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KERNELS OF MAXIMAL OPERATORS AND SIMPLICIAL CONES

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This paper is devoted to the analysis of two central problems in Choquet theory—the character of kernels of the ascents of maximal operators, and the problem of uniqueness of representing measures. The Choquet theory is usually constructed for Radon measures on compact spaces or in fundamentally similar situations [1]–[3]. Using the theory of Kantorovič spaces* [4], [5], we obtain general results on the structure of maximal operators on upper lattices. We note that all the results contain new information, even for the case of spaces of continuous functions. In particular, we establish conditions under which the traces of maximal operators on the complement of the Choquet boundary are anormal, which means that the Choquet theorems are valid in spaces of measurable functions. We also establish "independence" of the definition of a simplex from the range of the operator in question.

1. Let X be a K-lineal, Y a K-space, $L(X, Y)$ the set of regular operators from X into Y, and $L^+(X, Y)$ the positive cone in $L^+(X, Y)$.

In this paper we are always considering regularly ordered spaces. Recall that X is *regularly ordered* if X and $L(X, R)$, where R is the K-space of real numbers, are in duality under the form $(x, f) \to f(x)$, $x \in X$, $f \in L(X, R)$, and the cone X_+ of positive elements in X is closed in some (and hence, any) topology that is compatible with this duality.

A cone H is distinguished in X. Concerning H, it is always assumed that $\overline{H + X_+} = X$. The cone H determines an ordering $\succ_H$ in $L^+(X, Y)$; namely, $T_1 \succ_H T_2$ means that $T_1 h \geq T_2 h$, $h \in H$. We note that the existence of maximal operators (i.e. maximal elements in $(L^+(X, Y), \succ_H)$) is equivalent to the condition $\overline{H + H_+} = X$. If $P(H)$ is the largest upper lattice spanned by H (i.e., the cone of "H-convex polynomials"), then the ordering $\succ_H$ is called the *Choquet ordering induced by H.*

An operator $T \in L(X, Y)$ is said to be Choquet-bounding if $|T|$ is maximal in the Choquet ordering.

Theorem 1. *The set of all Choquet-bounding operators forms a K-space (in the structure induced by the K-space of regular operators).*

2. Now let X be a subspace of a K-space Z, and $\mathrm{Ch} = \mathrm{Ch}(H, X, Z)$ the Choquet boundary of the triple (H, X, Z) (see, in particular, [6]), i.e. the component on which

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**Editor's note.* In the Russian literature complete vector lattices are called Kantorovič spaces or K-spaces; a vector lattice is called a K-lineal.

the projection of the largest (in the Boolean algebra of projections) H-maximal projection P_{Ch} is realized. As usual, Ch^d is the disjunct complement of Ch. Let $\mathrm{Ker}(T)$ be the kernel of the operator T, and $N(T)$ the zero lineal of T, i.e., $N(T) = \{z \in Z: |z| \in \mathrm{Ker}(T)\}$. Let Ker be the common part of the kernels of the H-supermaximal operators defined on Z (an operator $T \in L^+(Z, Y)$ is called supermaximal if its restriction to X is maximal relative to H), and let N be the common part of the zero lineals of these operators.

Theorem 2. *The disjunct complement of* Ker *coincides with the Choquet boundary.*

Theorem 3. *If H is a coinitial of X or if $\overline{H - H} = X$, then the trace of a maximal operator on the component* Ch^d *is anormal.*

We obtain the following result, for example, from these theorems.

Theorem 4. *The following assertions are equivalent:*

a) Ker *is a component;*

b) $\mathrm{Ker} = N = \mathrm{Ch}^d$;

c) *for every H-supermaximal operator T we have* $TP_{\mathrm{Ch}^d} = 0$.

Thus, theorems from Choquet theory on the structure of maximal operators are valid in spaces of measurable functions. More precisely, we have the following result.

Theorem 5. *Suppose that the positive forms on Z are completely linear.*

a) *If H is a component of X, then an operator T is supermaximal in the Choquet ordering if and only if* $TP_{\mathrm{Ch}^d} = 0$.

b) *If $\overline{H - H} = X$, then any H-supermaximal operator reduces to zero on* Ch^d.

3. Let H be a cone in X. Then any image of the cone $L^+(X, Y)$ in the set of maximal operators which is maximal in the ordering $\succ_H$ is called a *sweep* Ψ_H generated by H. It is known that if H is a coincidental of X and $\overline{H - H} = X$, then a sweep exists. Each sweep generates an inversion formula

$$Th \leqslant \Psi_H(T)h, \quad h \in H, \quad T \in L^+(X, Y).$$

One of the central questions in Choquet theory is the uniqueness of a sweep.

A cone H in X is called *simplicial* if it generates a unique sweep in $L^+(X, Y)$ for every K-space Y.

The lemma on directedness of sprouts. *Let H be a coinitial, reproducing cone in X, and Y a K-space. The sprout $\mathrm{Spr}(T, H) = \{T' \in L^+(X, Y): T' \succ_H T\}$ is directed to the right in the ordering $\succ_H$ if and only if the operator $q_{H,T}: x \to \sup T(U_x^H)$ is additive on $-H$ (here $U_x^H = \{h \in H: h \leq x\}$ is the supporting H-convex set).*

The following theorem, whose proof uses the directedness lemma, gives an intrinsic characteristic of a simplicial cone.

Theorem 6. *The following assertions are equivalent:*

a) *the cone H is simplicial*;

b) *the sprout of each positive form is directed to the right*;

c) *for each* $h_1, h_2 \in -H$, *we have*

$$\overline{U^H_{h_1}+U^H_{h_2}-X_+}\supset U^H_{h_1+h_2}.$$

We note also the following assertions.

Proposition 1. *If H is a simplicial upper lattice, then the sweep is an additive operator.*

The next result is cited as an example.

Proposition 2. *If H is a subspace having the Riesz interpolation property* [2], *then the cone* $P(H)$ *is simplicial in* $\overline{P(H)-P(H)}$.

We note that if H is a closed subspace of the space $C(Q)$ of continuous functions on a compact Q, and H contains the constants and separates points, then the property of being simplicial for $P(H)$ is equivalent to the Riesz interpolation property in H. In this case the pair $(H, C(Q))$ is called a *Choquet simplex*.

Subspaces with the interpolation property are clearly connected with solvability of the Dirichlet problem on the Choquet boundary. We cite here only the simplest assertion along this line.

Definition. Let $H \subset X \subset Z$. An element $h_1 \in Z$ is called 1-*affine* if h_1 is the limit (in Z) of an increasing net of elements from H. An element $h_2 \in Z$ is called 2-*affine* if h_2 is the limit of a decreasing net of 1-affine elements.

Let $\overset{2}{H}$ denote the set of 2-affine elements.

Proposition 3. *Suppose that H has the Riesz interpolation property and majorizes X, and P is a projection such that* $P \leq P_{\mathrm{Ch}}$. *Then* $P(X) \subset P(\overset{2}{H})$.

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