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CHANGES OF VARIABLES IN THE YOUNG TRANSFORMATION

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Let X be a vector space, let Y be a K-space (conditionally complete vector lattice), and let $F: X \to Y \cup \{\infty\}$ be a convex operator. For a linear operator $A \in L(X, Y)$, we set

$$F^*A = \sup_{x \in X} (Ax - Fx).$$

The operator F^* is said to be the *Young transformation* of the operator F.

In this note, we announce the rules for changes of variables in the Young transformation. Almost all of the formulas presented are new even for scalar-valued functions. The results presented can be treated as duality theorems in the theory of extremal problems, including vector problems [1].

1. The composition of convex operators. Let $F: X \to Y \cup \{\infty\}$ be a convex operator acting into a partially ordered vector space Y, and let $G: Y \to Z \cup \{\infty\}$ be an increasing convex operator acting into a K-space Z. If the image $F[\mathrm{dom}(F)]$ of the effective domain $\mathrm{dom}(F)$ contains an interior point of $\mathrm{dom}(G)$, then the formula

$$(G \circ F)^* A = \inf\{(B \circ F)^* A + G^* B \colon B \in L^+(Y, Z)\}$$

holds for any $A \in L(X, Z)$. Moreover, it is an exact formula, i.e. the infimum in its right-hand side is attained.

2. The composition of a convex operator with a sublinear operator. If, under the conditions of § 1, the operator F is sublinear, then the following exact formula holds:

$$(G \circ F)^* A = \inf\{G^* B \colon A \in \partial(B \circ F);\ B \in L^+(Y, Z)\}.$$

3. The composition of a sublinear operator with a convex operator. If, under the conditions of § 1, the operator G is sublinear, then the following exact formula holds:

$$(G \circ F)^* = \inf_{B \in \partial(G)} (B \circ F)^*.$$

This fact is *the vector minimax theorem*. Indeed,

$$-(G \circ F)^* \mathbf{0} = \inf_{x \in \mathrm{dom}(F)} \sup_{B \in \partial(G)} B \circ Fx,$$

$$(B \circ F)^* \mathbf{0} = -\inf_{x \in \mathrm{dom}(F)} B \circ Fx.$$

Thus we have the equality

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$$\sup_{B\in\partial(G)} \inf_{x\in\mathrm{dom}(F)} B\circ Fx = \inf_{x\in\mathrm{dom}(F)} \sup_{B\in\partial(G)} B\circ Fx.$$

An important particular case of this assertion was recently communicated to the author by A. M. Rubinov.

4. **The composition of a convex operator with an affine operator.** Let X_1 and X be vector spaces, let Y be a K-space, and let $F\colon X \to Y \cup \{\infty\}$ be a convex operator whose effective domain contains an interior point of the image of the space X_1 under the affine mapping

$$A_x\colon x_1 \mapsto Ax_1 + x,$$

where $A \in L(X_1, X)$ and $x \in X$. Then the following exact formula holds for any $B \in L(X_1, X)$:

$$(F\circ A_x)^{*}B = \inf\{F^{*}C - Cx\colon B = C\circ A;\ C\in L(X, Y)\}.$$

5. **A sum of convex operators.** Let $F_1, \dots, F_n\colon X \to Y \cup \{\infty\}$ be convex operators acting into a K-space Y, where the cones $\mathrm{dom}(H_{F_1}), \dots, \mathrm{dom}(H_{F_n})$ are in general position. Here, $\mathrm{dom}(H_F)$ is the effective domain of the *Hörmander transformation* of the operator F, i.e.

$$\mathrm{dom}(H_F) = \{(x, t)\in X\times R^{+};\ x\in t\,\mathrm{dom}\,(F)\}.$$

Under the assumptions made above, the following formula holds:

$$(F_1 + \dots + F_n)^{*} = F_1^{*} \oplus \dots \oplus F_n^{*},$$

where $\oplus$ is the operation of inf-*convolution*, i.e.

$$F_1^{*}\oplus \dots \oplus F_n^{*}A = \inf\left\{\sum_{k=1}^{n} F_k^{*}A_k\colon A_k\in L(X, Y);\ \sum_{k=1}^{n} A_k = A\right\}.$$

6. **The maximum of convex operators.** Let $F_1, \dots, F_n\colon X \to Y \cup \{\infty\}$ be convex operators acting into a vector lattice Y and such that the cones $\mathrm{dom}(H_{F_1}), \dots, \mathrm{dom}(H_{F_n})$ are in general position. If Z is a K-space, and if $A \in L^{+}(Y, Z)$, then

$$(A\circ(F_1\vee \dots \vee F_n))^{*} = \inf\left\{\bigoplus_{k=1}^{n} (A_k\circ F_k)^{*}\colon A_k\in L^{+}(Y, Z);\quad \sum_{k=1}^{n} A_k = A\right\}.$$

This formula is also exact. In other words, the following system of conditions is compatible for any $B \in L(X, Z)$:

$$B_k\in L(X, Z),\quad A_k\in L^{+}(Y, Z),$$

$$B = \sum_{k=1}^{n} B_k,\quad A = \sum_{k=1}^{n} A_k,$$

$$(A\circ(F_1\vee \dots \vee F_n))^{*}B = \sum_{k=1}^{n} (A_k\circ F_k)^{*}B_k.$$

In this connection, one should also note that a sufficient condition for the cones $\mathrm{dom}(H_{F_1}), \dots, \mathrm{dom}(H_{F_n})$ to be in general position is that the intersection of $\mathrm{dom}(F_1), \dots, \mathrm{dom}(F_n)$ contain an interior point of each of these sets except, possibly, one.

7. The composition with a regular operator. Suppose the convex operator F is *regular*, i.e. $F = \epsilon_{\mathfrak{A}} \circ \langle \mathfrak{A} \rangle_y$, where $\mathfrak{A}$ is a weakly order bounded set in $L(X, Y)$, the element y belongs to the space $(Y^{\mathfrak{A}})_\infty$ of bounded Y-valued functions on A, and $\epsilon_{\mathfrak{A}}$ is the canonical operator [2]. Further, let $G: Y \to Z \cup \{\infty\}$ be an increasing convex operator acting into a K-space Z. If the image $F[X]$ contains an interior point of the effective domain $\operatorname{dom}(G)$, then the following exact formula holds for any $A \in L(X, Z)$:

$$(G \circ F)^* A = \inf \{G^*(B \circ \Delta_{\mathfrak{A}}) - By:\ B \circ \langle \mathfrak{A} \rangle = A;\ B \in L^+((Y^{\mathfrak{A}})_\infty, Z)\},$$

where $\Delta_{\mathfrak{A}}$ is the diagonal imbedding of Y into $(Y^{\mathfrak{A}})_\infty$.

In particular, the last formula disproves the conjecture on the structure of the cone normal to the Lebesgue set of a composition that one encounters in the literature. It is sufficient to consider the case where G is a Banach limit on the space l_∞ and $Fx = x^+$ for $x \in l_\infty$.

8. The Lagrange principle. Let X and X_1 be vector spaces, let Y_1 be a partially ordered Archimedean vector space, and let Y be a K-space. Let $A \in L(X, X_1)$ and $G: X \to Y_1$, $F: X \to Y$ be convex operators, for the sake of simplicity defined everywhere. Assume that the Slater condition holds, i.e. the point $-Gx^0$ is an interior point of Y_1^+ for some $x^0 \in X$, and that Y is a K-space of bounded elements. Consider the vector program

$$Ax = Ax^0, \quad Gx \leqslant 0, \quad Fx \to \inf$$

and let $y \in Y$ be the *value* of this program. We define the sublinear operator of *scalarization* $\tau: Y_1 \to Y$ by the relation

$$\tau y_1 = \inf \{t \in R:\ y_1 \leqslant -tGx^0\} 1,$$

where 1 is a strong unit in Y. Let

$$U = \{x \in X:\ Ax = Ax^0\}$$

and let F_U be the restriction of the operator F to this set.

We form the *Ioffe penalty*

$$\Phi:\ x \mapsto (F_U x - y) \vee \tau \circ Gx.$$

Clearly, Φ is a positive convex operator, and

$$0 = \inf_{x \in X} \Phi x = -\Phi^* 0.$$

Making use of the rules for changes of variables and of the Slater condition, we find operators $\alpha \in L^+(Y_1, Y)$ and $\beta \in L(X_1, Y)$ such that

$$y = \inf_{x \in X} (Fx + \alpha \circ Gx + \beta \circ (Ax - Ax^0)).$$

Thus the value of the vector program under consideration is the value of the unconditional program for the corresponding Lagrangian.

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BIBLIOGRAPHY

1. A. D. Ioffe and V. M. Tihomirov, *The theory of extremal problems*, "Nauka", Moscow, 1974. (Russian)

2. S. S. Kutateladze, Dokl. Akad. Nauk SSSR 230 (1976), 1029 = Soviet Math. Dokl. 17 (1976), 1428.

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