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POSITIVE LINEAR MINKOWSKI FUNCTIONALS OVER CONVEX SURFACES

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In the analysis of some extremal problems of isoperimetric type as well as in a number of other questions there arises the problem of representation of positive linear functionals relative to Minkowski operations over convex surfaces. In the present paper such functionals are described in terms of order relations similar to the so-called strong measure ordering given by Loomis. However, the idea of proof of a similar theorem of Cartier-Fell-Meyer [1], circumscribing the polar of a cone of convex functions, is unfortunately not suitable here. The present exposition uses certain properties of the space of convex sets.

Let $\mathfrak{V}_n$ be the set of all compact convex subsets of an n-dimensional arithmetic space R^n with euclidean norm $|\cdot|$. For any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{V}_n$ and $\alpha \geq 0$ the Minkowski operations are defined as

$$\mathfrak{x}+\mathfrak{y}=\{z\in R^n\colon z=x+y\ (x\in\mathfrak{x};\ y\in\mathfrak{y})\};$$
$$\alpha\mathfrak{x}=\{z\in R^n\colon z=\alpha x\ (x\in\mathfrak{x})\}.$$

Assigning to $\mathfrak{V}_n$ Hausdorff topology we obtain a topological semigroup with operators in the semigroup of nonnegative numbers R_+. Let $\mathfrak{V}O_n$ be the set of spatial convex compacta, i.e. the set of convex surfaces. Since $\mathfrak{V}O_n$ is everywhere dense in $\mathfrak{V}_n$, the set $\mathfrak{V}O_n^*$ of positive linear Minkowski functionals coincides with $\mathfrak{V}_n^*$—the set of continuous R_+-operator-homomorphisms of $\mathfrak{V}_n$ into R_+.

By $\mathrm{Sub}(R^n)$ we will denote the set of sublinear (convex, positive and homogeneous) functions defined on all of R^n. We equip this set with the ordinary algebraic structure as well as with the topology of uniform convergence on compact subsets of R^n.

Let ϕ be the mapping $\mathfrak{V}_n \to \mathrm{Sub}(R^n)$ which assigns to the convex compactum $\mathfrak{x}$ its support function $\phi(\mathfrak{x})$ defined by

$$\varphi(\mathfrak{x})(y)=\sup_{x\in\mathfrak{x}}(x,y)\quad(y\in R^n).$$

We have the well known

Theorem of Minkowski-Fenchel. *The mapping ϕ is an isomorphism of the algebraic and topological structures.*

Let us identify every sublinear function with its trace on the unit sphere $Z_n = \{x \in R^n\colon |x| = 1\}$. Then the elements of $\mathrm{Sub}(R^n)$ correspond to the points of the cone H_n determined by

$$H_n=\left\{h\in C(Z_n)\colon |x|\,h\left(\frac{x}{|x|}\right)+|y|\,h\left(\frac{y}{|y|}\right)-|x+y|\,h\left(\frac{x+y}{|x+y|}\right)\leqslant 0\qquad(x,y\in R^n)\right\}.$$

If in the last formula $z = 0$ then, by definition, $|z|h(z/|z|) = 0$; $C(Z_n)$ denotes the space of all functions on Z_n continuous in the Čebyšev norm.

From now on, $\mathfrak{B}_n$ will be used to denote each of the objects $\mathfrak{B}_n$, $\mathrm{Sub}(R^n)$ and H_n.

It follows from the Stone-Weierstrass theorem that $\mathfrak{B}_n$ is total in $C(Z_n)$. Consequently the problem reduces to the characterization of the conjugate cone

$$\mathfrak{B}_n{}^* = \{\mu \in C^*(Z_n):\ \mu(h) \geqslant 0\ \ (h \in \mathfrak{B}_n)\}.$$

Here $C^*(Z_n)$ denotes the conjugate space of $C(Z_n)$. We will also assume the identification of Radon measures with the Borel measures on Z_n.

Definition 1. Let $\mu, \nu \in C^*(Z_n)$; we call μ and ν *linearly equivalent* ($\mu \sim \nu$) if $\mu(z) = \nu(z)$ for any linear function $z \in \mathfrak{B}_n$.

Definition 2. For nonnegative measures $\mu, \nu \in C^*(Z_n)$ we call μ *linearly stronger* than ν ($\mu >> \nu$) if for any finite partition $\Sigma_{k=1}^s \nu_k = \nu$, $\nu_j \geq 0$, of the measure ν there exists a partition $\Sigma_{k=1}^s \mu_k = \mu$, $\mu_k \geq 0$, such that $\mu_k \sim \nu_k$ for $k = 1, 2, \cdots, s$.

It can be shown that the relation $>> 0$ is a partial order.

The fundamental result of this paper is the following

Theorem 1. *The difference of two nonnegative measures μ and ν is contained in $\mathfrak{B}_n^*$ if and only if μ is linearly stronger than ν.*

Sufficiency can be checked directly for functions $h = \sup_{1 \leq k \leq s} l_k$ where l_k is a linear functional over R^n. In addition, the set of such functions is everywhere dense in $\mathfrak{B}_n$.

Proof of necessity is based on two lemmas.

Lemma 1. *Let μ, ν, δ be nonnegative measures, where the support of δ is finite. If $\mu + \delta >> \nu + \delta$, then $\mu >> \nu$.*

Lemma 2. *Let $x_1, x_2, \cdots, x_p$, $y_1, y_2, \cdots, y_p$ be arbitrary vectors in R^n. If for any sublinear function $h \in \mathfrak{B}_n$ we have*

$$\sum_{k=1}^{p} h(x_k) \geqslant \sum_{k=1}^{p} h(y_k),$$

then the vector $(y_1, y_2, \cdots, y_p) \in R^p$ belongs to the image S of stochastic square matrices under the mapping

$$S \ni \|\alpha_k^s\| \to \Big(\sum_{k=1}^{p} \alpha_k^1 x_k,\ \sum_{k=1}^{p} \alpha_k^2 x_k,\ \ldots,\ \sum_{k=1}^{p} \alpha_k^p x_k\Big) \in (R^n)^p.$$

The proof of the theorem is completed as follows. It is clear that we have to prove the implication $\mu - \nu \subset \mathfrak{B}_n^* \Rightarrow \mu >> \nu$.

By Lemma 1 we can assume that the measure μ satisfies the well-known theorem of Aleksandrov [2] concerning the reconstruction of convex body from its surface function. Let $\mathfrak{x} \in \mathfrak{B}_n$ be such that $\mu = \mu(\mathfrak{x})$. Here μ: $\mathfrak{B}_n \to C^*(Z_n)$ is a mapping sending a convex compactum into its surface function. Let $\{\mathfrak{x}_m\}$ be a sequence of polyhedra approximating $\mathfrak{x}$ and such that $2\mathfrak{x} \supset \mathfrak{x}_m \supset \mathfrak{x}$. It is known that in this case $\{\mu(\mathfrak{x}_n)\}$ converges to $\mu(\mathfrak{x})$ with respect to the weak topology in the space $C^*(Z_n)$. Also, due to the monotonicity of the mixed volume, $\mu(\mathfrak{X}_m)(h) \geq \mu(\mathfrak{X})(h)$ for every $h \in \mathfrak{B}_n$. Therefore $\mu(\mathfrak{X}_n) - \gamma \in V_n^*$, and by Lemma 2 it follows that $\mu(\mathfrak{X}_m) >> \nu$. The required result can now be obtained by a limit process.

As a corollary we obtain

Theorem 2. *Let* $\mathfrak{X}_1, \mathfrak{X}_2, \cdots, \mathfrak{X}_{n-1}, \mathfrak{Y}_1, \mathfrak{Y}_2, \cdots, \mathfrak{Y}_{n-1}$ *be convex surfaces in* $\mathfrak{B}O_n$. *Then the inequality*

$$V(\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_{n-1}, \mathfrak{z}) \geqslant V(\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_{n-1}, \mathfrak{z})$$

holds for any convex surface $\mathfrak{z}$ *if and only if*

$$\mu(\mathfrak{x}_1, \mathfrak{x}_2, \ldots, \mathfrak{x}_{n-1}) \geqslant \mu(\mathfrak{y}_1, \mathfrak{y}_2, \ldots, \mathfrak{y}_{n-1}).$$

where $V(\cdot, \cdots, \cdot)$ *and* $\mu(\cdot, \cdots \cdot)$ *are mixed volume and mixed surface functions, respectively.*

Let us introduce a family of measures

$$NS = \{|x|\varepsilon_{x/|x|} + |y|\varepsilon_{y/|y|} - |x+y|\varepsilon_{(x+y)/|x+y|}\}_{x,\, y \in R^n},$$

where $\epsilon_{z/|z|}$ *for* $z \neq 0$ is the measure generated by a unit mass at the point $z/|z| \in Z_n$ and $\epsilon_{z/|z|} = 0$ for $z = 0$.

The next statement shows that no other inequalities between the sublinear functions can exist except those directly derived from the sublinearity.

Proposition. *The closure in the weak topology of* $C^*(Z_n)$ *of the conical hull* $K(NS)$ *of* NS *coincides with* $\mathfrak{B}_n^*$.

For the proof it is sufficient to consider the spaces $C(Z_n)$ and $C^*(Z_n)$ with weak and weakened topologies, respectively, as duals and to apply the separation theorem.

Remark. From the relation

$$K(NS) \subset \{\mu - \nu \in C^*(Z_n) \colon \mu \gg \nu\} \subset \mathfrak{B}_n^* \tag{1}$$

which is easy to prove, it is clear that for the proof of Theorem 1 it would be sufficient to show the weakened closure of the middle set in (1). The author did not succeed in proving this directly. We remark that Cartier, Fell and Meyer in an analogous situation (see [1]) also did not follow this approach.

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