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FORMULAS FOR COMPUTING SUBDIFFERENTIALS

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Let X be a vector space, and let $Y \cup \{+\infty\}$ be an ordered vector space Y with adjoined largest element ∞. We denote by $L(X, Y)$ the space of linear operators acting from X into Y. Consider a convex operator $F: X \to Y \cup \{+\infty\}$. Let the inequality $Fx^* < \infty$ hold for a point $x^* \in X$, i.e. x^* belongs to the *effective domain* $\operatorname{dom}(F)$ of F. The set $\partial_{x^*}(F)$ defined by the relation

$$\partial_{x^*}(F) = \{A \in L(X, Y): Ax - Ax^* \leq Fx - Fx^*,\ x \in X\}$$

is called the *subdifferential* of the operator F at the point x^*.

Subdifferentials of convex operators, which are a generalization of the notion of differential for nonsmooth convex mappings, play a key role in the modern theory of extremal problems. Nevertheless, there is an important gap in the list of rules for finding subdifferentials. Namely, there are practically no formulas for computing subdifferentials of complex convex operators (see [1] and the literature cited therin). The reason for this is that geometric methods of separation cannot be applied to vector-valued functions. In this paper, we present general formulas for computing subdifferentials of operators, which contain all the basic scalar variants. The method of deriving these formulas, based on the theory of ordered vector spaces [2], [3], is new in the scalar case also.

I. Formula for the subdifferential of a composition. Let $G: Y \to Z \cup \{+\infty\}$ be an increasing convex operator, where Z is a K-space (conditionally complete vector lattice) and $\operatorname{dom}(G) = Y$. Then the following representation holds for the composition $G \circ F$:

$$\partial_{x^*}(G \circ F) = \bigcup_{A \in \partial_{Fx^*}(G)} \partial_{x^*}(A \circ F).$$

II. Formula for the subdifferential of a sum. Let H_1 and H_2 be cones in a vector space X. The cones H_1 and H_2 are said to be in *general position* if $H_1 - H_2 = H_2 - H_1$.

A *system of cones* $H_1, \dots, H_n$ is said to be *in general position* if, for some permutation $\{i_1, \dots, i_n\}$ of the indices, the cones

$$H_{i_k}, \quad \bigcap_{s=k+1}^{n} H_{i_s}$$

are in general position for $k = 1, \dots, n-1$.

If $U_1, \dots, U_n$ are convex sets and $x^* \in U_1 \cap \dots \cap U_n$, then $U_1, \dots, U_n$ are said

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to be in *general position* (with respect to the point x^*) if the cones of admissible controls $\mathrm{Fd}_{x^*}(U_1), \ldots, \mathrm{Fd}_{x^*}(U_n)$ are in general position. Here,

$$\mathrm{Fd}_{x^*}(U) = \{h \in X: \exists \alpha > 0: x^* + \alpha h \in U\}.$$

Let $F_1, \ldots, F_n: X \to Y \cup \{+\infty\}$, where Y is a K-space and $F_1, \ldots, F_n$ are convex operators whose effective domains are in general position. Then

$$\partial_{x^*}(F_1 + \ldots + F_n) = \partial_{x^*}(F_1) + \ldots + \partial_{x^*}(F_n).$$

III. Formula for the subdifferential of a maximum. Let Y be a K-space. An operator $\alpha \in L(Y, Y)$ is said to be a *multiplier* if $0 \leq \alpha \leq I_Y$, i.e. if $0 \leq \alpha y \leq y$ for all $y \in Y^+$. The set of all multipliers is denoted by $\Lambda(Y)$.

Let $F_1, \ldots, F_n: X \to Y \cup \{+\infty\}$ be convex operators whose effective domains are in general position. Further, let $F_1 \vee \cdots \vee F_n: x \mapsto F_1 x \vee \cdots \vee F_n x$.

The following representation holds:

$$\partial_{x^*}(F_1 \vee \ldots \vee F_n) = \bigcup_{(\alpha_1, \ldots, \alpha_n) \in \Gamma(x^*)} (\partial_{x^*}(\alpha_1 \circ F_1) + \ldots + \partial_{x^*}(\alpha_n \circ F_n)),$$

where the union is taken over the set

$$\Gamma(x^*) = \left\{ (\alpha_1, \ldots, \alpha_n) \in \Lambda(Y)^n: \sum_{k=1}^{n} \alpha_k = I_Y, \quad \sum_{k=1}^{n} \alpha_k \circ F_k x^* = F_1 x^* \vee \ldots \vee F_n x^* \right\}.$$

IV. Formulas for the subdifferential of a composition at an interior point. If the set $\mathrm{dom}(F) - x^*$ is absorbing, i.e. if the point x^* is an interior point of $\mathrm{dom}(F)$, then the formulas we have presented can be substantially improved.

Let Y be a K-space, and let $\mathfrak{A}$ be a weakly order bounded subset of the space $L(X, Y)$. We denote by $(Y^{\mathfrak{A}})_\infty$ the space of bounded (in the order) Y-valued functions on $\mathfrak{A}$. Further, let $\epsilon_{\mathfrak{A}} = \epsilon_{\mathfrak{A}, Y}$ be a *canonical operator,*

$$\varepsilon_{\mathfrak{A}}: f \mapsto \sup \{f(A): A \in \mathfrak{A}\},$$

and let $\Delta_{\mathfrak{A}} = \Delta_{\mathfrak{A}, Y}$ be the identification of Y with the subspace of constant functions in $(Y^{\mathfrak{A}})_\infty$. Moreover, we define the operator $\langle \mathfrak{A} \rangle \in L(X, (Y^{\mathfrak{A}})_\infty)$ by the relation

$$(\langle \mathfrak{A} \rangle x)(A) = Ax, \quad x \in X, \quad A \in \mathfrak{A}.$$

For the operator $F: X \to Y \cup \{+\infty\}$, we denote by $F'(x^*)$ the *directional derivative* of F at the point $x^* \in \mathrm{dom}(F)$, i.e.

$$F'(x^*)x = (o)\text{-}\lim_{\alpha \downarrow 0} (F(x^* + \alpha x) - Fx^*)/\alpha.$$

Now, if $G: Y \to Z \cup \{+\infty\}$, where Z is also a K-space, G is a convex operator, $\mathrm{dom}(G) = Y$, and G is increasing, then G and F are said to be *compatible* when $(G \circ F)'(x^*) = G'(Fx^*) \circ F'(x^*)$. It can be shown that G and F are compatible, e.g., if F is arbitrary and G is (0)-continuous.

If the operators G and F are compatible, then the following representation holds:

$$\partial_{x^*}(G \circ F) = \{A \circ \langle \partial_{x^*}(F) \rangle: A \circ \Delta_{\partial_{x^*}(F)} \in \partial_{Fx^*}(G); A \in L^+((Y^{\partial_{x^*}(F)})_\infty, Z)\},$$

where $L^+((Y^{\partial_{x^*}(F)})_\infty, Z)$ is the cone of positive operators.

In connection with the last representation, we note that the following relation holds for every operator $A \in L^+((Y^{\mathfrak{A}})_\infty, Z)$:

$$A=\alpha\circ\langle A\circ\Delta_{\mathfrak{A},Y}\circ\partial_0(\varepsilon_{\mathfrak{A},Y})\rangle$$

for some $\alpha\in\partial_0(\epsilon_{\mathfrak{A},Z})$.

It follows that the relation

$$\partial_{x^*}(G\circ F)=\bigcup_{A\in\partial_{Fx^*}(G)}\ \bigcup_{\alpha\in\partial_0(\varepsilon_{\partial_{x^*}(F)},Z)}\alpha\circ\langle A\circ\partial_{x^*}(F)\rangle$$

holds for compatible operators G and F.

The last relation also holds under weaker assumptions concerning the space Y.

In conclusion, we apply the formulas obtained for an analysis of a convex programming problem in the following form.

Let $F, G: X\to Y\cup\{+\infty\}$ be convex operators, where, for simplicity, $\mathrm{dom}(F)=\mathrm{dom}(G)=X$. Assume that either $Gx\leq 0$ or $Gx\geq 0$ for every $x\in X$, and that, moreover, the element $-Gx^0$ is a weak order unit in Y for a point x^0.

Consider the *convex program* $Gx\leq 0$, $Fx\to\inf$. An admissible point x^* is optimal if and only if the following system of conditions is compatible:

$$\alpha^{\bullet},\beta^{\bullet}\in\Lambda(Y);$$
$$\mathrm{Ker}\,(\alpha^{\bullet})=\{0\};$$
$$\alpha^{\bullet}+\beta^{\bullet}=I_Y;$$
$$\beta^{\bullet}\circ Gx^{\bullet}=0;$$
$$0\in\alpha^{\bullet}\circ\partial_{x}^{\bullet}(F)+\beta^{\bullet}\circ\partial_{x}^{\bullet}(G).$$

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