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SUPPORT SETS FOR SUBLINEAR OPERATORS

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1. Let X be a vector space, Y a K-space (conditionally complete vector lattice) and A a weakly order bounded set in the space of linear operators $L(X, Y)$. In other words, the sublinear operator

$$P_{\mathfrak{A}}\colon x \mapsto \sup\{Ax\colon A \in \mathfrak{A}\}$$

is given, defined on the whole space X. Consider the set

$$\operatorname{cop}(\mathfrak{A}) = \{A \in L(X, Y)\colon Ax \leq P_{\mathfrak{A}}x\ (x \in X)\}.$$

This set is called the *support hull* of the set $\mathfrak{A}$. It is clear that $\mathfrak{A} = \operatorname{cop}(\mathfrak{A})$ if and only if $\mathfrak{A}$ is the support set of the everywhere defined operator $P_{\mathfrak{A}}$, i.e. its subdifferential (at zero) $\partial(P_{\mathfrak{A}})$.

One of the basic questions of the theory of sublinear operators is the problem of explicit description of the natural Minkowski duality, i.e. the problem of intrinsic description of the support hulls. This question has been studied in a series of papers [1]–[5]. There, as a rule, attempts were made to represent the closure operator in the sense of Moore cop: $\mathfrak{A} \to \operatorname{cop}(\mathfrak{A})$ in the form of a superposition of an analogous algebraic operator and topological closure (usually for the weak operator topology). However, with exception of the case $Y = R$, there are no factorizations of that kind in good operator topologies.

In this paper the reasons for this phenomenon are explained and a general approach to the construction of the desired decompositions is formulated. As examples we cite natural factorizations of the support hull operator for the foundations** of spaces of continuous functions on Stone compacts. The basic idea is that for the study of arbitrary sublinear operators one has to learn to deal with "one" canonical operator, through which one may omit any sublinear operator so that the corresponding remainder will be linear. This canonical operator is defined by a system of projections onto a simple proper subspace, and using them one constructs the desired factorizations.

2. Our first goal is to find an explicit integral description of the support hulls. To do that the following constructions will be needed.

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* *Editor's note.* The present translation incorporates corrections made by the author.

** *Translator's note.* In the Western literature a foundation is sometimes called a quasi-order dense ideal (see, for example, B. Z. Vulih, *Introduction to the theory of partially ordered spaces* (English translation), Wolters-Noordhoff, Groningen, 1967, p. 96; W. A. J. Luxemburg and A. C. Zaanen, *Riesz spaces*, vol. I, North-Holland, Amsterdam-London; American Elsevier, New York, 1971, p. 15).

Let $\Delta_{\mathfrak{A}}: Y \to Y^{\mathfrak{A}}$ be the embedding of Y into the diagonal of the space $Y^{\mathfrak{A}}$, i.e. $\Delta_{\mathfrak{A}}(x) = (x)_{A \in \mathfrak{A}}$. We denote by $(Y^{\mathfrak{A}})_\infty$ the foundation

$$(\Delta_{\mathfrak{A}}\langle Y\rangle + Y_+^{\mathfrak{A}}) \cap (\Delta_{\mathfrak{A}}\langle Y\rangle - Y_+^{\mathfrak{A}})$$

of the K-space $Y^{\mathfrak{A}}$.

On the space $(Y^{\mathfrak{A}})_\infty$ the *canonical sublinear operator* $\epsilon_{\mathfrak{A}}: (Y^{\mathfrak{A}})_\infty \to Y$ acts by the rule

$$\varepsilon_{\mathfrak{A}}: (y_A)_{A\in\mathfrak{A}} \mapsto \sup\{y_A: A \in \mathfrak{A}\}.$$

The set $\partial(\epsilon_{\mathfrak{A}})$ is denoted by $\Lambda_{\mathfrak{A}}$. It is obvious that the operator α from $L((Y^{\mathfrak{A}})_\infty, Y)$ belongs to $\Lambda_{\mathfrak{A}}$ if and only if α is positive (i.e. $\alpha \in \mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Y)$) and, in addition, $\alpha\Delta_{\mathfrak{A}} = I_Y$, where I_Y is the identity transformation of Y onto itself. In the case when $\mathfrak{A}$ is a weakly order bounded subset in $L(X, Y)$, we will also need the natural linear operator $[\mathfrak{A}]: X \to (Y^{\mathfrak{A}})_\infty$, defined by the relation

$$[\mathfrak{A}]x = (Ax)_{A\in\mathfrak{A}}.$$

In the sequel, the index $\mathfrak{A}$ in the notation will occasionally be omitted.

Proposition 1. *Let Z be a K-space and $P: Y \to Z$ an increasing sublinear operator. Then*

$$\partial(P\circ\varepsilon_{\mathfrak{A}}) = \{A \in \mathscr{L}_+((Y^{\mathfrak{A}})_\infty, Y): A\Delta_{\mathfrak{A}} \in \partial(P)\}.$$

Theorem 1. *For any weakly order bounded set A in the space* $L(X, Y)$

$$\operatorname{cop}(\mathfrak{A}) = \{\alpha[\mathfrak{A}]: \alpha \in \Lambda_{\mathfrak{A}}\}.$$

Remark. One should consider Theorem 1 as an analog of theorems of Choquet type on integral representations. Indeed, if $Y = R$, then $(Y^{\mathfrak{A}})_\infty = l_\infty(\mathfrak{A})$. Thus if f_0 is a linear functional on X and $\mathfrak{A}$ is a weakly bounded set of such functionals, then the conclusion of Theorem 1 means that $f_0(x) \leq \sup\{f(x): f \in \mathfrak{A}\}$ for all $x \in X$ if and only if

$$f_0(x) = \int_{\mathfrak{A}} f(x)\,d\alpha \quad (x \in X)$$

for some finitely additive probability measure on the algebra of subsets of $\mathfrak{A}$.

As an example of application of Theorem 1 let us compute some subdifferentials.

Proposition 2. *Let $P_1: X \to Y$ be a sublinear operator and $P_2: Y \to Z$ an increasingly sublinear operator. Then*

$$\partial(P_2\circ P_1) = \{A[\partial(P_1)]: A\Delta \in \partial(P_2),\quad A \in \mathscr{L}_+((Y^{\partial(P_1)})_\infty, Y)\}.$$

Moreover, if $\partial(P_1) = \operatorname{cop}(\mathfrak{A}_1)$ and $\partial(P_2) = \operatorname{cop}(\mathfrak{A}_2)$, then

$$\partial(P_1 \circ P_2) = \{A[\mathfrak{A}_1]: \exists \alpha_2 \in \Lambda_{\mathfrak{A}_2}: A\Delta = \alpha_2[\mathfrak{A}_2]\}.$$

It is also possible to compute other subdifferentials in a similar way. As a consequence of independent interest let us note

The operator T is a lattice homomorphism of X into Y if and only if for every operator T' such that $0 \leq T' \leq T$ there in an operator α: $Y \to Y$ such that $0 \leq \alpha \leq I_Y$ and, in addition, $T' = \alpha T$.

3. Let us apply the results obtained above to the question of factorization of the support hull operator. Theorem 1 shows that to split off the topological closure in the well-known sense it is necessary and sufficient to select an everywhere dense subset in the subdifferential of the canonical sublinear operator. In this section we will assume that the K-space Y is a foundation in the product of lines $R^{\mathfrak{B}}$ or a foundation in the K-space of continuous functions on the Stone compact $\mathfrak{B}$.

Definition. A weakly order bounded set $\mathfrak{A}$ in the space $L(X, Y)$ is called *strongly operator convex* if for every (o)-summable family $(\alpha_\xi)_{\xi \in \Xi}$ of operators α_ξ: $Y \to Y$ such that $0 \leq \alpha_\xi \leq I_Y$ and $\Sigma_{\xi \in \Xi} \alpha_\xi = I_Y$, and an arbitrary family $(A_\xi)_{\xi \in \Xi}$ of operators in $\mathfrak{A}$, $\Sigma_{\xi \in \Xi} \alpha_\xi A_\xi \in \mathfrak{A}$ holds.

It is easy to see that $\operatorname{cop}(\mathfrak{A})$ is a strongly operator convex set. Thus, for each $\mathfrak{A}$ the set $\operatorname{stop}(\mathfrak{A})$, the *strong operator convex hull* of $\mathfrak{A}$ is defined; it is the smallest strongly operator convex set containing $\mathfrak{A}$. One may show that

$$\operatorname{stop}(\mathfrak{A}) = \left\{ \sum_{\xi \in \Xi} \alpha_\xi A_\xi : 0 \leqslant \alpha_\xi \leqslant I_Y, \ \sum_{\xi \in \Xi} \alpha_\xi = I_Y; \ A_\xi \in \mathfrak{A} \right\}.$$

In the literature, as a rule, one considers operator convex sets (that is, those for which the families considered in the above definition are finite) and the corresponding Moore closure operator $\mathfrak{A} \mapsto \operatorname{op}(\mathfrak{A})$, the *operator convex hull.* It is clear that for each $\mathfrak{A}$

$$\operatorname{op}(\mathfrak{A}) \subset \operatorname{stop}(\mathfrak{A}) \subset \operatorname{cop}(\mathfrak{A}),$$

where, in general, the inclusions are strict. Moreover, the closure of $\operatorname{op}(\mathfrak{A})$, in the weak operator topology for example, does not necessarily coincide with $\operatorname{cop}(\mathfrak{A})$ or even contain $\operatorname{stop}(\mathfrak{A})$.

The following example is a key one.

Example. Let Y be a foundation in the product of lines $R^{\mathfrak{B}}$. Note that

$$(Y^{\mathfrak{A}})_\infty = \{ y \in R^{\mathfrak{A} \times \mathfrak{B}} : \sup_{A \in \mathfrak{A}} |y(A, \cdot)| \in Y \}.$$

Consider the set $\mathfrak{P} = \{p_A : A \in \mathfrak{A}\}$, where the p_A are the coordinate projections, i.e. $p_A : (x_A)_{A \in \mathfrak{A}} \to x_A$. Obviously $\epsilon_{\mathfrak{A}} = P_{\mathfrak{P}}$. The operator α: $(Y^{\mathfrak{A}})_\infty \to Y$ belongs to $\Lambda_{\mathfrak{A}}$ if and only if

$$\alpha y(B) = \int_{\mathfrak{A}} y(\cdot, B) \, d\mu_B,$$

where μ_B is a finitely additive probability measure on the algebra of subsets of $\mathfrak{A}$. One may verify that $\alpha_0 \in \operatorname{stop}(\mathfrak{P})$ if and only if there exist numbers α_A^B such that

$$0 \leqslant \alpha_A{}^B \leqslant 1; \quad \sum_{A \in \mathfrak{A}} \alpha_A{}^B = 1;$$

$$\alpha_0 y(B) = \sum \alpha_A{}^B y(A, B).$$

At the same time, to the elements in op($\mathfrak{P}$) there correspond the families (α_A^B) for which $\alpha_{A_k}^B \neq 0$ is possible only for a fixed finite subset $\{A_1, \ldots, A_n\} \subset \mathfrak{A}$.

Under our hypotheses on the space Y, for each vector space X the space $L(X, Y)$ may be considered as a subspace of $L(X, R^{\mathfrak{B}})$. Thus we have in $L(X, Y)$ the *simple operator topology*, i.e. by definition the topology in $L(X, Y)$ induced by the weak operator topology of the space $L(X, R^{\mathfrak{B}})$. Analogously, in the case of foundations of spaces of continuous functions, one may also speak of the *strong operator topology*.

Examples of corresponding factorizations of the operator cop are given in

Theorem 2. (1) *A set of operators is a support set if and only if it is weakly order bounded, operator convex, and closed in the simple operator topology.*

(2) *A set of operators is a support set if and only if it is weakly order bounded, strongly operator convex, and closed in the strong operator topology.*

In the continuous case, the above applies to compactly generated (in the strong operator topology) sets $\mathfrak{A}$, *i.e. such that* $P_{\mathfrak{A}} = P_{\mathfrak{C}}$ *for some compact* $\mathfrak{C}$.

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Translated by S. IVANOV

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