mathscr{E}$ is an additive subgroup of the field $\mathscr{R}$. In addition, we shall assume that $1 \in \mathscr{E}$; otherwise $\mathscr{E}$ can be replaced by the group $e^{-1}\mathscr{E}$, $0 < e \in \mathscr{E}$, which is isomorphic to $\mathscr{E}$. The multiplication $\odot$ represents a $\mathbb{R}^{\wedge}$-bilinear continuous mapping from $\mathbb{R}^{\wedge} \times \mathscr{E}$ into $\mathscr{E}$. Let $\beta\colon \mathscr{R} \times \mathscr{R} \to \mathscr{R}$ be its extension by continuity. Then β is $\mathscr{R}$-bilinear and $\beta(1,1) = 1^{\wedge} \odot 1 = 1$. Consequently,

β coincides with the usual multiplication in $\mathscr{R}$; i.e., $\mathscr{E}$ is a vector sublattice of the field $\mathscr{R}$, regarded as a vector lattice over $\mathbb{R}^\wedge$. Thereby $E' \subset \mathscr{R}\downarrow$.

The minorant property of E' in $\mathscr{R}\downarrow$ evidently follows from the fact that $[\![\mathscr{E}$ is dense in $\mathscr{R}]\!] = 1$. We shall prove that E is minorant in E'.

From the properties of the isomorphism χ (see 3.2) it is clear that

$$\chi(b)\imath x = 0 \leftrightarrow j(b) \le \{x\}^\perp \leftrightarrow x \in j(b)^\perp$$

for any $b \in B$ and $x \in E_+$. Thus $\chi(b)$ is the projection onto the band generated in $\mathscr{R}\downarrow$ by the set $\imath(j(b))$. Besides, if $\chi(b)x = 0$ for all $x \in E_+$, then $b = \{0\}$. So, for any $b \in B$ there can be found a strictly positive element $y \in E$, for which $y = \chi(b)y$. Now we shall take $0 < z \in E'$. The representation $z = \operatorname{mix}(b_\xi x_\xi)$, where (b_ξ) is a partition of unity in B and $(x_\xi) \subset E_+$, holds. Apparently, $\chi(b_\xi)x_\xi \ne 0$ for at least one index ξ. Let $\pi := \chi(b_\xi) \circ \chi([\![x_\xi \ne 0]\!])$ and y be a strictly positive element in E, for which $y = \pi y$. Then for $x_0 := y \wedge x_\xi$ we have $0 < x_0 \le \pi x_\xi \le \chi(b_x i)x_\xi \le z$ and $x_0 \in E$. Therefore, E is minorant in E'.

3.7. The element $\mathscr{E} \in V^{(B)}$ from Theorem 3.6 is called the *Boolean realization* of E. Thus, vector sublattices of the field of real numbers $\mathscr{R}$ regarded as vector lattices over the field $\mathbb{R}^\wedge$ serve as Boolean realizations of Archimedean vector lattices.

Now we shall note several corollaries from 3.2 and 3.6 keeping the same notations B, E, E', $\mathscr{E}$, $\imath$, $\mathscr{R}$.

(1) For every $x' \in E'$ there exists a set $(x_\xi) \subset E$ and a partition of unity (π_ξ) in $\mathscr{P}(\mathscr{R}\downarrow)$ such that

$$x' = o\text{-}\sum_{\xi\in\Xi} \pi_\xi \imath x_\xi.$$

(2) For any $x \in \mathscr{R}\downarrow$ and $\varepsilon > 0$ there exists $x_\xi \in E'$ such that $|x - x_\xi| \le \varepsilon 1$.

(3) If $h\colon E \to \mathscr{R}\downarrow$ is a lattice isomorphism and for every $b \in B$ the projection onto the band generated by the set $h(j(b))$ in $\mathscr{R}\downarrow$ coincides with $\chi(b)$, then there exists an $a \in \mathscr{R}\downarrow$, for which $hx = a \cdot \imath(x)$ $(x \in E)$.

(4) If E contains the order unit $\mathbf{1}$, then the isomorphism $\imath$ is uniquely defined by the additional requirement $\imath\mathbf{1} = 1$.

(5) If E is a K-space, then $\mathscr{E} = \mathscr{R}$, $E' = \mathscr{R}\downarrow$, and $\imath(E)$ is a foundation of the K-space $\mathscr{R}\downarrow$. Moreover, $\imath^{-1} \circ \chi(b) \circ \imath$ is the projection onto the band of $j(b)$ for every $b \in B$.

(6) The image $\imath(E)$ coincides with all of $\mathscr{R}\downarrow$ if and only if E is an extended K-space.

(7) Extended K-spaces are isomorphic if and only if their bases are isomorphic.

(8) Let E be an extended K-space with unit $\mathbf{1}$. Then the there exists a unique multiplication in E such that E is a faithful f-algebra with the multiplication unit $\mathbf{1}$.

3.8. We shall dwell on questions of extension and completion of Archimedean vector lattices.

By a *maximal extension* of an Archimedean vector lattice E we understand a K-space $mE := \mathscr{R}\downarrow$, where $\mathscr{R}$ is a field of real numbers in the model $V^{(B)}$, $B := \mathfrak{B}(E)$. It is clear from the Theorem 3.6 that there exists an isomorphism $\imath\colon E \to mE$; moreover, the sublattice $\imath(E)$ is minorant in mE and $\imath(E)^{\perp\perp} = mE$. The maximal extension is defined up to isomorphism by these properties. To be more precise, the following statements are valid.

(1) Let E be an Archimedean vector lattice and F an extended K-space. We assume that an isomorphism h from E onto a minorant sublattice of F is given and $h(E)^{\perp\perp} = F$. Then there exists an isomorphism κ from F onto mE such that $\iota = \kappa \circ h$.

From the above conditions it is easy to derive that $j\colon b \mapsto j(b) := h(b)^{\perp\perp}$ is an isomorphism from $B := \mathfrak{B}(E)$ into $\mathfrak{B}(F)$. According to 3.6(5),(6) there exists an isomorphism k from F onto mE, for which $k^{-1} \circ \chi(b) \circ k$ is the projection onto the band $j(b)$ (for each $b \in B$). We shall apply 3.6(3) to $F_0 := h(E)$ and $g := \iota \circ h^{-1}\colon F_0 \to \mathscr{R}\downarrow$. There can be found an element $a \in \mathscr{R}\downarrow$ such that $g(x) = a \cdot k(x)$ $(x \in F_0)$. Now set $\kappa(x) := a \cdot k(x)$ $(x \in F)$. Then $\iota = \kappa \circ h$.

(2) For any Archimedean vector lattice E there exists a K-space ${}_{\circ}E$, unique up to isomorphism and an o-continuous lattice isomorphism $\iota\colon E \to {}_{\circ}E$ such that

$$\sup\{\iota x\colon x \in E,\ \iota x \le y\} = y = \inf\{\iota x\colon x \in E,\ \iota x \ge y\}$$

for every element $y \in {}_{\circ}E$.

Let F be a K-space and $A \subset F$. We denote by dA the set of all $x \in F$ that can be represented in the form $o\text{-}\sum_{\xi\in\Xi} \pi_\xi a_\xi$, where $(a_\xi) \subset A$ and (π_ξ) is a partition of unity in $\mathfrak{P}(F)$. Let rA be the set of all elements $x \in F$ of the form $x = r\text{–}\lim_n a_n$, where (a_n) is an arbitrary regular convergent sequence in A.

(3) For an Archimedean vector lattice E the formula ${}_{\circ}E = rdE$ holds.

3.9. Interpreting the notion of convergent numerical net in $V^{(B)}$ and invoking 3.4(3), 3.7(5) one can obtain useful tests for o-convergence in a K-space E with unit $\mathbf{1}$.

Theorem. *Let $(x_\alpha)_{\alpha\in A}$ be an order bounded net in E and $x \in E$. The following statements are equivalent*:

(1) *the net (x_α) o-converges to the element x*;

(2) *for any number $\varepsilon > 0$ the net of unit elements $(e_\varepsilon^{y(\alpha)})_{\alpha\in A}$, where $y(\alpha) := |x - x_\alpha|$, o-converges to zero*;

(3) *for any number $\varepsilon > 0$ there exists a partition of unity $(\pi_\alpha)_{\alpha\in A}$ in the Boolean algebra $\mathfrak{P}(E)$ such that*

$$\pi_\alpha|x - x_\beta| < \varepsilon\mathbf{1} \qquad (\alpha, \beta \in A);$$

(4) *for any number $\varepsilon > 0$ there exists an increasing net of projections $(\rho_\alpha)_{\alpha\in A}$ $\subset \mathfrak{P}(E)$ such that*

$$\rho_\alpha|x - x_\beta| < \varepsilon\mathbf{1} \qquad (\alpha, \beta \in A;\ \beta \ge \alpha).$$

3.10. Comments. (a) The Boolean status of K-spaces is established by the Gordon Theorem 3.2 (see **[8]**). This fact can be formulated in the following way: an extended K-space is an interpretation of the field of real numbers in a suitable Boolean-valued model. In addition, it turns out that any theorem (within the framework of ZF theory) on real numbers has its analogue in the corresponding K-space. Conversion of one kind of theorems into others is realized by certain precisely defined procedures: ascent, descent, canonical imbedding; i.e., as a matter of fact it is realized algorithmically. Therefore, the Kantorovich statement "the elements from a K-space are generalized numbers" finds a precise mathematical formulation in Boolean-valued analysis. On the other hand, Boolean-valued analysis turns the heuristic transfer principle, which played an auxiliary guiding role in most investigations of the pre-Boolean theory of the K-spaces, into a precise research method.

(b) If in 3.2 B is the σ-algebra of measurable sets modulo sets of measure zero for a measure μ, then $\mathscr{R}\downarrow$ is isomorphic to the extended K-space of measurable functions $L^0(\mu)$. This fact (for the Lebesgue measure on the interval) was already known to Scott and Solovay (see **[43]**). If B is the complete Boolean algebra of projections in a Hilbert space, then $\mathscr{R}\downarrow$ is isomorphic to the space of those self-adjoint operators which have a spectral function acting in B. The two special cases of the Gordon theorem noted above were effectively used by G. Takeuti; see **[45]**, and also the bibliography in **[21]**. The object $\mathscr{R}\downarrow$ for general Boolean algebras was also considered by T. Jech **[33, 34]** who essentially rediscovered the Gordon theorem. The difference is that in **[33]** a (complex) extended K-space with unit is defined by another system of axioms and is called a complete Stone algebra. The interconnections from 3.4, 3.5 between properties of numerical objects and corresponding objects in a K-space $\mathscr{R}\downarrow$ were obtained essentially by Gordon **[8, 9]**.

(c) The Realization Theorem 3.6 was obtained by Kusraev **[19]**. There is a closely related result (formulated in other terms) in **[35]**, where a Boolean interpretation of the theory of linearly ordered sets is developed. Corollaries 3.7(7),(8) are well known (see **[7, 14]**). The concept of maximal extension for a K-space was introduced by Pinsker in a different way. He also proved the existence of a maximal extension unique up to isomorphism, for an arbitrary K-space. Theorem 2.8(2) on order completion of an Archimedean vector lattice was stated by Yudin. Corresponding references are in **[7, 14]**. Statement 2.6(3) was obtained by Veksler **[5]**.

The tests for o-convergence 3.9(2) and 3.9(4) (for sequences) were established by Kantorovich and Vulikh, respectively (see **[14]**). In 3.8 it is shown that these tests are, essentially, just interpretations of convergence properties of numerical nets (sequences).

(d) As was noted in 2.14(a), the first attempts to formalize the heuristic Kantorovich principle led to theorems on preservation of relations (see **[7, 14]**). Modern forms of theorems on preservation of relations, which use the method of Boolean-valued models, can be found in **[9, 35]** (see also **[21]**).

(e) Boolean realizations (not only of Archemedean vector spaces) provide subsystems of the field $\mathscr{R}$ (see 3.6(1)). For example, the following statements are formulated in **[19]**: (1) a Boolean realization of an Archimedean lattice ordered groups is a subgroup of the additive group of $\mathscr{R}$; (2) an Archimedean f-ring contains two mutually complementary bands, one of which has zero multiplication and is realized as (1), and the other is realized as a subring of $\mathscr{R}$; (3) an Archimedean f-algebra contains two mutually complementary bands, one of which is realized as in 3.6, and the other is a sublattice and a subalgebra of the field $\mathscr{R}$ considered as a lattice ordered algebra over the field $\mathbb{R}^\wedge$ (see also **[35]**).

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