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SOME THEOREMS ON CONVERGENCE OF OPERATORS

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In this article we will investigate certain questions concerning convergence of sequences of operators and concerning the problem of defining an operator in some class. An analogous question on convergence of positive operators to the identity was studied in the survey [1], in the papers [2–8], and in our articles [9], [10], which were the starting point for these further developments. We will use arguments from the theory of Kantorovič spaces as presented in [11].

Let X be an ordered vector space. Let Y be a K-space.* Let $T \in \mathfrak{L}^+(X, Y)$ (that is, T is a positive linear operator from X into Y). A cone (= convex cone) H in X will be called a *supremal generator of X relative to the operator T* if it is minorizing (that is, for each x in X the set $\{h \in H: h \leq x\}$ is nonempty) and

$$Tx = \sup_{h \leq x,\, h \in H} Th.$$

If $X \subset Y$ and $T = E$, where E is the inclusion operator from X into Y, then the above definition coincides with the standard definition of a supremal generator for a K-space; if Y is the K-space of real numbers R, the definition coincides with the definition of a generator relative to a functional [9].

In the situation which we have described we have the following theorem (see [10]) which should be thought of as a special case of the Hahn-Banach-Kantorovič theorem.

Theorem 1. *Let H be a minorizing cone in X. The following assertions are equivalent*:

1) *H is a supremal generator of X relative to T.*

2) *For each sequence of operators (T_n) with $T_n \in \mathfrak{L}^+(X, Y)$ such that $\underline{\lim}_n T_n h \geq Th$ for all h in H we have (o)-$\lim_n T_n x = Tx$ for all x in X.*

3) $\mathrm{Spr}(T, H) = \{T' \in \mathfrak{L}^+(X, Y) : T'h \geq Th (h \in H)\} = \{T\}$.

In the case of a solid cone of positive elements in X, the assumption that H is minorizing can be dropped.

We take Y, for example, to be the K-space H_G of functions which are harmonic and bounded in a bounded region of (a numerical space) G with a compact boundary ∂G. For X we take $C(\partial G)$, where $C(Q)$ denotes the space of continuous functions on Q. Let $HC_{\overline{G}}$ be the space of functions which are harmonic in G and continuous in $\overline{G} = G \cup \partial G$. Let HC_G be the space of traces of functions in $HC_{\overline{G}}$ on G, and let $H_{\partial G}$ be

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**Editor's note.* "K-space" is the Russian term for "complete vector lattice".

the space of traces on ∂G. It is clear that HC_G and $H_{\partial G}$ are naturally isomorphic and can be identified. Suppose further that $T: C(\partial G) \to H_G$ is the operator which takes each function in $C(\partial G)$ to the corresponding solution of the generalized Dirichlet problem. Then the theorem of Keldyš [12] can be expressed as $\operatorname{Spr}(T, H_{\partial G}) = \{T\}$. From Theorem 1 we obtain for $f \in C(\partial G)$ the representation

$$Tf = \sup \{h \in HC_G \colon h(x) \leqslant f(x),\ x \in \partial G\}.$$

It is clear that, in turn, the theorem of Keldyš is a simple corollary of the above representation.

Theorem 1 can be applied to the study of operators $T : V \to Y$ (where V is a normed space and Y is a K-space) having an abstract norm, that is, operators for which the set $TS = \{|Tx| : \|x\| \leq 1\}$ is bounded (the element $\sup TS$ is called the abstract norm of T and is written $|T|$). The transition to the case of positive operators can be made with the help of an order suspension (see, for example, [13]). We recall that by an order suspension of a normed space V we mean the space $V \times R$ ordered by the solid cone $\{(x, t) \in V \times R : \|x\| \leq t\}$—above the graph of the functional $x \mapsto \|x\|$.

Theorem 2. *Let H be a cone in V and let $T: V \to Y$ be an operator which has an abstract norm. The following assertions are equivalent:*

1) *The cone $\widetilde{H} = H \times (-R_+)$, where $R_+ = \{t \in R : t \geq 0\}$, is a supremal generator of the order suspension of V relative to the operator $(T, |T|) : (x, t) \mapsto Tx + t|T|$.*

2) *For each x in V we have the representation*

$$Tx = \sup_{h \in H} (Th - |T| \|x - h\|).$$

3) *For each sequence (T_n) with $T_n : V \to Y$ such that $\overline{\lim}_n |T_n| \leq |T|$ and $\underline{\lim}_n T_n h \geq Th$ for all h in H, we have $(o)\text{-}\lim_n T_n x = x$ for all $x \in V$.*

4) *For each operator $T' : V \to Y$ such that $|T'| \leq |T|$ and $T'h \geq Th$ ($h \in H$), we have $T' = T$.*

In the case $Y = R$ this theorem is analogous to a theorem of Šmul'jan [14]; see also [15]. An analogous assertion is also valid for the case where V is normed by means of an arbitrary K-ideal.

Of special interest are finite generators, which under natural assumptions [10] exist only in K-linear manifolds of bounded elements. Hereafter we will pay particular attention to the case where V is a KN-linear manifold of bounded elements. For a cone H in V we will let $\widetilde{H}$ denote the conical hull of the element $(-1, -1)$, where 1 is the identity in V, and the cone $\{(h, -h) \in V \times V : h \in H\}$ in the space $V \times V$.

We see the effect of "duplication of the generator", that is, we have

Theorem 3. *The following assertions are equivalent:*

1) *$\widetilde{H}$ is a supremal generator of the order suspension of V relative to the operator $(E, 1)$, where E is the inclusion operator from V into Y.*

2) *H is a supremal generator of $V \times Y$ relative to the operator $\widetilde{E} : (x_1, x_2) \mapsto x_1$.*

3) *For each sequence (T_n), where $T_n : V \to Y$, $\overline{\lim}_n |T_n| \leq 1$ and $\underline{\lim}_n T_n h \geq h$ for $h \in H$, we have $(o)\text{-}\lim_n T_n x = x$ for all $x \in V$.*

4) *For each operator* $T: V \to Y$ *such that* $|T| \leq 1$ *and* $Th \geq h$, $h \in H$, *we have* $T = E$.

In applying this theorem we should keep in mind the fact that a subspace H in V satisfies the property that $\tilde{H}$ is a supremal generator relative to the operator $\tilde{E}$ if and only if $\tilde{H}$ is a supremal generator of $V \times V$ relative to the K-space $Y \times Y$.

We introduce a typical application of Theorem 3. We consider a compact metric space Q with a positive Baire measure μ. We let $S_\mu(Q)$ denote the corresponding space of measurable functions. We will assume that Q is realized in the conjugate space $C'(Q)$, that is, we identify a point x in Q with the Dirac measure $\epsilon_x: f \mapsto f(x)$. We set $\hat{Q} = Q \cup (-Q)$ and we define on $\hat{Q}$ the measure $\hat{\mu}$ by setting

$$\hat{\mu}(e) = \mu(e \cap Q) + \mu(-(e \cap (-Q)))$$

for each Baire set e in $\hat{Q}$. If H is a subspace of $C(Q)$, we let $\hat{H}$ denote the cone in $C(\hat{Q})$ spanned by the function -1 and the set of all functions $\hat{h}$, where for h in H and x in Q we set

$$\hat{h}(\varepsilon_x) = h(x),\ \hat{h}(-\varepsilon_x) = -h(x)$$

(we assume that a topology is induced in $\hat{Q}$).

Theorem 3 leads to the following result.

Theorem 4. *The following assertions are equivalent*:

1) *The measure* $\hat{\mu}$ *is maximal in the Choquet ordering generated by the cone* $\hat{H}$.

2) *The cone* H *is a supremal generator of the space* $C(Q) \times C(Q)$ *relative to the* K-*space*.

3) *If* (T_n) *is a sequence of operators from* $C(Q)$ *into* $S_\mu(Q)$ *such that* $|T_n| 1 \leq 1$ *and for each* h *in* H *the sequence* $(T_n h)$ *converges to* h *almost everywhere* (*respectively, in measure*), *then for each function* f *in* $C(Q)$ *the sequence* $(T_n f)$ *converges to* f *almost everywhere* (*respectively, in measure*).

It would be interesting to replace the abstract norm by the "usual" norm in the above theorems, for then we could look at many specific cases. Unfortunately, simple examples show that in Theorems 2–4 it is impossible to replace the abstract norm by the usual norm (even when Y is a KB-space with an additive norm). We note, however, that for $Y = B(Q)$, where $B(Q)$ is the space of bounded functions on Q, we have $|T| \leq 1 \Leftrightarrow \|T\| \leq 1$. Here if Q is compact, in Theorem 3 we can consider uniform convergence. In particular, for $V = C(Q)$ the result implies the corresponding result for nonexpanding operators in [8].

It is also possible to replace the abstract norm by the usual norm in some sense for compact operators with values in the space of continuous functions on a compact space Q. Here we use a theorem of Michael [16].

Theorem 5. *Let* V *be a normed space. Let* T *be a compact operator* $T: V \to C(Q)$. *Let* H *be a cone in* V. *Let* ϵ *be a positive number. Then we have the implications* (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) *and* (4) $\Rightarrow$ (1), *where*:

1) *The cone* $\tilde{H}$ *is a supremal generator of the order suspension of* V *relative to the functional* $(T_x, \|T\|)$, $T_x: v \mapsto (Tv)(x)$ *for each* x *in* Q.

2) *For each operator* $T': V \to B(Q)$ *such that* $\|T'\| \leq \|T\|$ *and* $T'h \geq Th$, $h \in H$, *we have* $T' = T$.

3) *For an arbitrary sequence* (T_n) *of operators* $T_n: V \to C(Q)$ *such that* $\lim_n \|T_n\| \leq \|T\|$ *and the uniform* $\lim_n T_n h \geq Th$ *for all* h *in* H *it follows that* (T_n) *converges to* T *in the strong operator topology.*

4) *For each compact operator* $T': V \to C(Q)$ *such that* $\|T'\| \leq (1+\epsilon)\|T\|$ *and* $T'h \geq Th$, $h \in H$, *we have* $T' = T$.

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