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LAGRANGE MULTIPLIERS IN VECTOR OPTIMIZATION PROBLEMS

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Let X and X_1 be vector spaces, and let Y and Y_1 be ordered vector spaces, where Y is a K-space (conditionally complete vector lattice). There are given a linear operator G_1: $X \to X_1$, an element $x_1 \in X_1$, and convex operators G_2: $X \to Y_1$ and F: $X \to Y$ defined on a convex set $U_0 \subset X$. Moreover, let U be a convex subset of U_0.

The following problem is called a *vector optimization problem*, or, more precisely, a *convex vector program*: find an element $x^* \in U$ such that

$$G_1x^*=x_1, \quad G_2x^*\leqslant 0;$$
$$Fx^*=\inf\{Fx\colon x\in U,\ G_1x=x_1,\ G_2x\leqslant 0\}.$$

This problem is expressed conventionally in the form

$$x\in U,\ G_1x=x_1,\ G_2x\leqslant 0,\ Fx\to\inf.$$

As usual, an element x that satisfies the system of constraints is called an *admissible element*, and the element x^* is called an *optimal element* or a *solution of the vector optimization problem*.

Most important examples of vector optimization problems are problems of optimal operation of multipurpose large systems and their continuous analogues. Moreover, it can be shown that the search for "limiting factors"—obstacles to solving a vector optimization problem—is, in turn, a convex vector program.

The purpose of this note is to obtain general optimality criteria for vector optimization problems without narrowing a priori the class of spaces under consideration. There are essential difficulties on the way to this goal which are related, first of all, to the fact that geometrical separation methods loose their meaning in vector programming. Most essential among these difficulties are the following: the absence of a vector analogue of the Moreau-Rockafellar theorem, the absence of a vector analogue of the Dubovickiĭ-Miljutin decomposition theorem, and the lack of a description of the cone normal to a level set of a convex operator; in other words, the lack of simplest sufficient conditions of Slater type which would guarantee the existence of penalty functions in problems with a sufficiently qualified structure of constraints [1] and [2].

On the basis of methods of the theory of ordered vector spaces and, in particular, of the theory of subdifferentials of convex operators acting into arbitrary K-spaces [3]–[7], we remove the first two of the difficulties mentioned. In so doing there obviously arise general optimality criteria for nonqualified description of the admissible set. As far as the question of the existence of penalty functions is concerned, we must be satisfied with less, namely, we state only the equivalence of the existence of a penalty and the existence of a most favorable (from the computational standpoint) vector variant of the Lagrange multiplier method.

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For simplicity, we restrict ourselves in the sequel to cases wherein X_1 and Y_1 are subspaces of Y, which, in fact, does not decrease the generality.

1. We shall need the following auxiliary information and results.

Let X be a vector space, let Y be a K-space, and let $F: X \to Y \cup \{\infty\}$ be a convex operator. We set

$$\mathrm{dom}(F)=\{x\in X:\ Fx<+\infty\}.$$

This set is called the *effective domain* of the operator F. Let $L(X, Y)$ be the set of linear operators from X into Y, and let

$$\partial_{x^*}(F)=\{A\in L(X, Y): Ax-Ax^*\leqslant Fx-Fx^*,\ x\in X\}$$

be the *subdifferential* of F at the point x^*.

Proposition 1. *Let* $F_1, \ldots, F_n: X \to Y \cup \{\infty\}$ *be convex operators, where* $\mathrm{dom}(F_1) = \cdots = \mathrm{dom}(F_n)$ *and* $\partial_{x^*}(F_1 + \cdots + F_n) \neq \varnothing$ *for a point* $x^* \in \mathrm{dom}(F_1)$.

Then

$$\partial_{x^*}(F_1+\ldots+F_n)=\partial_{x^*}(F_1)+\ldots+\partial_{x^*}(F_n).$$

If the set $\mathrm{dom}(F_1)$ *absorbs with respect to the point* x^*, *then*

$$\partial_{x^*}(F_1\vee\ldots\vee F_n)=\bigcup_{(\alpha_1,\ldots,\alpha_n)\in\Gamma(x^*)}(\alpha_1\circ\partial_{x^*}(F_1)+\ldots+\alpha_n\circ\partial_{x^*}(F_n)),$$

where the union is taken over the set

$$\Gamma(x^*)=\Big\{(\alpha_1,\ldots,\alpha_n):\alpha_h\geqslant 0;\ \sum_{h=1}^{n}\alpha_h=I_Y;\ I_Y y=y,\ y\in Y;$$

$$\sum_{h=1}^{n}\alpha_h\circ F_h x^*=F_1x^*\vee\ldots\vee F_n x^*\Big\}.$$

The proofs of these assertions are essentially different from the proofs of their classical scalar analogues, the Moreau-Rockafellar and Dubovickiĭ-Miljutin theorems.

Proposition 2. *Let* $F: X \to Y \cup \{\infty\}$ *be a convex operator with* $x^* \in U \subset \mathrm{dom}(F)$, *where the set* $\mathrm{dom}(F)$ *is absorbing with respect to* x^*. *Let the mappings* F_U *and* $\delta_Y(U)$ *be defined by the relations*

$$\delta_Y(U)x=\begin{cases}0, & x\in U,\\ +\infty, & x\notin U;\end{cases}\qquad F_U x=\begin{cases}Fx, & x\in U,\\ +\infty, & x\notin U.\end{cases}$$

Then

$$\partial_{x^*}(F_U)=\partial_{x^*}(F)+\partial_{x^*}(\delta_Y(U));$$
$$\partial_{x^*}(\delta_Y(U))=\{A\in L(X, Y): Ah\leqslant 0,\ h\in \mathrm{Fd}_{x^*}(U)\},$$

where $\mathrm{Fd}_{x^*}(U)$ *is the cone of feasible directions to* U *at* x^*.

The assertions presented are sufficient for an analysis of the simplest programs. As an example, we consider the following situation.

Let X_0 be a subspace of X with $x_0 \in X$, and let F be a convex operator with $\mathrm{dom}(F) = X$. Consider the problem $x \in X_0 + x_0$, $Fx \to \inf$. Clearly, an optimality criterion for this problem consists in the requirement that there exist an operator $\mathfrak{A} \in L(X/X_0, Y)$ such that $\mathfrak{A} \circ \phi \in \partial_{x^*}(F)$, where ϕ is the canonical homomorphism of X onto X/X_0. Hence it can be seen that optimality criteria in vector optimization problems from the Lagrange theory standpoint differ qualitatively from the scalar case. Therefore, the question of separating the problems that admit qualified Lagrange multipliers becomes of particular importance. We turn to an analysis of the simplest versions of this question.

2. Consider a vector optimization problem with equality constraints. First, we remind the reader that a function $G: U_0 \to Y$ is said to be a *penalty* for the problem $x \in U$, $Fx \to \inf$, if a solution of this problem is a solution of the problem $x \in U_0$, $(F + G)x \to \inf$.

Let $A_k \in L(X, Y)$, $y_k \in Y$, $k = 1, \ldots, n$, and let $x^* \in U \subset \operatorname{dom}(F)$ be given parameters.

Problem 1. $x \in U$; $A_k x = y_k$, $k = 1, \ldots, n$, $Fx \to \inf$.

Proposition 3. *The following assertions are equivalent:*

(i) *The function G_U is a penalty for the problem*

$$x \in U \cap \left(\bigcap_{k=1}^{n} \{x\colon A_k x = y_k\}\right); \quad Fx \to \inf,$$

where $Gx = |A_1 x - y_1| \vee \cdots \vee |A_n x - y_n|$.

(ii) *An admissible point x^* is a solution of Problem 1 if and only if there exist operators $\gamma_k^* \in L(X, Y)$ such that the system of conditions*

$$0 \in \partial_{x^*}(F_U) + \sum_{k=1}^{n} \gamma_k^* \circ A_k + \partial_{x^*}(\delta_Y(U));$$

$$\gamma_k^* = \alpha_k^* - \beta_k^*; \quad \alpha_k^*, \beta_k^* \geqslant 0; \quad \sum_{k=1}^{n} (\alpha_k^* + \beta_k^*) = I_Y$$

is compatible.

(iii) *An admissible point x^* is a solution of Problem 1 if and only if there exist*

$$\gamma_k^* = \alpha_k^* - \beta_k^*; \quad \alpha_k^*, \beta_k^* \geqslant 0, \quad \sum_{k=1}^{n} (\alpha_k^* + \beta_k^*) = I_Y$$

and the Lagrangian

$$L(x; \gamma_1, \ldots, \gamma_n) = Fx + \sum_{k=1}^{n} \gamma_k \circ (y_k - A_k x)$$

has a saddle point on the set $U \times \Gamma$ at $(x^*; \gamma_1^*, \ldots, \gamma_n^*)$, where

$$\Gamma = \left\{ (\gamma_1, \ldots, \gamma_n)\colon \gamma_k = \alpha_k - \beta_k;\ \alpha_k, \beta_k \geqslant 0,\ \sum_{k=1}^{n} (\alpha_k + \beta_k) = I_Y \right\}.$$

3. Consider a vector optimization problem with inequality constraints.

Let $F\colon X \to Y \cup \{\infty\}$ be a convex operator, let $x^* \in U \subset \operatorname{dom}(F)$, and let the convex operators $G_1, \ldots, G_n\colon X \to Y \cup \{\infty\}$ be such that $U' = \operatorname{dom}(G_1) \cap \cdots \cap \operatorname{dom}(G_n) \supset U$, with U' an absorbing set relative to the point x^*.

Problem 2. $x \in U$, $G_k x \leq 0$, $k = 1, \ldots, n$, $Fx \to \inf$.

Proposition 4. *The following assertions are equivalent:*

(i) *The function* G_U *is a penalty in the problem*

$$x \in U \cap \left(\bigcap_{k=1}^{n} \{x: G_k x \leq 0\} \right), \quad Fx \to \inf,$$

where $Gx = G_1 x \vee \cdots \vee G_n x \vee 0$.

(ii) *An admissible point* x^* *is a solution of Problem 2 if and only if there exist operators* $\alpha_k^* \in L(Y, Y)$ *such that* $\alpha_k^* \geq 0$, $\Sigma_{k=1}^n \alpha_k^* \leq I_Y$, *where*

$$0 \in \partial_{x^*}(Fu) + \sum_{k=1} \alpha_k^* \circ \partial_{x^*}(G_k) + \partial_{x^*}(\delta_Y(U));$$

$$\sum_{k=1}^{n} \alpha_k^* \circ G_k x^* = 0.$$

(iii) *An admissible point* x^* *is a solution of Problem 2 if and only if there exist operators* $\alpha_k^* \in L(Y, Y)$ *such that* $\alpha_k^* \geq 0$, $\Sigma_{k=1}^n \alpha_k^* \leq I_Y$, *where the Lagrangian*

$$L(x; \alpha_1, \ldots, \alpha_n) = Fx + \sum_{k=1}^{n} \alpha_k \circ G_k x$$

has a saddle point on the set $U \times \Lambda$ *at* $(x^*; \alpha_1^*, \ldots, \alpha_n^*)$, *where*

$$\Lambda = \left\{ (\alpha_1, \ldots, \alpha_n) : \alpha_k \geq 0, \ \sum_{k=1}^{n} \alpha_k \leq I_Y \right\}.$$

4. For vector optimization problems there arise specific problems, some of which are trivial in the scalar case. In particular, an analysis of meaningful statements of multipurpose planning shows the essential importance of so-called generalized solutions. Namely, a set $U^* \subset U$ is said to be a *generalized solution* of the convex vector program $x \in U$, $Fx \to \inf$ if the relation

$$\inf \{Fx: x \in U\} = \inf \{Fx^*: x^* \in U^*\}$$

holds. Making use of the factorization method for sublinear operators, one can obtain criteria for generalized solutions in subdifferential form. Here we present only the simplest criterion, since the general case requires quite cumbersome constructions, which, by the way, do not contain difficulties of fundamental nature.

Proposition 5. *Let the set* U *be contained in* $\operatorname{dom}(F)$, *and suppose* $\operatorname{dom}(F)$ *absorbs with respect to every point* $x_1^*, \ldots, x_n^*$. *The set* $U^* = \{x_1^*, \ldots, x_n^*\}$ *is a generalized solution of the problem* $x \in U$, $Fx \to \inf$ *if and only if the system of conditions*

$$\alpha_1^* \geq 0, \ldots, \alpha_n^* \geq 0; \quad \sum_{k=1}^{n} \alpha_k^* = I_Y;$$

$$0 \in \alpha_k^* \circ \partial_{x_k^*}(F) + \partial_{x_k^*}(\delta_Y(U));$$

$$\sum_{k=1}^{n} \alpha_k^* \circ Fx_k^* = Fx_1^* \wedge \ldots \wedge Fx_n^*$$

is compatible.

In particular, the last result means that the problem of finding a generalized solution, "limiting factors", reduces to a certain convex vector program.

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